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INVESTIGATION OF THE CLASSICAL SOLUTION OF A BOUNDARY VALUE PROBLEM FOR A SECOND ORDER PARABOLIC EQUATION WITH NON-CLASSICAL BOUNDARY CONDITIONS

Abstract

The existence and uniqueness of the classic solution is proved for a second order parabolic equation with non-classical boundary conditions.

In the domain $D_T = \{(x,t) : 0 \leq x \leq 1, 0 \leq t \leq T\}$ consider the equation

$$a_1(t)\, u_t(x,t) + a_0(t)\, u(x,t) = u_{xx}(x,t) + f(x,t) \tag{1}$$

with boundary conditions:

$$u(x,0) + \delta u(x,T) = \varphi(x) \quad (0 \leq x \leq 1), \tag{2}$$

$$u_x(0,t) = 0, \quad u_x(1,t) + d u_{xx}(1,t) = 0 \quad (0 \leq t \leq T), \tag{3}$$

where $d > 0$, $\delta \geq 0$ are given numbers, $a_0(t)$, $a_1(t) > 0$, $\varphi(x)$, $f(x,t)$ are given functions, $u(x,t)$ is a desired function, and under the classic solution of problem (1)-(3) we understand the function $u(x,t)$ continuous in the closed domain D_T together with all its derivatives contained in equation (1) and satisfying all the conditions (1)-(3) in the ordinary sense. The following lemma is valid.

Lemma 1. *Let* $\delta \geq 0$, $0 < a_1(t)$, $a_0(t) \in C[0,1]$

$$\varphi(x) \in C[0,1], \quad d\varphi(1) + \int_0^1 \varphi(x)\, dx = 0, \tag{4}$$

$$f(x,t) \in C^{1,0}_{x,t}(D_T), \quad df(1,t) + \int_0^1 f(x,t)\, dx = 0 \quad (0 \leq t \leq T). \tag{5}$$

Then

$$du(1,t) + \int_0^1 u(x,t)\, dx = 0 \quad (0 \leq t \leq T). \tag{6}$$

is fulfilled for the classical solution of problem (1)-(3).

Proof. Allowing for (3) and (5), from equation (1) we have:

$$a_1(t) \frac{d}{dt}\left[du(1,t) + \int_0^1 u(x,t)\, dx \right] +$$

$$+ a_0(t) \left[du(1,t) + \int_0^1 u(x,t)\, dx \right] = 0 \quad (0 \leq t \leq T).$$

Accept the denotation

$$y(t) \equiv du(1,t) + \int_0^1 u(x,t)\,dx = 0 \quad (0 \le t \le T), \tag{7}$$

and rewrite the last relation in the form:

$$a_1(t)\,y'(t) + a_0(t)\,y(t) = 0 \quad (0 \le t \le T). \tag{8}$$

Allowing for (2) and (4), from (7) it is easy to see that

$$y(0) + y(T) = d\varphi(1) + \int_0^1 \varphi(x)\,dx = 0. \tag{9}$$

Obviously, the general solution of (8) is of the form:

$$y(t) = ce^{-\int_0^t \frac{a_0(\tau)}{a_1(\tau)}d\tau} \qquad (0 \le t \le T). \tag{10}$$

Hence, allowing for (9), we get:

$$c\left(1 + \delta e^{-\int_0^T \frac{a_0(\tau)}{a_1(\tau)}d\tau}\right) = 0. \tag{11}$$

By $\delta \ge 0$, from (11) we get $c = 0$ and substituting it into (10), we deduce $y(t) = 0 \quad (0 \le t \le T)$. Consequently, from (7) it is clear that condition (6) is also fulfilled. The lemma is proved.

Now, for investigating the classical solution of problem (1)-(3), give some known facts and establish some new auxiliary facts.

Consider the spectral problem [3]

$$y''(x) + \lambda y(x) = 0 \quad (0 \le x \le 1), \tag{12}$$

$$y'(0) = 0, \quad y'(1) = d\lambda y(1) \quad (d > 0), \tag{13}$$

that has only eigen functions $y_k(x) = \sqrt{2}\cos\left(\sqrt{\lambda_k}x\right)$, $k = 0,1,...$, with positive eigen-values λ_k from the equation $tg\sqrt{\lambda} = -d\sqrt{\lambda}$. We assign zero index to any eigen function, and enumerate the remaining ones in the order of increasing of eigen values.

