APPLIED PROBLEMS OF MATHEMATICS AND MECHANICS

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INVESTIGATION OF THREE-DIMENSIONAL STRESS-STRAIN STATE OF A SMALL THICKNESS RADIALLY-INHOMOGENEOUS TRANSVERSALLY-ISOTROPIC SPHERE

Abstract

The tree-dimensional stress-strain state of a small thickness radially-inhomogeneous transversally-isotropic sphere is investigated by the method of asymptotic integration of elasticity theory equations.

Assuming that the load given on the lateral surfaces is sufficiently smooth, the inhomogeneous solutions are constructed. Then homogeneous solutions are constructed and asymptotic expansions of homogeneous solutions are obtained. The analysis of stress-strain states corresponding to different types of homogeneous solutions is carried out.

It is shown that the stress-strain state consists of three types: internal stress state, simple fringe effect, and boundary layer. Some boundary layer solutions have no damping properties and they may cover all the domain occupied by the shell.

1. Consider an axi-symmetric problem of elasticity theory for a small thickness radially-inhomogeneous, transversally-isotropic hollow sphere. In the spherical system of coordinates, denote the domain occupied by a spherical shell by $\Gamma = \{r \in [r_1; r_2], \ \theta \in [\theta_1; \theta_2], \ \varphi \in [0; 2\pi]\}$. Assume that the shell contains none of the poles $0; \pi$.

The equilibrium equations in displacements have the form [1]:

$$\left\{ \begin{array}{l} \dfrac{\partial}{\partial p}\left\{e^{-\varepsilon\rho}\left[b_{11}\dfrac{\partial u_\rho}{\partial\rho}+\varepsilon b_{12}\left(u_\theta ctg\theta+2u_\rho+\dfrac{\partial u_\theta}{\partial\theta}\right)\right]\right\}+ \\ +\varepsilon e^{-\varepsilon\rho}\left[\varepsilon\left(2b_{12}-b_{22}-b_{23}-b_{44}\right)\left(\dfrac{\partial u_\theta}{\partial\theta}+u_\theta ctg\theta\right)+\right. \\ +2\left(b_{11}-b_{12}\right)\dfrac{\partial u_\rho}{\partial\rho}+2\left(2b_{12}-b_{22}-b_{23}\right)\varepsilon u_\rho+ \\ \left.+b_{44}\left(\dfrac{\partial^2 u_\theta}{\partial\rho\partial\theta}+\varepsilon\dfrac{\partial^2 u_\rho}{\partial\theta^2}+\dfrac{\partial u_\theta}{\partial\rho}ctg\theta+\varepsilon\dfrac{\partial u_\rho}{\partial\theta}ctg\theta\right)\right]=0 \\ \dfrac{\partial}{\partial\rho}\left\{b_{44}e^{-\varepsilon\rho}\left(\dfrac{\partial u_\theta}{\partial\rho}+e\dfrac{\partial u_\rho}{\partial\theta}-\varepsilon u_\theta\right)\right\}+\varepsilon e^{-\varepsilon\rho}\left[b_{12}\dfrac{\partial^2 u_\rho}{\partial\rho\partial\theta}+\right. \\ +\varepsilon\left(b_{22}+b_{23}+3b_{44}\right)\dfrac{\partial u_\rho}{\partial\theta}-\varepsilon\left(b_{23}+3b_{44}\right)u_\theta+\varepsilon b_{22}\times \\ \left.\times\left(\dfrac{\partial^2 u_\theta}{\partial\theta^2}+\dfrac{\partial u_\theta}{\partial\theta}ctg\theta-u_\theta ctg^2\theta\right)+3b_{44}\dfrac{\partial u_\theta}{\partial\rho}\right]=0 \end{array} \right. \tag{1.1}$$

Here $u_\theta = u_\theta(\rho;\theta)$, $u_\rho = u_\rho(\rho;\theta)$ are the displacement vector components: $\rho = \dfrac{1}{\varepsilon}\ln\left(\dfrac{r}{r_0}\right)$ is a new radial variable; $\varepsilon = \dfrac{1}{2}\ln\left(\dfrac{r_2}{r_1}\right)$is a small parameter, characterizing the sphere's thickness; $r_0 = \sqrt{r_1 r_2}$, $\rho \in [-1; 1]$; $b_{ij} = b_{ij}(\rho)$ are clastic characteristics considered as arbitrary piecewise-continuous functions of variable ρ.

Assume that the following boundary conditions are given on the lateral surfaces:

$$\left.\sigma_{\rho\rho}\right|_{\rho=\pm1}=\frac{e^{-\varepsilon\rho}}{\varepsilon r_0}\left[b_{11}\frac{\partial u_\rho}{\partial\rho}+\varepsilon b_{12}\left(u_\theta ctg\theta+2u_\rho+\frac{\partial u_\theta}{\partial\theta}\right)\right]\Bigg|_{\rho=\pm1}=f^{\pm}(\theta)\,, \quad (1.2)$$

$$\left.\sigma_{\rho\theta}\right|_{\rho=\pm1}=\frac{b_{44}e^{-\varepsilon\rho}}{\varepsilon r_0}\left[\frac{\partial u_\theta}{\partial\rho}+\varepsilon\frac{\partial u_\rho}{\partial\theta}-\varepsilon u_\theta\right]\Bigg|_{\rho=\pm1}=t^{\pm}(\theta)$$

We assume that the loads $f^{\pm}(\theta)$ or $t^{\pm}(\theta)$ given on the face are sufficiently smooth functions, and have order $O(1)$ with respect to ε.

