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"KADETS $\frac{1}{4}$ - THEOREM" AND MULTIPLIERS OF TYPE (p,p)

Abstract

In the present paper we prove analogues of the well-known "Kadets $\frac{1}{4}$ – theorem" for perturbated system of exponents cosines and sines in Lebesgue spaces.

1. Introduction

A famous classical theorem of Paley-Wiener [6] claims that if $\{\lambda_n\}$ is a sequence of real numbers such that $d \equiv \sup_n |\lambda_n - n| < \frac{1}{\pi^2}$, then the system of exponent

$$\left\{e^{i\lambda_n t}\right\}, \quad n \in \mathbb{Z}, \tag{1}$$

forms a basis in $L_2(-\pi,\pi)$ and is isomorphic to the system $\left\{e^{int}\right\}_{n\in\mathbb{Z}}$. They raised a question to replace the constant $\frac{1}{\pi^2}$ with larger one such that the system (1) would still form a basis in $L_2(-\pi,\pi)$. This question was completely solved by M. I. Kadets [7], who proved that the above assertion holds under the weaker condition $d < \frac{1}{4}$ and the constant $\frac{1}{4}$ is optimal.

It is natural to ask whether there is an analogy of Kadets' result in the space $L_p(-\pi,\pi)$ with $p \neq 2$. This question was the topic of investigation of Bilalov [1]–[3]. In some particular cases he has obtained affirmative answer to this and some other relative questions. In the case when λ_n has perturbation ($\lambda_n = n + \alpha \operatorname{sign} n$) has been completely solved in works of Moiseev [9] and Sedletskii [10].

We use the notation $\|\{\cdot\}\|_2$ to denote the multiplier of type (2,2). It is known that $(2,2) \equiv l_\infty$ and

$$\left\|\{\lambda_n - n\}_{n\in\mathbb{Z}}\right\|_2 = \left\|\{\lambda_n - n\}_{n\in\mathbb{Z}}\right\|_{l_\infty} = \sup_n |\lambda_n - n|.$$

Thus, the theorem of Paley-Wiener essentially claims that there exists $\delta > 0$ such that if $\{\lambda_n - n\}_{n\in\mathbb{Z}} \in (2,2)$ and $\left\|\{\lambda_n - n\}_{n\in\mathbb{Z}}\right\|_2 < \delta$, then the system (1) forms Riesz basis in $L_2(-\pi,\pi)$. In the present paper we extend this assertion to any $L_p(-\pi,\pi)$.

2. Auxiliary statements

In what follows, we use the abbreviation $\{a_n\} \equiv \{a_n\}_{n\in\mathbb{Z}}$. Given $f \in L_1(-\pi,\pi)$ by $\{F_n(f)\}$ we denote its Fourier coefficients

$$F_n(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int} dt, \quad n \in \mathbb{Z}.$$

We shall say that $\{\delta_n\}$ is a multiplier of type (p,q), i.e. $\{\delta_n\} \in (p,q)$, if for any $f \in L_p$ there exists $g \in L_q$ such that $F_n(g) = \delta_n F_n(f)$ for all $n \in \mathbb{Z}$. It is known that if $\{\delta_n\} \in (p,q)$, $1 \le p,q \le +\infty$, then there exists $\delta_{pq} > 0$ such that the inequality

$$\left\| \sum \delta_n c_n e^{int} \right\|_q \le \delta_{pq} \left\| \sum c_n e^{int} \right\|_p \tag{2}$$

holds for any finite sum $\sum$, where $\|\cdot\|_p$ - is the standard norm in $L_p(-\pi,\pi)$. The quantity

$$\inf \left\{ \delta_{pq} : \left\| \sum \delta_n c_n e^{int} \right\|_q \le \delta_{pq} \left\| \sum c_n e^{int} \right\|_p \right\}$$

is called the norm of the multiplier $\{\delta_n\}$ and is denoted by $\|\{\delta_n\}\|_{p,q}$. Clearly, if $\{\delta_n\} \in (p,q)$, then for any real number δ,

$$\{\delta\delta_n\} \in (p,q), \qquad \|\{\delta\delta_n\}\|_{p,q} = |\delta| \, \|\{\delta_n\}\|_{p,q}.$$