It is known [3] that beginning with some number N, the following estimations hold:

$$\left|\sqrt{\lambda_k} - \pi/2 - (k-1)\pi\right| < \frac{1}{(d\pi k)}. \tag{14}$$

Compare the system $\{y_k(x)\}$ without the function $y_0(x)$ with the known system $\{v_k(x)\}$, $v_k(x) = \sqrt{2}\cos\sqrt{\mu_k}x$, where $\sqrt{\mu_k} = \frac{\pi}{2} + \pi(k-1)$, $k = 1,2,...$, that is an orthonormalized basis in $L_2(0,1)$. Similarly [1], for $k \ge N$ allowing for (14) the following relations are true:

$$\|y_k(x) - v_k(x)\|^2_{L_2(0,1)} < \frac{2}{3(dk\pi)^2}.$$

Thus,

$$\sum_{k=1}^{\infty}\left\|y_k(x)-v_k(x)\right\|^2_{L_2(0,1)}<\frac{1}{9d^2}, \tag{15}$$

hence the convergence of the series from the left hand side of this inequality holds.

The following lemma is proved similar to [1].

Lemma 2. *The biorthogonally conjugated system* $\{z_k(x)\}$, $k=1,2,...$, *is determined by the formula*

$$z_k(x)=\sqrt{2}\left(\cos\left(\sqrt{\lambda_k}x\right)-\cos\left(\sqrt{\lambda_k}\right)\right)/\left(1+d\cos^2\left(\sqrt{\lambda_k}\right)\right).$$

The following theorem is valid.

Theorem 1 [3]. *The system* $\{y_k(x)\}$, $k=1,2,...$ *forms a Riesz basis in the space* $L_2(0,1)$.

Let now $\eta_k(x)=\sqrt{2}\sin\left(\sqrt{\lambda_k}x\right)$, $\xi_k(x)=\sqrt{2}\sin\left(\sqrt{\mu_k}x\right)$, $k=1,2,....$ Then similar to (15), the following inequalities are true

$$\left\|\eta_k(x)-\xi_k(x)\right\|^2_{L_2(0,1)}<\frac{1}{3(dk\pi)^2},\ k\geq N,$$

$$\sum_{k=1}^{\infty}\left\|\eta_k(x)-\xi_k(x)\right\|^2_{L_2(0,1)}\leq\frac{1}{9d^2}. \tag{16}$$

Assume that $g(x)\in L_2(0,1)$. Then, allowing for (15), (16), similar to [4] we get

$$\left(\sum_{k=1}^{\infty}\left(\int_0^1 g(x)\,y_k(x)\,dx\right)^2\right)^{1/2}\leq M\left\|g(x)\right\|_{L_2(0,1)}, \tag{17}$$

$$\left(\sum_{k=1}^{\infty}\left(\int_0^1 g(x)\,\eta_k(x)\,dx\right)^2\right)^{1/2}\leq M\left\|g(x)\right\|_{L_2(0,1)}, \tag{18}$$

where

$$M=\left\{N(1+N)+2+\frac{1}{9d^2}\right\}^{1/2}. \tag{19}$$

Since the system of functions $\{y_k(x)\}$, $k=1,2,...$ is a Riesz basis in the space $L_2(0,1)$, then for any function $g(x)\in L_2(0,1)$ it is valid

$$g(x)=\sum_{k=1}^{\infty}g_k y_k(x), \tag{20}$$

where

$$g_k=(g(x),z_k(x))=\int_0^1 g(x)\,z_k(x)\,dx.$$

Multiply (20) scalarly by $g(x)$, use (17) and Cauchy-Bunyakovsky inequalities. Then we have:

$$\|g(x)\|^2_{L_2(0,1)} \leq \left(\sum_{k=1}^{\infty} g_k^2\right)^{1/2} M \|g(x)\|_{L_2(0,1)}$$

or

$$M^{-1}\|g(x)\|_{L_2(0,1)} \leq \left(\sum_{k=1}^{\infty} g_k^2\right)^{1/2}. \tag{21}$$