Assume that the arbitrary boundary conditions that bring the sphere about equilibrium are given on the ends of the spherical shell.

2. Consider the construction of special solutions of (1.1) satisfying boundary conditions (1.2), i.e. unhomogeneous solutions.

We look for the inhomogeneous solutions in the form

$$\begin{aligned} u_\rho&=\varepsilon^{-1}\left(u_{\rho0}+\varepsilon u_{\rho1}+...\right),\\ u_\theta&=\varepsilon^{-1}\left(u_{\theta0}+\varepsilon u_{\theta1}+...\right). \end{aligned} \quad (2.1)$$

Substitution of (2.1) in (1.1), (1.2) reduces to the system whose sequential integration gives relations for the coefficients of the expansion (2.1)

$$u_{\rho0}=d_1(\theta)\,;\quad u_{\theta0}=d_2(\theta)\,,$$

$$u_{\rho1}=-\int_0^\rho\frac{b_{12}}{b_{11}}dx\left(2d_1(\theta)+d_2(\theta)\,ctg\theta+d_2'(\theta)\right)+d_3(\theta)\,,$$

$$u_{\theta1}=\rho d_2(\theta)+d_4(\theta)\,,$$

where

$$\begin{cases} p_0\left(d_2''(\theta)+d_2'(\theta)\,ctg\theta\right)+\left(b_{23}^{(0)}-b_{22}^{(0)}-\frac{p_0}{\sin^2\theta}\right)d_2(\theta)-\\ \quad-\left(3b_{44}^{(0)}+g_0\right)d_1'(\theta)=t(\theta)\\ b_{44}^{(0)}\left(d_1''(\theta)+d_1'(\theta)\,ctg\theta\right)-g_0\left(d_2'(\theta)+d_2(\theta)\,ctg\theta\right)-\\ \quad-2g_0d_1(\theta)=-f(\theta) \end{cases}$$

$$f(\theta)=f^{+}(\theta)-f^{-}(\theta)\,;\quad t(\theta)=t^{+}(\theta)-t^{-}(\theta)\,;$$

$$p_k=\int_{-1}^{1}\frac{\rho^k}{e_0(\rho)}d\rho;\quad g_k=\int_{-1}^{1}h_0(\rho)\,\rho^k d\rho;$$

$$b_{44}^{(k)}=\int_{-1}^{1}b_{44}\rho^k d\rho;\quad b_{23}^{(k)}=\int_{-1}^{1}b_{23}\rho^k d\rho;\quad b_{22}^{(k)}=\int_{-1}^{1}b_{22}\rho^k d\rho;\quad e_0(\rho)=\frac{b_{11}}{b_{12}^2-b_{11}b_{22}};$$

$$h_0(\rho)=\frac{b_{11}b_{22}+b_{11}b_{23}-2b_{12}^2}{b_{11}}.$$

3. Any solution of equations (1.1) satisfying the no stress condition on lateral surfaces is said to be an homogeneour solution.

Construct the homogeneous solution. Assume in (1.2) $f^{\pm}(\theta) = t^{\pm}(\theta) = 0$:

$$\sigma_{\rho\rho}\big|_{\rho=\pm1} = \sigma_{\rho\theta}\big|_{\rho=\pm1} = 0. \tag{3.1}$$

Look for the solution of (1.1), (3.1) in the form:

$$u_{\rho}(\rho;\theta) = a(\rho)\, m(\theta)\,; \quad u_{\theta}(\rho;\theta) = c(\rho)\, m'(\theta)\,, \tag{3.2}$$

where the function $m(\theta)$ satisfies the Legendre equation [2]:

$$m''(\theta) + ctg\theta \cdot m'(\theta) + \left(z^2 - \frac{1}{4}\right) m(\theta) = 0. \tag{3.3}$$

Substituting (3.2) into (1.1), (3.1), allowing for (3.3) we get following boundary value problems :

$$\left\{ \begin{array}{l} \dfrac{d}{d\rho}\left\{ e^{-\varepsilon\rho}\left[b_{11}\dfrac{da}{\partial\rho} + \varepsilon b_{12}\left(2a - \left(z^2 - \dfrac{1}{4}\right)c\right)\right]\right\} + \\ +\varepsilon e^{-\varepsilon\rho}\left\{ 2\left(b_{11} - b_{12}\right)\dfrac{da}{d\rho} + 2\left(2b_{12} - b_{22} - b_{23}\right)\varepsilon a - \left(z^2 - \frac{1}{4}\right) \times \right. \\ \left. \times\left[\varepsilon\left(2b_{12} - b_{22} - b_{23} - b_{44}\right)c + \varepsilon b_{44}a + b_{44}\dfrac{dc}{d\rho}\right]\right\} = 0 \\ \dfrac{d}{d\rho}\left\{ b_{44}e^{-\varepsilon\rho}\left(\dfrac{dc}{d\rho} + \varepsilon\left(a - c\right)\right)\right\} + \\ +\varepsilon e^{-\varepsilon\rho}\left[b_{12}\dfrac{da}{\partial\rho} + \varepsilon\left(b_{22} + b_{23} + 3b_{44}\right)a + \right. \\ \left. +3b_{44}\dfrac{dc}{\partial\rho} - \varepsilon\left(b_{23} + 3b_{44}\right)c + \varepsilon b_{22}\left(\dfrac{5}{4} - z^2\right)c\right] = 0; \end{array} \right. \tag{3.4}$$