In particular, it follows that there are multipliers with arbitrarily small norms. Let us denote $\|\{\delta_n\}\|_p \equiv \|\{\delta_n\}\|_{p,p}$ and let

$$\|\{a_n\}\|_{V_r} = \sup_{\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}} \left\{ \sum_{k=1}^{m} \left| a_{n_{k+1}} - a_{n_k} \right|^r \right\}^{1/r}, \quad V_r \equiv \left\{ \{a_n\} : \|\{a_n\}\|_{v_r} < +\infty \right\}.$$

We shall need the following statement Hirschman which can be found in [4].

Theorem H. *Let $\delta_n = O\left(|n|^{-\alpha}\right)$ as $|n| \to \infty$. Then the following statements hold:*

1) *If $\alpha > 0$ and $\{\delta_n\} \in V_r$ for some $r > 2$, then $\{\delta_n\} \in (p,p)$ for any $p \in \left(\dfrac{2r}{r+2}, \dfrac{2r}{r-2}\right)$. But if $\{\delta_n\} \in V_r$ for $r \in [1,2)$, then $\{\delta_n\} \in (p,p)$ for any $p \in [1,+\infty)$;*

2) *If $\alpha \in \left(0, \dfrac{1}{2}\right]$, then $\{\delta_n\} \in (p,p)$ for any $p \in \left(\dfrac{2}{1+2\alpha}, \dfrac{2}{1-2\alpha}\right)$.*

3. Main results. Basis of exponents

Let $\lambda_n = n + \delta_n$, $n \in \mathbb{Z}$. The first result of our present paper is as follows:

Theorem 1. *Let $\{\delta_n\}_{n\in\mathbb{Z}} \in (p,p)$, $1 < p < +\infty$, be such that $\|\{\delta_n\}\|_p < \dfrac{\ln 2}{\pi}$. Then the system (1) forms a basis in $L_p(-\pi,+\pi)$ and is isomorphic to $\left\{e^{int}\right\}_{n\in\mathbb{Z}}$.*

Proof. Let $\sum\limits_n f_n \left(e^{i\lambda_n t} - e^{int}\right)$ be any finite sum. In view of the identity

$$e^{i\lambda_n t} - e^{int} = \left(e^{i\lambda_n t} - 1\right) e^{int} = \sum_{k=1}^{\infty} \frac{(i\delta_n t)^k}{k!} e^{int} = \sum_{k=1}^{\infty} \frac{(it)^k}{k!} \delta_n^k e^{int},$$

we have

$$\left\| \sum_n f_n \left(e^{i\lambda_n t} - e^{int}\right) \right\|_p = \left\| \sum_n f_n \sum_{k=1}^{\infty} \frac{(it)^k}{k!} \delta_n^k e^{int} \right\|_p =$$

$$= \left\| \sum_{k=1}^{\infty} \frac{(it)^k}{k!} \sum_n f_n \delta_n^k e^{int} \right\|_p \le \sum_{k=1}^{\infty} \frac{\pi^k}{k!} \left\| \sum_n \delta_n^k f_n e^{int} \right\|_p . \tag{3}$$

Let $\{\delta_n\}_{n\in\mathbb{Z}} \in (p,p)$, and denote $\delta_p = \|\{\delta_n\}\|_p$. Since $f(t) \equiv \sum_n f_n e^{int} \in L_p(-\pi,\pi)$, we have

$$\left\| \sum_n \delta_n f_n e^{int} \right\|_p \le \delta_p \left\| \sum_n f_n e^{int} \right\|_p .$$