Further, it is easy to see that

$$|g_k| \leq \left|\int_0^1 g(x)\, y_k(x)\, dx\right| + \frac{1}{d\sqrt{\lambda_k}}\left|\int_0^1 g(x)\, dx\right|,$$

hence we find:

$$\left(\sum_{k=1}^{\infty} g_k^2\right)^{1/2} \leq M_0 \|g(x)\|_{L_2(0,1)}, \tag{22}$$

where

$$M_0 = 2\left[M + \frac{1}{d}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{1/2}\right]. \tag{23}$$

From inequalities (21) and (22) we deduce:

$$M^{-1}\|g(x)\|_{L_2(0,1)} \leq \left(\sum_{k=1}^{\infty} g_k^2\right)^{1/2} \leq M_0 \|g(x)\|_{L_2(0,1)}, \tag{24}$$

where M and M_0 are determined by relations (19) and (23), respectively.

Assume $g(x) \in W_2^1(0,1)$ and $I(g) \equiv dg(1) + \int_0^1 g(x)\, dx = 0$. Then we have:

$$g_k = \frac{\sqrt{2}}{\alpha_k}\int_0^1 g(x)\left(\cos\left(\sqrt{\lambda_k} x\right) - \cos\left(\sqrt{\lambda_k}\right)\right) dx =$$

$$= -\frac{\sqrt{2}}{\alpha_k}\frac{1}{\sqrt{\lambda_k}}\int_0^1 g'(x)\sin\left(\sqrt{\lambda_k} x\right) dx, \tag{25}$$

where

$$\alpha_k = 1 + d\cos^2\sqrt{\lambda_k} > 1.$$

Hence, allowing for (18) we find:

$$\left(\sum_{k=1}^{\infty}\left(\sqrt{\lambda_k}\,|g_k|\right)^2\right)^{1/2} \leq M\left\|g'(x)\right\|_{L_2(0,1)}. \tag{26}$$

Let $g(x) \in W_2^2(0,1)$, $I(g) = 0$, $g'(0) = 0$. Then, from (25) we find:

$$g_k = \frac{-\sqrt{2}}{\alpha_k}\frac{1}{\lambda_k}\left(g'(1)\cos\left(\sqrt{\lambda_k}\right) + \int_0^1 g''(x)\cos\left(\sqrt{\lambda_k}x\right)dx\right). \tag{27}$$

Hence, allowing for (17), we find:

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k |g_k|\right)^2\right)^{1/2} \le 2m_0\left|g'(1)\right| + \sqrt{2}M\left\|g''(x)\right\|_{L_2(0,1)}, \tag{28}$$

where

$$m_0 = \frac{1}{d}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{1/2}. \tag{29}$$

Assume now that $g(x) \in W_2^3(0,1)$, $I(g) = 0$, $g'(0) = 0$, $g'(1) + dg''(1) = 0$. Then from (27) we have:

$$g_k = -\frac{\sqrt{2}}{\alpha_k}\frac{1}{\sqrt{\lambda_k}}\int_0^1 g'''(x)\cos\left(\sqrt{\lambda_k}x\right)dx. \tag{30}$$

Allowing for (18), hence we get:

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}|g_k|\right)^2\right)^{1/2} \le M\left\|g'''(x)\right\|_{L_2(0,1)}. \tag{31}$$

Further, let $g(x) \in W_2^4(0,1)$, $I(g) = 0$, $g'(0) = 0$, $g'(1)+dg''(1) = 0$, $g'''(0) = 0$. Then from (30) we find:

$$g_k = \frac{\sqrt{2}}{\alpha_k}\frac{1}{\lambda_k^2}\left(\frac{1}{d\sqrt{\lambda_k}}g'''(1)\sin\left(\sqrt{\lambda_k}\right) + \int_0^1 g^{(4)}(x)\cos\left(\sqrt{\lambda_k}x\right)dx\right). \tag{32}$$

Allowing for (27), hence we have:

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^2|g_k|\right)^2\right)^{1/2} \le 2m_0\left|g'''(1)\right| + \sqrt{2}M\left\|g^{(4)}(x)\right\|_{L_2(0,1)}. \tag{33}$$