$$\left\{ \begin{array}{l} \left.\left[b_{11}a'(\rho) + \varepsilon b_{12}\left(2a(\rho) - \left(z^2 - \dfrac{1}{4}\right)c(\rho)\right)\right]\right|_{\rho=\pm1} = 0 \\ b_{44}\left[c'(\rho) + \varepsilon\left(a(\rho) - c(\rho)\right)\right]\big|_{\rho=\pm1} = 0 \end{array} \right. \tag{3.5}$$

For solving (3.4),(3.5) as $\varepsilon \to 0$ we use the asymptotic method based on three interative processes [3]. We can obtain the homogeneous solutions corresponding to the first iterative process, from (2.2), (2.3) if we put $f^{\pm}(\theta) = t^{\pm}(\theta) = 0$. We have:

$$\begin{aligned} u_{\rho}^{(1)}(\rho;\theta) &= B\left(\cos\theta \cdot \ln\left(ctg\frac{\theta}{2}\right) - 1\right), \\ u_{\theta}^{(1)}(\rho;\theta) &= B\left(\sin\theta \cdot \ln\left(ctg\frac{\theta}{2}\right) + ctg\theta\right). \end{aligned} \tag{3.6}$$

The eigen values $z_0^{\pm} = \pm\dfrac{3}{2}$ correspond to these solutions.

The stresses corresponding to these solutions are of the form:

$$\sigma_{\rho\rho}^{(1)} = \sigma_{\rho\theta}^{(1)} = 0; \quad \sigma_{\theta\theta}^{(1)} = -\sigma_{\varphi\varphi}^{(1)} = \frac{\left(b_{22} - b_{23}\right)e^{-\varepsilon\rho}}{r_0\sin^2\theta}B. \tag{3.7}$$

Now, appeal to the second iterative process. We'll look for the solution in the form:

$$a^{(2)}(\rho) = a_{20}(\rho) + \varepsilon a_{21}(\rho) + ...,$$

$$c^{(2)}(\rho)=\varepsilon\left(c_{20}(\rho)+\varepsilon c_{21}(\rho)+...\right), \tag{3.8}$$
$$z=\varepsilon^{-1/2}\left(\alpha_0+\varepsilon\alpha_1+...\right).$$

Substituting (3.8) in (3.4),(3.5) , after some transformations we get:

$$u_\rho^{(2)}(\rho;\theta)=\sum_{j=1}^{4}T_j a_j^{(2)}(\rho)\,m_j(\theta),$$

$$u_\theta^{(2)}(\rho;\theta)=\sum_{j=1}^{4}T_j c_j^{(2)}(\rho)\,m'_j(\theta), \tag{3.9}$$

$$a_j^{(2)}(\rho)=1+\varepsilon\left[\frac{\alpha_{0j}^2p_1+b_{22}^{(0)}-b_{23}^{(0)}}{p_0}\int_0^\rho\frac{b_{12}}{b_{11}}dx-\alpha_{0j}^2\int_0^\rho\frac{b_{12}}{b_{11}}xdx\right]+O\left(\varepsilon^2\right),$$

$$c_j^{(2)}(\rho)=\varepsilon\left[\frac{\alpha_{0j}^2p_1+g_0}{\alpha_{0j}^2\;p_0}-\rho+O(\varepsilon)\right].$$

For determining α_{0j} we obtain the biquadratic equation:

$$\left(p_0p_2+b_{22}^{(1)}(p_1+t_1)\right)\alpha_{0j}^4+(g_1p_0-g_0p_1)\,\alpha_{0j}^2+2g_0p_0-g_0^2=0 \tag{3.10}$$

The stresses corresponding to solutions (3.9) accept the form:

$$\sigma_{\rho\rho}^{(2)}=\frac{\varepsilon}{r_0}\sum_{j=1}^{4}T_j\left\{\frac{g_0+\alpha_{0j}^2p_1}{p_0}\left[-\alpha_{0j}^2\int_{-1}^{\rho}\left(\int_{-1}^{y}\frac{1}{e_0(x)}dx\right)dy-\int_{-1}^{\rho}h_0(x)\,dx\right]+\right.$$
$$+\alpha_{0j}^4\int_{-1}^{\rho}\left(\int_{-1}^{y}\frac{1}{e_0(x)}xdx\right)dy+\alpha_{0j}^2\left[\int_{-1}^{\rho}\left(\int_{-1}^{y}h_0(x)\,dx\right)dy+\int_{-1}^{\rho}h_0(x)\,xdx\right]+$$
$$\left.+2\int_{-1}^{\rho}h_0(x)\,dx+O(\varepsilon)\right\}m_j(\theta), \tag{3.11}$$

$$\sigma_{\rho\theta}^{(2)}=\frac{\varepsilon}{r_0}\sum_{j=1}^{4}T_j\left\{-\int_{-1}^{\rho}h_0(x)\,dx+\alpha_{0j}^2\int_{-1}^{\rho}\frac{1}{e_0(x)}xdx-\right.$$
$$\left.-\frac{g_0+\alpha_{0j}^2p_1}{p_0}\int_{-1}^{\rho}\frac{1}{e_0(x)}dx+O(\varepsilon)\right\}m'_j(\theta),$$