Therefore,

$$\left\| \sum_n \delta_n^k f_n e^{int} \right\|_p \le \delta_p^k \left\| \sum_n f_n e^{int} \right\|_p .$$

Combining this with (), we obtain

$$\left\| \sum_n f_n \left(e^{i\lambda_n t} - e^{int} \right) \right\|_p \le \left(e^{\pi\delta_p} - 1 \right) \left\| \sum_n f_n e^{int} \right\|_p . \tag{4}$$

Now let $f \in L_p(-\pi;\pi)$. In (4) we define $\{f_n\}$ to be the Fourier coefficients of the function f. From (4) it follows that the series

$$\sum_{-\infty}^{\infty} f_n (e^{i\lambda_n t} - e^{int})$$

converges in $L_p(-\pi,\pi)$. Consider the operator T defined as

$$Tf = \sum_{-\infty}^{-\infty} f_n (e^{i\lambda_n t} - e^{int}).$$

Then from (4) we get $\|Tf\|_p \le \left(e^{\pi\delta_p} - 1\right) \|f\|_p$, i.e.

$$\|T\| \le e^{\pi\delta_p} - 1.$$

Since $\delta_n < \dfrac{\ln 2}{\pi}$, it follows that $\|T\| < 1$. Therefore, the operator $I+T$ is invertible (here $I : L_p \to L_p$ denotes the identity operator). On the other hand, $(I+T)\left[e^{int}\right] = e^{i\lambda_n t}$ for any $n \in \mathbb{Z}$. Thus, the systems (1) and $\left\{e^{int}\right\}_{n\in\mathbb{Z}}$ are isomorphic in $L_p(-\pi,\pi)$. This finishes the proof of theorem 1.

Remark. In the case $p = 2$ it is known that $\sup_n |\delta_n| = \|\{\delta_n\}\|_2$, i.e. $\|\{\delta_n\}\|_{l_\infty} = \|\{\delta_n\}\|_2$. Thus, taking $p = 2$ from theorem 1 we obtain a known result.

Using the above mentioned result of Hirschman, from theorem 1 we obtain the following consequence.

Corollary 1. *Let $\delta_n = \delta \cdot \tilde{\delta}_n$ for any $n \in \mathbb{Z}$. Then the following statements hold:*

1) If $\tilde{\delta}_n = O\left(|n|^{-\alpha}\right)$ as $|n| \to \infty$ for some $\alpha > 0$ and if $\left\{\tilde{\delta}_n\right\}_{n\in\mathbb{Z}} \in V_r, r > 2$, then there exists $\delta_p > 0$ such that for any $\delta \in [0,\delta_p)$ the system (1) forms a basis for $L_p(-\pi,\pi)$ and is isomorphic to $\left\{e^{int}\right\}_{n\in\mathbb{Z}}$ for any $p \in \left(\frac{2r}{r+2}, \frac{2r}{r-2}\right)$. Besides, *if* $\left\{\tilde{\delta}_n\right\}_{n\in\mathbb{Z}} \in V_r$, $1 \le r < 2$, *then the preceding assertion takes place for any* $p \in (1,+\infty)$*;*

2) *If* $\tilde{\delta}_n = O\left(|n|^{-\alpha}\right)$ *as* $|n| \to \infty$ *and if* $\alpha \in \left(0, \frac{1}{2}\right)$, *then there exists* $\delta_p > 0$ *such that for any* $\delta \in [0,\delta_p)$ *the assertion* 1) *holds for any* $p \in \left(\frac{2}{1+2\alpha}, \frac{2}{1-2\alpha}\right)$.