Now, let $g(x) \in W_2^2(0,1)$, $I(g) = 0$, $g'(0) = 0$. Then it holds the expansion

$$g''(x) = \sum_{k=1}^{\infty}\left(g''(x), z_k(x)\right)y_k(x), \tag{34}$$

where

$$\left(g''(x), z_k(x)\right) = \int_0^1 g''(x)z_k(x)\,dx. \tag{35}$$

It is easy to see that

$$\int_0^1 g''(x)\, z_k(x)\, dx = -\frac{\lambda_k\sqrt{2}}{\alpha_k}\int_0^1 g(x)\left(\cos\left(\sqrt{\lambda_k}x\right) - \cos\left(\sqrt{\lambda_k}\right)\right) dx$$

or

$$\left(g''(x), z_k(x)\right) = -\lambda_k\left(g(x), z_k(x)\right). \tag{36}$$

Substituting (36) into (34), we find:

$$g''(x) = -\sum_{k=1}^{\infty}\lambda_k\left(g(x), z_k(x)\right) y_k(x). \tag{37}$$

Similar to (21) we find:

$$\left\|g''(x)\right\|_{L_2(0,1)} \le M\left(\sum_{k=1}^{\infty}\left(\lambda_k g_k\right)^2\right)^{1/2}. \tag{38}$$

Since by theorem 1 [3] the system $\{y_k(x)\}$ $(k = 1, 2, ...)$ forms a Riesz basis, it is obvious that each classical solution of problem (1)-(3) is of the form:

$$u(x,t) = \sum_{k=1}^{\infty}u_k(t)\, y_k(x), \tag{39}$$

where

$$u_k(t) = \left(u(x,t), z_k(x)\right) = \int_0^1 u(x,t)\, z_k(x)\, dx,$$

moreover,

$$y_k(x) = \sqrt{2}\cos\left(\sqrt{\lambda_k}x\right), \quad z_k(x) =$$

$$= \sqrt{2}\left(\cos\left(\sqrt{\lambda_k}x\right) - \cos\left(\sqrt{\lambda_k}\right)\right) / \left(1 + d\cos^2\left(\sqrt{\lambda_k}\right)\right).$$

Assume that (4) and (5) are fulfilled. Then, allowing for lemma 1, from (37) we have:

$$u_{xx}(x,t) = -\sum_{k=1}^{\infty}\lambda_k u_k(t)\, y_k(x). \tag{40}$$

Applying the method of separation of variables for determining the desired $u_k(t)$ $(k = 1, 2, ...)$, allowing for (35) and (40) we find:

$$a_1(t)\, u_k'(t) + \lambda_k u_k(t) = f_k(t) - a_0(t)\, u_k(t) \quad (0 \le t \le T), \tag{41}$$

$$u_k(0) + \delta u_k(T) = \varphi_k \quad (k = 1, 2, ...), \tag{42}$$

where

$$f_k(t) = \int_0^1 f(x,t)\, z_k(x)\, dx, \quad \varphi_k = \int_0^1 \varphi(x)\, z_k(x)\, dx \quad (k = 1, 2, ...).$$

By solving problem (41), (42), we have:

$$u_k(t)=\frac{\varphi_k e^{-\int\limits_0^t \frac{\lambda_k ds}{a_1(s)}}}{1+\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}}+\int\limits_0^t \frac{F_k(\tau;u)}{a_1(\tau)} e^{-\int\limits_\tau^t \frac{\lambda_k ds}{a_1(s)}} d\tau-$$

$$-\frac{\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}}{1+\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}}\int\limits_0^T \frac{F_k(\tau;u)}{a_1(\tau)} e^{-\int\limits_\tau^t \frac{\lambda_k ds}{a_1(s)}} d\tau \quad (k=1,2,...), \tag{43}$$

where

$$F_k(t;u)=f_k(t)-a_0(t)\,u_k(t).$$

Further, allowing for (43), from (41) we have:

$$u_k'(t)=-\frac{\lambda_k+a_0(t)}{a_1(t)}\left\{\frac{\varphi_k e^{-\int\limits_0^t \frac{\lambda_k ds}{a_1(s)}}}{1+\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}}+\int\limits_0^t \frac{F_k(\tau;u)}{a_1(\tau)} e^{-\int\limits_\tau^t \frac{\lambda_k ds}{a_1(s)}} d\tau-\right.$$

$$\left.-\frac{\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}}{1+\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}}\int\limits_0^T \frac{F_k(\tau;u)}{a_1(\tau)} e^{-\int\limits_\tau^t \frac{\lambda_k ds}{a_1(s)}} d\tau\right\}+\frac{f_k(t)}{a_1(t)} \quad (k=1,2,...). \tag{44}$$