$$\sigma_{\varphi\varphi}^{(2)}=\frac{1}{r_0}\sum_{j=1}^{4}T_j\left\{\left[\frac{g_0+\alpha_{0j}^2p_1}{p_0e(\rho)}+h_0(\rho)-\alpha_{0j}^2\frac{\rho}{e(\rho)}+O(\varepsilon)\right]\cdot m_j(\theta)+\right.$$
$$\left.+\varepsilon\left[(b_{22}-b_{23})\left(\frac{g_0+\alpha_{0j}^2p_1}{\alpha_{0j}^2p_0}-\rho\right)ctg\theta+O(\varepsilon)\right]m'_j(\theta)\right\},$$

$$\sigma_{\theta\theta}^{(2)}=\frac{1}{r_0}\sum_{j=1}^{4}T_j\left\{\left[\frac{g_0+\alpha_{0j}^2p_1}{p_0e(\rho)}+h_0(\rho)-\alpha_{0j}^2\frac{\rho}{e_0(\rho)}+O(\varepsilon)\right]m_j(\theta)+\right.$$

$$\left.+\varepsilon\left[(b_{23}-b_{22})\left(\frac{g_0+\alpha_{0j}^2p_1}{\alpha_{0j}^2p_0}-\rho\right)ctg\theta+O(\varepsilon)\right]m_j'(\theta)\right\},$$

where $e(\rho)=\dfrac{b_{12}^2-b_{11}b_{23}}{b_{11}}$.

For the second iterative process, the principal term of the asymptotic solution of the equation (3.3) has the form:

$$m_k(\theta)=\begin{cases}\dfrac{1}{\sqrt{\sin\theta}}\dfrac{1}{\sqrt[4]{-\alpha_{0k}^2}}\exp\left[-\varepsilon^{-1/2}\sqrt{-\alpha_{0k}^2}(\theta-\theta_1)\right]\left(1+O\left(\varepsilon^{1/2}\right)\right); \\ \text{in the neighborhood } \theta=\theta_1 \\ \dfrac{1}{\sqrt{\sin\theta}}\dfrac{1}{\sqrt[4]{-\alpha_{0k}^2}}\exp\left[\varepsilon^{-1/2}\sqrt{-\alpha_{0k}^2}(\theta-\theta_2)\right]\left(1+O\left(\varepsilon^{1/2}\right)\right); \\ \text{in the neighborhood } \theta=\theta_2\end{cases} \tag{3.12}$$

According to the third iterative process, the solution of (3.4), (3.5) is found in the form:

$$a^{(3)}(\rho)=a_{30}(\rho)+\varepsilon a_{31}(\rho)+...,$$

$$c^{(3)}(\rho)=\frac{\varepsilon}{\beta_0}\left(c_{30}(\rho)+\varepsilon c_{31}(\rho)+...\right), \tag{3.13}$$

$$z=i\varepsilon^{-1}\left(\beta_0+\varepsilon\beta_1+...\right).$$

After substituting in (3.13) in (3.4), (3.5) for the first terms we get:

$$T(\beta_0)\overline{q}_0=\left\{d(\beta_0)\overline{q}_0;\ \ l(\beta_0)\overline{q}_0=\overline{0}\text{ for }\rho=\pm1\right\}=\overline{0} \tag{3.14}$$

$$d(\beta_0)\overline{q}_0=\left(B_0+\beta_0B_1+\beta_0^2B_2\right)\overline{q}_0,\ \ l(\beta_0)\overline{q}_0=(C_0+\beta_0C_1)\overline{q}_0,$$

$$B_0=\begin{Vmatrix}\partial(b_{11}\partial) & 0\\ 0 & \partial(b_{44}\partial)\end{Vmatrix};\ B_1=\begin{Vmatrix}0 & b_{44}\partial+\partial(b_{12})\\ \partial(b_{44})+b_{12}\partial & 0\end{Vmatrix}$$

$$B_2=\begin{Vmatrix}b_{44} & 0\\ 0 & b_{22}\end{Vmatrix};\ C_0=\begin{Vmatrix}b_{11}\partial & 0\\ 0 & b_{44}\partial\end{Vmatrix};\ C_1=\begin{Vmatrix}0 & b_{12}\\ b_{44} & 0\end{Vmatrix},\overline{q}_0=(a_{30},\ c_{30})^T.$$

The spectral problem (3.14) describes the potential solution of a transversally-isotropic plate inhomogeneous in thickness. Unlike the isotropic case, purely imaginary values [4] also may be accepted for a transversally-isotropic plate β_{0k} inhomogeneous in thickness. By substitution

$$a_{30}=-\beta_0^{-3}\left(e_0\psi''\right)'+\beta_0^{-1}b_{44}^{-1}\psi'+\beta_0^{-1}(e_1\psi)'$$

$$c_{30}=\beta_0^{-2}e_0\psi''-e_1\psi$$

$$e_0=b_{11}\chi,\ \ e_1=b_{12}\chi,\ \ e_2=b_{22}\chi,\ \ \chi=\left(b_{12}^2-b_{11}b_{22}\right)^{-1}$$

the spectral problem (3.14) is reduced to the following problem:

$$\begin{cases} \left(e_0\psi''(\rho)\right)'' - \beta_0^2\left[\left(e_1\psi(\rho)\right)'' + e_1\psi''(\rho) + \left(b_{44}^{-1}\psi'(\rho)\right)'\right] + \beta_0^4 e_2\psi(\rho) = 0 \\ \psi'(\rho)\big|_{\rho=\pm1} = 0, \quad \beta_0\psi(\rho)\big|_{\rho=\pm1} = 0 \end{cases} \tag{3.15}$$

(3.15) is the generalization of P.A. Papkovich's [4-7] spectral problem for the inhomogeneous transversally-isotropic case.