Let us now consider the following Riesz properties [5]:

$$\left\|\sum_{-N_1}^{0} a_n e^{int}\right\|_p + \left\|\sum_{1}^{N_2} a_n e^{int}\right\|_p \le M_p \left\|\sum_{-N_1}^{N_2} a_n e^{int}\right\|_p, \quad 1 < p < +\infty, \tag{5}$$

where $N_1 \ge 0$; $N_2 \ge 1$ are integers and M_p is a constant that depends only on p. Using this property it is not difficult to show that if $\{\delta_n\} \in (p,p)$, then $\left\{\tilde{\delta}_n\right\}$ also belongs to the class (p,p), where $card\left\{n : \delta_n \ne \tilde{\delta}_n\right\} < +\infty$. Consider the following example: let

$$\delta_n = \begin{cases} \beta^+, & if \quad n \ge n_1, \\ \beta^-, & if \quad n \le n_2, \end{cases}$$

where n_1, n_2 are any integers. All the cases are easily reduced to the case $n_1 = 0$, $n_2 = -1$, so let us assume that $n_1 = 0$; $n_2 = -1$. Then from (5) it follows that $\{\delta_n\} \in (p,p)$, and besides $\|\{\delta_n\}\|_p \le M_p\beta$, where $\beta = \max\left\{\left|\beta^+\right|; \left|\beta^-\right|\right\}$. From theorem 1 we obtain that for any $\beta_i \in (-\delta_p, \delta_p)$, $i = 1,2$, where $\delta_p = \frac{\ln 2}{\pi M_p}$, the system (1) forms a basis in $L_p(-\pi,\pi)$ isomorphic to $\left\{e^{int}\right\}_{n\in\mathbb{Z}}$.

4. Bases of cosines and sines

Let us mention the following Lemma, which is easy to establish:

Lemma 1. *The system* $\left\{e^{\pm i\lambda_n t}\right\}_{n\in\mathbb{N}}$ *(or the system* $1 \cup \left\{e^{\pm i\lambda_n t}\right\}_{n\in\mathbb{N}}$*) forms a basis in* $L_p(-\pi,\pi)$ *if and only if systems* $\{\sin\lambda_n t\}_{n\in\mathbb{N}}$ *and* $\{\cos\lambda_n t\}_{n\in\mathbb{N}}$ *(respectively, systems* $\{\sin\lambda_n t\}_{n\in\mathbb{N}}$ *and* $1 \cup \{\cos\lambda_n t\}_{n\in\mathbb{N}}$*) simultaneously form bases in* $L_p(0,\pi)$*.*

Let all the hypothesis of theorem 1 be satisfied. According to the results of Levinson [8], we can assume that $\lambda_0 = 0$. Then from lemma 1 we get that if $\|\{\delta_n\}\|_p < \delta_p$, then the systems $\{\sin\lambda_n t\}_{n\in\mathbb{N}}$ and $1 \cup \{\cos\lambda_n t\}_{n\in\mathbb{N}}$ form bases in $L_p(0,\pi)$.

It is easy to note that if $\{\delta_n\} \in (p,p)$, then $\{\delta_n + c\}$ also belongs to (p,p) for any real c, and besides $\|\{\delta_n + c\}\|_p \le \|\{\delta_n\}\|_p + |c|$.

Now consider the system $\left\{e^{\pm i\lambda_n t}\right\}_{n\in\mathbb{N}}$. It is absolutely obvious that this system forms a basis in $L_p(-\pi,\pi)$ only under such case if system $\left\{e^{i\alpha t} \cdot e^{\pm i\lambda_n t}\right\}_{n\in\mathbb{N}}$ forms a basis in $L_p(-\pi,\pi)$, where α is any real number. Let

$$\mu_n = \begin{cases} -\lambda_{|n|} + \alpha, & if \quad n \le -1, \\ \lambda_{n+1} + \alpha, & if \quad n \ge 0, \end{cases}$$

We have

$$\tilde{\delta}_n = \mu_n - n = -\lambda_{|n|} + \alpha - n = -\left(\lambda_{|n|} - |n|\right) + \alpha = -\delta_{|n|} + \alpha \quad \text{if} \quad n \le -1;$$