Substituting (43) into (39), we find:

$$u(x,t)=\sum_{k=1}^{\infty}\left\{\frac{\varphi_k e^{-\int\limits_0^t \frac{\lambda_k ds}{a_1(s)}}}{1+\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}}+\int\limits_0^t \frac{F_k(\tau;u)}{a_1(\tau)} e^{-\int\limits_\tau^t \frac{\lambda_k ds}{a_1(s)}} d\tau-\right.$$

$$\left.-\frac{\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}}{1+\delta e^{-\int\limits_0^T \frac{\lambda_k ds}{a_1(s)}}}\int\limits_0^T \frac{F_k(\tau;u)}{a_1(\tau)} e^{-\int\limits_\tau^t \frac{\lambda_k ds}{a_1(s)}} d\tau\right\} y_k(x). \tag{45}$$

Proceeding from definition of the classical solution of problem, (1)-(3), we prove the following lemma.

Lemma 3. *Let all the conditions of lemma 1 be fulfilled. If $u(x,t)$ is any classical solution of problem (1)-(3), the functions*

$$u_k(t)=\int\limits_0^1 u(x,t)\,z_k(x)\,dx \quad (k=1,2,...)$$

satisfy on $[0,T]$ the system (43).

Proof. Let $u(x,t)$ be any classical solution of problem (1)-(3). Then, it is obvious that

$$\int_0^1 u_t(x,t)\,z_k(x)\,dx = \frac{d}{dt}\left(\int_0^1 u(x,t)\,z_k(x)\,dx\right) = u'_k(t) \quad (k=1,2,...), \tag{46}$$

moreover $u_k(t) \in C^1[0,T] \quad (k=1,2,...)$.

Further, by lemma 1 and allowing for (36), we have:

$$\int_0^1 u_{xx}(x,t)\,z_k(x)\,dx = -\lambda_k \int_0^1 u(x,t)\,z_k(x)\,dx \quad (k=1,2,...). \tag{47}$$

Now, having multiplied the both hand sides of equation (1) by the function $z_k(x)$, integrating with respect to x the obtained equality from 0 to 1, and using relations (46), (47), we get:

$$a_1(t)\,u'_k(t) + a_0(t)\,u_k(t) = -\lambda_k u_k(t) + f_k(t) \quad (k=1,2,...,0 \le t \le T). \tag{41}$$

Further, multiply the both hand sides of (2) by $z_k(x)$ and integrate the obtained equality with respect to x from 0 to 1. Then we have:

$$u_k(0) + \delta u_k(T) = \varphi_k \quad (k=1,2,...). \tag{42}$$

Thus, $u_k(t) \quad (k=1,2,...)$ is the solution of problem (41), (42). Hence, as it was said before obtaining system (43), it directly follows that the functions $u_k(t)$ $(k=1,2,...)$ satisfy on $[0,T]$ the system (43). The lemma is proved.

By $B^{3/2}_{2,T}$ denote the aggregate of all the functions of the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\,y_k(x),$$

considered in D_T, where each of the functions $u_k(t)$ $(k=1,2,...)$ is continuous on $[0,T]$ and

$$J(u) \equiv \left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{1/2} < +\infty.$$

Define the norm in this set as follows:

$$\|u\|_{B^{3/2}_{2,T}} = J(u).$$

It is known [5] that $B^{3/2}_{2,T}$ is a Banach space.

Consider in the space $B^{3/2}_{2,T}$ the operator Φ:

$$\Phi(u(x,t)) = \widetilde{u}(x,t) = \sum_{k=1}^{\infty} u_k(t)\,y_k(x), \tag{48}$$

where $u_k(t)$ $(k=1,2,...)$ equals the right hand side of (43).