In the following stage we get a boundary value problem for determining $\overline{q}_1 = (a_{31}, c_{31})^T$ and β_1 :

$$\begin{cases} \left(B_0 + \beta_0 B_1 + \beta_0^2 B_2\right)\overline{q}_1 = \left(\rho\left(B_0 + \beta_0 B_1 + \beta_0^2 B_2\right) - 2\beta_0\beta_1 B_2 + \right. \\ \qquad \left. + A_0 + \beta_0 A_1 + \beta_1 A_2\right)\overline{q}_0; \\ \left(C_0 + \beta_0 C_1\right)\overline{q}_1\big|_{\rho=\pm1} = \left(\rho\left(C_0 + \beta_0 C_1\right) + \beta_1 C_3 + C_4\right)\overline{q}_0\big|_{\rho=\pm1}; \end{cases} \tag{3.16}$$

$$A_0 = \left\| \begin{matrix} -2\partial(b_{12}) + (2b_{12} - b_{11})\partial & 0 \\ 0 & \partial(b_{44}) - 2b_{44}\partial \end{matrix} \right\|$$

$$A_1 = \left\| \begin{matrix} 0 & (b_{44} + b_{23} + b_{22} - b_{12}) \\ -(b_{22} + b_{23} + 2b_{44}) & 0 \end{matrix} \right\|$$

$$A_2 = \left\| \begin{matrix} 0 & -2\left(\partial(b_{12}) + b_{44}\partial\right) \\ 0 & 0 \end{matrix} \right\|$$

$$C_3 = \left\| \begin{matrix} 0 & -2b_{12} \\ 0 & 0 \end{matrix} \right\|, \quad C_4 = \left\| \begin{matrix} -2b_{12} & 0 \\ 0 & b_{44} \end{matrix} \right\|$$

The solvability condition of (3.16) is the orthogonality of the right side of the solution of the conjugated problem:

$$T^*(\beta_0)\,\overline{q}_0^* = T\left(-\overline{\beta}_0\right)\overline{q}_0^* = \overline{0} \tag{3.17}$$

From (3.17) for β_1 we have:

$$\beta_1 = \frac{M_2}{2M_1}$$

$$M_2 = \int_{-1}^{1}\left\{\beta_0\left[\frac{d}{d\rho}(\rho b_{12}c_{30})\,\overline{a}_{30}^* + \rho b_{44}\frac{dc_{30}}{d\rho}\overline{a}_{30}^* + \frac{d}{d\rho}(\rho b_{44}a_{30})\,\overline{c}_{30}^* + \right.\right.$$

$$\left. + \rho b_{12}\frac{da_{30}}{d\rho}\overline{c}_{30}^* + (b_{44} + b_{23} + b_{22} - 2b_{12})\,c_{30}\overline{a}_{30}^* - (b_{22} + b_{23} + 3b_{44})\,a_{30}\overline{c}_{30}^*\right] +$$

$$+ \beta_0^2\rho\left(b_{44}a_{30}\overline{a}_{30}^* + b_{22}c_{30}\overline{c}_{30}^*\right) + \overline{c}_{30}^*\frac{d}{d\rho}(b_{44}c_{30}) + \frac{d}{d\rho}\left(\rho b_{11}\frac{da_{30}}{d\rho}\right)\overline{a}_{30}^* +$$

$$\left. + \overline{c}_{30}^*\frac{d}{d\rho}\left(\rho b_{44}\frac{dc_{30}}{d\rho}\right) - 3b_{44}\frac{dc_{30}}{d\rho}\overline{c}_{30}^* - 2(b_{11} - b_{12})\frac{da_{30}}{d\rho}\overline{a}_{30}^* - 2\overline{a}_{30}^*\frac{d}{d\rho}(b_{12}a_{30})\right\}d\rho;$$

$$M_1 = \int_{-1}^{1}\left\{\frac{d}{d\rho}(b_{12}c_{30})\,\overline{a}_{30}^* + b_{44}\frac{dc_{30}}{d\rho}\overline{a}_{30}^* + \beta_0\left(b_{44}a_{30}\overline{a}_{30}^* + b_{22}c_{30}\overline{c}_{30}^*\right)\right\}d\rho.$$

The solutions corresponding to the third iterative process are of the form:

$$u_\rho(\rho;\theta)=\sum_{k=1}^{\infty}D_k\left[-\beta_{0k}^{-3}\left(e_0\psi_k''\right)'+\beta_{0k}^{-1}b_{44}^{-1}\psi_k'+\beta_{0k}^{-1}\left(e_1\psi_k\right)'+O(\varepsilon)\right]m_k(\theta) \quad (3.18)$$

$$u_\rho(\rho;\theta)=\varepsilon\sum_{k=1}^{\infty}D_k\left[\beta_{0k}^{-3}e_0\psi_k''-\beta_{0k}^{-1}e_1\psi_k+O(\varepsilon)\right]m_k'(\theta).$$