$\tilde{\delta}_n = \mu_n - n = \lambda_{n+1} + \alpha - n = \lambda_{n+1} - (n+1) + \alpha + 1 = \delta_{n+1} + \alpha + 1 \quad \text{if} \quad n \geq 0.$

Thus,

$$\tilde{\delta}_n = \begin{cases} \delta_{n+1} + 1 + \alpha, & n \geq 0\,; \\ -\delta_{|n|} + \alpha, & n \leq -1. \end{cases}$$

Using the property Riesz (5) once again we can easily show that if $\{\delta_n\} \in (p,p)$, where $1 < p < +\infty$, then $\left\{\tilde{\delta}_n\right\}$ also belongs to (p,p). Thus, if $\left\|\left\{\tilde{\delta}_n\right\}\right\|_p < \delta_p$ (δ_p – is a constant from theorem 1), then the system $\left\{e^{\pm i\lambda_n t}\right\}_{n\in\mathbb{N}}$ forms a basis in $L_p(-\pi,\pi)$, and, moreover, each system of $\{\cos\lambda_n t\}_{n\in\mathbb{N}}$, $\{\sin\lambda_n t\}_{n\in\mathbb{N}}$ forms a basis in $L_p(0,\pi)$. Taking $\alpha = -\frac{1}{2}$, we eventually obtain that if at least one of the two conditions

1) $\|\{\delta_n\}\|_p < \delta_p$

2) $\left\|\left\{\delta_n + \frac{1}{2}\right\}\right\|_p < \delta_p$

is satisfied, then the system of sines $\{\sin\lambda_n t\}_{n\in\mathbb{N}}$ forms a basis in $L_p(0,\pi)$. On the other hand, if condition 1) holds, then the system $1 \cup \{\cos\lambda_n t\}_{n\in\mathbb{N}}$ also forms a basis in $L_p(0,\pi)$.

Let the condition 2) take place. Then it is clear that the system $\{\cos\lambda_n t\}_{n\in\mathbb{N}}$ forms a basis in $L_p(0,\pi)$. Denote $\mu_n = \lambda_{n+1}$. For any $n \geq 0$ we have

$$\mu_n - n = \lambda_{n+1} - n = \lambda_{n+1} - (n+1) + 1 \Rightarrow \mu_n - n - \frac{1}{2} = \delta_{n+1} + \frac{1}{2}.$$

Thus if $\left\|\left\{\mu_n - n - \frac{1}{2}\right\}\right\|_p = \left\|\left\{\delta_{n+1} + \frac{1}{2}\right\}\right\|_p < \delta_p$, then the system $\{\cos\mu_n t\}_{n\geq 0}$ forms a basis in $L_p(0,\pi)$. In accordance to the result of Levinson, we can assume that $\mu_0 = 0$. As a result we obtain that if $\left\|\left\{\mu_n - n - \frac{1}{2}\right\}\right\|_p < \delta_p$, then the system $1 \cup \{\cos\mu_n t\}_{n\in\mathbb{N}}$ forms a basis in $L_p(0,\pi)$.

Thus, we have proved the following theorem:

Theorem 2. *Let $\{\delta_n\} \in (p,p)$. If 1) $\|\{\delta_n\}\|_p < \frac{\ln 2}{\pi}$ or 2) $\left\|\left\{\delta_n + \frac{1}{2}\right\}\right\|_p < \frac{\ln 2}{\pi}$, then the system of sines $\{\sin\lambda_n t\}_{n\in\mathbb{N}}$ forms a basis in $L_p(0,\pi)$ isomorphic to $\{\sin nt\}_{n\in\mathbb{N}}$. If condition 1) holds or if $\left\|\left\{\delta_n - \frac{1}{2}\right\}\right\|_p < \frac{\ln 2}{\pi}$, then the system of cosines $1 \cup \{\cos\lambda_n t\}_{n\in\mathbb{N}}$ forms a basis in $L_p(0,\pi)$ isomorphic to $\{\cos nt\}_{n\geq 0}$.*

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