It is easy to see that

$$\left|1+\delta e^{-\int_0^T \frac{\lambda_k}{a_1(s)}ds}\right|^{-1} \le \left|1-|\delta|\, e^{-\int_0^1 \frac{\lambda_1}{a_1(s)}ds}\right|^{-1} \equiv \rho(T).$$

Then from (43) and (44) we have

$$|u_k(t)| \le \rho(T)\,|\varphi_k| + \left\|\frac{1}{a_1(t)}\right\|_{C[0,T]} (1+|\delta|\,\rho(T))\sqrt{T}\left(\int_0^T |F_k(\tau;u)|^2 d\tau\right)^{1/2},$$

$$|u_k'(t)| \le \left\|\frac{1+a_0(t)}{a_1(t)}\right\|_{C[0,T]} \lambda_k\times$$

$$\times\left\{\rho(T)\,|\varphi_k| + \left\|\frac{1}{a_1(t)}\right\|_{C[0,T]} (1+|\delta|\,\rho(T))\sqrt{T}\left(\int_0^T |F_k(\tau;u)|^2 d\tau\right)^{1/2}\right\} + \left|\frac{f_k(t)}{a_1(t)}\right|.$$

Hence we find:

$$\|\widetilde{u}\|_{B^{3/2}_{2,T}} = \left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{1/2} \le \sqrt{2}\rho(T)\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,|\varphi_k|\right)^2\right)^{1/2} +$$

$$+\sqrt{2}\left\|\frac{1}{a_1(t)}\right\|_{C[0,T]} (1+|\delta|\,\rho(T))\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,|F_k(\tau;u)|\right)^2 d\tau\right)^{1/2}, \qquad (49)$$

$$\left(\sum_{k=1}^{\infty}\left(\sqrt{\lambda_k}\,\|u_k'(t)\|_{C[0,T]}\right)^2\right)^{1/2} \le \sqrt{3}\left\|\frac{1+a_0(t)}{a_1(t)}\right\|_{C[0,T]}\times$$

$$\times\left\{\rho(T)\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,|\varphi_k|\right)^2\right)^{1/2}+\right.$$

$$\left.+\left\|\frac{1}{a_1(t)}\right\|_{C[0,T]} (1+|\delta|\,\rho(T))\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,|F_k(\tau;u)|\right)^2 d\tau\right)^{1/2}\right\}+$$

$$+\sqrt{3}\left\|\frac{1}{a_1(t)}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\sqrt{\lambda_k}\,\|f_k(t)\|_{C[0,T]}\right)^2\right)^{1/2}. \qquad (50)$$

Let the data of problem (1)-(3) satisfy the following conditions:

1) $\varphi(x)\in W_2^3(0,1)$,

$d\varphi(1)+\int_0^1 \varphi(x)\,dx=0,\quad \varphi'(0)=0, \varphi'(1)+d\varphi''(1)=0;$

2) $f(x,t)\in C_{x,t}^{2,0}(D_T)$, $f_{xxx}(x,t)\in L_2(D_T)$,

$df(1,t)+\int_0^1 f(x,t)\,dx=0,\quad f_x(0,t)=0,$

$f_x(1,t)+df_{xx}(1,t)=0 \quad (0\le t\le T);$

3) $a_0(t),\ 0<a_1(t)\in C[0,T],\delta\ge 0,\ 1-|\delta|\, e^{-\int\limits_0^T \frac{\lambda_1}{a_1(s)}ds}\ne 0.$