The stresses corresponding to the third iterative process have the form:

$$\sigma_{\rho\rho}^{(3)}=\frac{1}{\varepsilon r_0}\sum_{k=1}^{\infty}\left[-\beta_{0k}\psi_k+O(\varepsilon)\right]m_k(\theta);$$

$$\sigma_{\rho\theta}^{(3)}=\frac{1}{r_0}\sum_{k=1}^{\infty}\left[\beta_{0k}^{-1}\psi_k'+O(\varepsilon)\right]m_k'(\theta);$$

$$\sigma_{\theta\theta}^{(3)}=\frac{1}{\varepsilon r_0}\sum_{k=1}^{\infty}\left[-\beta_{0k}^{-1}\psi_k''+O(\varepsilon)\right]m_k(\theta); \quad (3.19)$$

$$\sigma_{\varphi\varphi}^{(3)}=\frac{1}{\varepsilon r_0}\sum_{k=1}^{\infty}\left[\beta_{0k}^{-1}\chi\left(b_{11}b_{23}-b_{12}^2\right)\psi_k''+\beta_{0k}\chi b_{12}\left(b_{22}-b_{23}\right)\psi_k+O(\varepsilon)\right]m_k(\theta).$$

For the third iterative process, the principal term of the asymptotic solution of the equation (3.3) is of the form:

$$m_k(\theta)=\begin{cases}\dfrac{1}{\sqrt{\sin\theta}}\beta_{0k}^{-1/2}\exp\left[-\varepsilon^{-1}\sqrt{\beta_{0k}^2}\left(\theta-\theta_1\right)\right]\left(1+O(\varepsilon)\right); \\ \text{in the neighborhood } \theta=\theta_1 \\ \dfrac{1}{\sqrt{\sin\theta}}\beta_{0k}^{-1/2}\exp\left[\varepsilon^{-1}\sqrt{\beta_{0k}^2}\left(\theta-\theta_2\right)\right]\left(1+O(\varepsilon)\right); \\ \text{in the neighborhood } \theta=\theta_2\end{cases} \quad (3.20)$$

4. Represent the displacements in the form:

$$u_\rho(\rho;\theta)=u_\rho^{(1)}(\rho;\theta)+\sum_{k=1}^{\infty}E_k a_k(\rho)\,m_k(\theta);$$

$$u_\theta(\rho;\theta)=u_\theta^{(1)}(\rho;\theta)+\sum_{k=1}^{\infty}E_k c_k(\rho)\,m_k'(\theta). \quad (4.1)$$

The second term contains the displacements determined by the second and third group of solutions.

For the stresses we have;

$$\sigma_{\theta\theta}=\sigma_{\theta\theta}^{(1)}+\sum_{k=1}^{\infty}E_k\left(\sigma_{1k}^{(1)}(\rho)\,m_k(\theta)+\sigma_{1k}^{(2)}(\rho)\,m_k'(\theta)\,ctg\theta\right); \quad (4.2)$$

$$\sigma_{\rho\theta}=\sum_{k=1}^{\infty}E_k\sigma_{2k}(\rho)\,m_k'(\theta);$$

$$\sigma_{1k}^{(1)}(\rho)=\frac{e^{-\varepsilon\rho}}{\varepsilon r_0}\left[b_{12}a_k'(\rho)+\varepsilon(b_{22}+b_{23})a_k(\rho)-\varepsilon b_{22}\left(z_k^2-\frac{1}{4}\right)c_k(\rho)\right];$$

$$\sigma_{1k}^{(2)}(\rho)=\frac{e^{-\varepsilon\rho}}{r_0}(b_{23}-b_{22})c_k(\rho);$$

$$\sigma_{2k}(\rho)=\frac{e^{-\varepsilon\rho}}{\varepsilon r_0}b_{44}\left(c_k'(\rho)+\varepsilon(a_k(\rho)-c_k(\rho))\right).$$

Investigate the relation of homogeneous relations with the principal vector P of stresses acting in the section $\theta=const$. We have:

$$P=2\pi\int_{r_1}^{r_2}(\sigma_{r\theta}\cos\theta-\sigma_{\theta\theta}\sin\theta)\,r\sin\theta dr=$$

$$=2\pi\varepsilon r_0^2\sin\theta\int_{-1}^{1}(\sigma_{\rho\theta}\cos\theta-\sigma_{\theta\theta}\sin\theta)\,e^{2\varepsilon\rho}d\rho. \tag{4.3}$$

Substituting (4.2) in (4.3), we get:

$$P=-4\pi\varepsilon r_0B\int_{-1}^{1}G(\rho)\,e^{\varepsilon\rho}d\rho+$$

$$+2\pi\varepsilon r_0^2\sin\theta\sum_{k=1}^{\infty}E_k\left[d_{2k}m_k'(\theta)\cos\theta-d_{1k}m_k(\theta)\sin\theta\right], \tag{4.4}$$

where

$$b_{22}-b_{23}=2G,\quad d_{1k}=\int_{-1}^{1}\sigma_{1k}^{(1)}(\rho)\,e^{2\varepsilon\rho}d\rho;\quad d_{2k}=\int_{-1}^{1}\left[\sigma_{2k}(\rho)-\sigma_{1k}^{(2)}(\rho)\right]e^{2\varepsilon\rho}d\rho.$$

Prove that $d_{1k}=d_{2k}=0\quad(k=1,2,...)$. For that we consider the following boundary value problem:

$$\begin{cases}\sigma_{\theta\theta}|_{\theta=\theta_j}=\sigma_{1k}^{(1)}(\rho)\,m_k(\theta_j)+\sigma_{1k}^{(2)}(\rho)\,m_k'(\theta_j)\,ctg\theta_j\\ \sigma_{\rho\theta}|_{\theta=\theta_j}=\sigma_{2k}(\rho)\,m_k'(\theta_j)\end{cases}\quad(j=1,2) \tag{4.5}$$

The $''k''$-th terms in the sums (4.2) is the solution of this problem.

The principal vector that corresponds to the stress state of the problem (4.5) in the section $\theta=const$ is reduced to the following form:

$$P_k=2\pi\varepsilon r_0^2\sin\theta\left[d_{2k}m_k'(\theta)\cos\theta-d_{1k}m_k(\theta)\sin\theta\right]. \tag{4.6}$$

According to the solvability condition of the elasticity theory problem, the principal vector P_k should be independent of the variable θ. However in (4.6) the right side depends on θ. Hence it follows that $P_k=0$, i.e.

$$d_{2k}m_k'(\theta)\cos\theta-d_{1k}m_k(\theta)\sin\theta=0. \tag{4.7}$$

By the linear independence of $m_k(\theta)\sin\theta$ and $m'_k(\theta)\cos\theta$ it follows from (4.7) that $d_{1k} = d_{2k} = 0$ for any $''k''$. Thus, for the main vector P we get.

$$P = -4\pi\varepsilon r_0 B \int\limits_{-1}^{1} Ge^{\varepsilon\rho}d\rho, \tag{4.8}$$

The stress state corresponding to the second and third groups of solutions is self-balanced in each section $\theta = const$. The solution of (3.6) corresponding to the first asymptotic process determines the internal stress-strain state of the spherical shell.

The stress states corresponding to the solutions of (3.9) are fringe effects in applied theory of shells. The third asymptotic process determines the solutions (3.18) that are of boundary layer character. The first terms of (3.18) are equivalent to Saint-Venant's fringe effect of an inhomogeneous transversally isotropic plate.

For imaginary β_{0k} the Saint-Venant boundary layer damps very weakly, and the solution (3.18) should be attached to permeating solutions. In this case the stress-strain state of a transversally-isotropic and isotropic shells differ qualitatively [4].

5. Assume that the following boundary conditions are given on the ends of the spherical shell:

$$\sigma_{\theta\theta}|_{\theta=\theta_j} = f_{1j}(\rho)\,; \quad \sigma_{\rho\theta}|_{\theta=\theta_j} = f_{2j}(\rho)\,, \tag{5.1}$$

where $f_{1j}(\rho)\,,\ f_{2j}(\rho)$ $(j = 1, 2)$ are sufficiently smooth functions satisfying the equilibrium conditions.

As it was shown, the not self-balanced part of stresses may be removed by means of the permeating solution (3.6). The relation of the constant B with the principal vector P is given by the equality (4.8).

We'll look for the solution in the form (4.1). For determining the constants E_k, we'll use the Lagrange variational principle [8].

Since the homogeneous solutions satisfy the equilibrium equation and boundary conditions on the lateral surface, the variational principle accepts the following form:

$$\sum_{j=1}^{2}\int\limits_{-1}^{1} \left[(\sigma_{\theta\theta} - f_{1j})\,\delta u_\theta + (\sigma_{\rho\theta} - f_{2j})\,\delta u_\rho\right]\Big|_{\theta=\theta_j} \cdot e^{2\varepsilon\rho}d\rho = 0. \tag{5.2}$$

Assuming δE_k as independent variations, from (5.2) we get an infinite system of linear algebraic equations:

$$\sum_{j=1}^{\infty} F_{jk}E_k = t_{0j} \quad \left(j = \overline{1,\infty}\right) \tag{5.3}$$

$$F_{jk} = \int\limits_{-1}^{1} \sigma_{1k}^{(1)}(\rho)\, c_j(\rho)\, e^{2\varepsilon\rho}d\rho \cdot \sum_{s=1}^{2} m_k(\theta_s)\, m'_j(\theta_s) +$$

$$+\int\limits_{-1}^{1} \sigma_{1k}^{(2)}(\rho)\, c_j(\rho)\, e^{2\varepsilon\rho}d\rho \cdot \sum_{s=1}^{2} m'_k(\theta_s)\, m'_j(\theta_s)\, ctg\theta_s +$$

$$+\int_{-1}^{1}\sigma_{2k}(\rho)\,a_j(\rho)\,e^{2\varepsilon\rho}d\rho\cdot\sum_{s=1}^{2}m'_k(\theta_s)\,m_j(\theta_s)\,,$$

$$t_{0j}=\sum_{s=1}^{2}\left[m'_j(\theta_s)\int_{-1}^{1}f^*_{1s}(\rho)\,c_j(\rho)\,e^{2\varepsilon\rho}d\rho+m_j(\theta_s)\int_{-1}^{1}f_{2s}(\rho)\,a_j(\rho)\,e^{2\varepsilon\rho}d\rho\right];$$

$$f^*_{1s}(\rho)=f_{1s}(\rho)-\frac{2BG(\rho)}{r_0e^{\varepsilon\rho}\sin^2\theta_s}.$$

The solvability and convergence of the reduction method for the system (5.3) is proved in [9].