Then allowing for (26) and (27), from (49) and (50) we have

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{1/2}\le\sqrt{2}\rho(T)\,M\,\|\varphi'''(x)\|_{L_2(0,1)}+$$
$$+2\left\|\frac{1}{a_1(t)}\right\|_{C[0,T]}(1+|\delta|\,\rho(T))\sqrt{T}\left[\|f_{xxx}(x,t)\|_{L_2(D_T)}+\right.$$
$$\left.+\sqrt{T}\,\|a_0(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{1/2}\right], \tag{51}$$

$$\left(\sum_{k=1}^{\infty}\left(\sqrt{\lambda_k}\,\|u'_k(t)\|_{C[0,T]}\right)^2\right)^{1/2}\le\sqrt{3}\left\|\frac{1+a_0(t)}{a_1(t)}\right\|_{C[0,T]}\left\{M\rho(T)\,\|\varphi'''(x)\|_{L_2(0,1)}+\right.$$
$$+\sqrt{2}\left\|\frac{1}{a_1(t)}\right\|_{C[0,T]}(1+|\delta|\,\rho(T))\sqrt{T}\left[\|f_{xxx}(x,t)\|_{L_2(D_T)}+\right.$$
$$\left.\left.+\sqrt{T}\,\|a_0(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{1/2}\right]\right\}+$$
$$+\sqrt{3}\left\|\frac{1}{a_1(t)}\right\|_{C[0,T]}\left\|\|f_x(x,t)\|_{L_2(0,1)}\right\|_{C[0,T]}. \tag{52}$$

Accept the denotation

$$A(T)=\sqrt{2}\rho(T)\,M\,\|\varphi'''(x)\|_{L_2(0,1)}+$$
$$+2\left\|\frac{1}{a_1(t)}\right\|_{C[0,T]}(1+|\delta|\,\rho(T))\sqrt{T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)},$$
$$B(T)=2\left\|\frac{1}{a_1(t)}\right\|_{C[0,T]}(1+|\delta|\,\rho(T))\,\|a_0(t)\|_{C[0,T]}.$$

Then it is clear from (50) that

$$\|\widetilde{u}\|_{B^{3/2}_{2,T}}\le A(T)+B(T)\,T\,\|u(x,t)\|_{B^{3/2}_{2,T}}. \tag{53}$$

So, we can prove the following theorem.

Theorem 2. *Let 1-3 be fulfilled, and*

$$(A(T)+2)\,B(T)\,T<1. \tag{54}$$

Then problem (1)-(3) has a unique classical solution in the ball

$$K=K_R\left(\|u\|_{B^{3/2}_{2,T}}\le R=A(T)+2\right)\ \mathit{from}\ B^{3/2}_{2,T}.$$

Proof. Write equation (45) in the form

$$u = \Phi u, \tag{55}$$

where the operator Φ is determined from relation (48). Consider the operator Φ in the ball $K = K_R \left(\|u\|_{B^{3/2}_{2,T}} \leq R = A(T) + 2 \right)$ from the space $B^{3/2}_{2,T}$. It is seen from estimation (53) that under the conditions of theorem 2, for any $u \in K_R$ it holds the inequality

$$\|\Phi u\|_{B^{3/2}_{2,T}} \leq A(T) + B(T)\, T\, \|u\|_{B^{3/2}_{2,T}} \tag{56}$$

and for any $u_1, u_2 \in K_R$ we have:

$$\|\Phi u_1 - \Phi u_2\|_{B^{3/2}_{2,T}} \leq B(T)\, T\, \|u_1 - u_2\|_{B^{3/2}_{2,T}}. \tag{57}$$

From inequalities (56) and (57) it follows that under the conditions of theorem 2, the operator Φ acts in the ball $K = K_R$ and is contractive. Therefore in the ball $K = K_R$ the operator Φ has a unique fixed point $\{u\}$ that is the solution of (55).

As an element of the space, $B^{3/2}_{2,T}$ the function $u(x,t)$ has continuous derivatives $u_x(x,t)$, $u_{xx}(x,t)$.

It is obvious that

$$|u_t(x,t)| \leq \left(\sum_{k=1}^{\infty} \lambda_k^{-1} \right)^{1/2} \left(\sum_{k=1}^{\infty} \left(\sqrt{\lambda_k}\, \|u'_k(t)\|_{C[0,T]} \right)^2 \right)^{1/2}. \tag{58}$$

Allowing for (52) from (56) it follows that the function $u_t(x,t)$ is continuous in D_T.

It is easily verified that equation (1) and conditions (2), (3) are easily satisfied in the ordinary sense. So, $u(x,t)$ is the classical solution of problem (1)-(3) and by lemma 3 it is unique. The theorem is proved.

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