Minding that $\sigma^{(2)}_{\rho\theta}=O(\sqrt{\varepsilon})\,,\ \sigma^{(2)}_{\theta\theta}=O(1)\,,$ revise the suppositions for the external load. Assume $f_{2j}=O(1)\,.$ Expand the tangential stresses given on the ends, in the form:

$$f_{2j}=f^{(1)}_{2j}+f^{(2)}_{2j}$$

$$f^{(1)}_{2j}=\int_{-1}^{1}f_{2j}(\rho)\,d\rho;\quad f^{(2)}_{2j}=f_{2j}-f^{(1)}_{2j}$$

Note that $f^{(1)}_{2j}=O(\sqrt{\varepsilon})\,;\ f^{(2)}_{2j}=O(1)$ and $f^{(1)}_{2j}\sim\varepsilon^{1/2}f^*_{2j}$.

We look for the unknown constants T_j, D_k in the form:

$$T_j=T_{j0}+\varepsilon T_{j1}+...; \tag{5.4}$$

$$D_k=\varepsilon D_{k0}+\varepsilon^2D_{k1}+...; \tag{5.5}$$

After substituting (5.4),(5.5) in (5.3), we have:

$$\sum_{j=1}^{4}q_{kj}T_{j0}=\tau'_k;\quad\left(k=\overline{1,4}\right) \tag{5.6}$$

$$\sum_{j=1}^{\infty}M_{kj}D_{k0}=\tau''_j;\quad j=\overline{1,\infty}; \tag{5.7}$$

$$q_{kj}=\frac{\sqrt{-\alpha^2_{0j}}-\sqrt{-\alpha^2_{0k}}}{\sqrt[4]{\alpha^2_{0k}\alpha^2_{0j}}}\cdot\frac{\alpha^2_{0j}\left(p_1^2-p_0p_2\right)+\left(p_1g_0-p_0g_1\right)}{p_0}$$

$$\tau'_k=\frac{r_0\sin\theta_1\sin\theta_2}{\sin\theta_1-\sin\theta_2}\sum_{s=1}^{\infty}\frac{1}{\sqrt{\sin\theta_s}}\times$$

$$\left[(-1)^s\frac{\sqrt{-\alpha^2_{0k}}}{\sqrt[4]{-\alpha^2_{0k}}}\int_{-1}^{1}f^*_{1s}(\rho)\left(\frac{\alpha^2_{0k}p_1+g_0}{\alpha^2_{0k}p_0}-\rho\right)d\rho+\frac{1}{\sqrt[4]{-\alpha^2_{0k}}}f^*_{2s}\right];$$

$$M_{kj}=\frac{1}{\sqrt{\beta_{0j}\beta^3_{0k}}}\int_{-1}^{1}\left[-e_1\psi_k\psi''_j+b^{-1}_{44}\psi'_j\psi'_k+\beta^2_{0j}e_2\psi_k\psi_j\right]d\rho+$$

$$+\frac{1}{\sqrt{\beta_{0k}\beta_{0j}^{3}}}\int_{-1}^{1}\left[e_1\psi_k\psi_j''-\beta_{0j}^{2}e_2\psi_k\psi_j\right]d\rho;$$

$$\tau_j''=\frac{r_0\sin\theta_1\sin\theta_2}{(\sin\theta_1-\sin\theta_2)\sqrt{\beta_{0j}}}\left\langle-\frac{1}{\sqrt{\sin\theta_1}}\int_{-1}^{1}f_{11}^{*}(\rho)\left[\beta_{0j}^{-2}e_0\psi_j''-e_1\psi_j\right]d\rho+\right.$$

$$+\frac{1}{\sqrt{\sin\theta_2}}\int_{-1}^{1}f_{12}^{*}(\rho)\left[\beta_{0j}^{-2}e_0\psi_j''-e_1\psi_j\right]d\rho+\frac{1}{\sqrt{\sin\theta_1}}\int_{-1}^{1}f_{21}(\rho)\left[-\beta_{0j}^{-3}\left(e_0\psi_j''\right)'+\right.$$

$$\left.+\beta_{0j}^{-1}b_{44}^{-1}\psi_j'+\beta_{0j}^{-1}\left(e_1\psi_j\right)'\right]d\rho+$$

$$\left.+\frac{1}{\sqrt{\sin\theta_2}}\int_{-1}^{1}f_{22}(\rho)\left[-\beta_{0j}^{-3}\left(e_0\psi_j''\right)'+\beta_{0j}^{-1}b_{44}^{-1}\psi_j'+\beta_{0j}^{-1}\left(e_1\psi_j\right)'\right]d\rho\right\rangle$$

The definition of T_{jp}, D_{kp} $(p=1,2,...)$ is invariably reduced to the inversion of one and the same matrices that coincide with the matrices of the system (5.6),(5.7).

The general solution of homogeneous problem on definition of stress-strain state of a shell will be a superposition of solutions corresponding to the three iterative processes described above.

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