Transactions of NAS of Azerbaijan, 2010, vol. XXX, No 4, pp. 17-24. ___________ 17

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WEIGHTED MORREY A PRIORI ESTIMATES FOR POISSON EQUATION

Abstract

Let Ω a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^2$ and let u be a solution of the classical Poisson problem in Ω; i.e.,

$$\begin{cases} -\triangle u = f & in\ \Omega, \\ u = 0 & on\ \ \partial\Omega, \end{cases}$$

where $f \in M_{p,\kappa}(\Omega, w)$, $1 \leq p < \infty$, $0 \leq \kappa < 1$ and ω is a weight in A_p.

The main goal of this paper is to prove the following a priori estimate

$$\|u\|_{W^2_{p,\kappa}(\Omega,w)} \leq C\|f\|_{M_{p,\kappa}(\Omega,w)},$$

and to give some applications for weights given by powers of the distance to the boundary.

1. Introduction

We will use the standard notation for Sobolev spaces and for derivatives, namely, if α is a multi-index, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{Z}^n_+$ we denote $|\alpha| = \sum\limits_{j=1}^{n} \alpha_j$, $D^\alpha = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n}$ and

$$W^k_p(\Omega) = \{v \in L_p(\Omega) : D^\alpha v \in L^p(\Omega),\ \forall\ |\alpha| \leq k\}.$$

Let Γ be the standard fundamental solution of the Laplacian operator, namely,

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} \log |x|^{-1} & n = 2, \\ \frac{1}{n(n-2)\omega_n} |x|^{2-n} & n \geq 3, \end{cases}$$

with ω_n the area of the unit sphere in $\mathbb{R}^n$.

Given a function $f \in C^\infty_0(\mathbb{R}^n)$, it is a classic result that the potential u given by

$$u(x) = \int \Gamma(x-y)f(y)dy$$

is a solution of $-\triangle u = f$ in $\mathbb{R}^n$ and satisfies the estimate

$$\|u\|_{W^2_p(\mathbb{R}^n)} \leq C\|f\|_{L_p(\mathbb{R}^n)} \tag{1.1}$$

for $1 < p < \infty$. Indeed, this estimate is a consequence of the Calderón-Zygmund theory of singular integrals (see for example [18]).

Since the work by Komori and Shirai [12], many results on weighted Morrey estimates for maximal functions and singular integral operators have been obtained.

In particular, generalizations of (1.1) to weighted Morrey norms are known to hold for weights in the class A_p (see for example [19]).

On the other hand, a priori estimates like (1.1) for solutions of the Dirichlet problem

$$\begin{cases} -\triangle u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

on smooth bounded domains Ω are also well known (see for example the classic paper by Agmon, Douglis and Nirenberg [3] where a priori estimates for general elliptic problems are proved).

Therefore, it is a natural question whether weighted a priori estimates are valid also for the solution of the Dirichlet problem (1.2). In this paper we give a positive answer to this question, namely, we prove that

$$\|u(x)\|_{W^2_{p,\kappa}(\Omega,w)} \le C\|f\|_{M_{p,\kappa}(\Omega,w)},$$

for $\omega \in A_p$, $1 < p < \infty$, $0 \le \kappa < 1$, where the constant C depends only on Ω and on the weight ω.

As an application we obtain weighted Morrey a priori estimates for weights given by powers of the distance to $\partial\Omega$. Estimates of this type are of interest in the analysis of some non-linear problems and were derived using different arguments (see [20]).

2. Preliminaries on weighted Morrey spaces

In the study of local properties of solutions to of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$ play an important role, see [10], [16], introduced in 1938 by C. Morrey [17].

Let Ω a bounded domain in $\mathbb{R}^n$ and $d = diam\Omega$. For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r.

Definition 2.1. *Let $1 \le p < \infty$ and $0 \le \lambda < n$. We denote by $M_{p,\lambda}(\Omega)$ the Morrey space as the set of locally integrable functions $f(x)$, $x \in \Omega$ with the finite norm*

$$\|f\|_{M_{p,\lambda}(\Omega)} = \sup_{x\in\Omega,\, 0<r\le d} \left(r^{-\lambda} \int_{B(x,r)\cup\Omega} |f(x)|^p dx \right)^{1/p}.$$

We recall the definition of the A_p class for $1 < p < \infty$. A non-negative locally integrable function ω belongs to A_p if there exists a constant C such that

$$\left(\frac{1}{|Q|} \int_Q \omega(x) dx \right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{-1/(p-1)} dx \right)^{p-1} \le C$$

for any cube $Q \subset \mathbb{R}^n$.

Definition 2.2. *Let $1 \le p < \infty$, $0 \le \kappa < 1$ and w be a weight function. We denote by $L_{p,\kappa}(\Omega, w)$ the weighted Morrey space as the set of locally integrable* *functions $f(x)$, $x \in \Omega$ with the finite norm*

$$\|f\|_{M_{p,\kappa}(\Omega,w)} = \sup_Q \left(\frac{1}{w(Q)^\kappa} \int_{Q \cup \Omega} |f(x)|^p w(x) dx \right)^{1/p},$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

Remark 2.1. Alternatively, we could define the weighted Morrey spaces with balls instead of cubes. Hence we shall use these two definitions of weighted Morrey spaces appropriate to calculation.

Remark 2.2. (1) If $w \equiv 1$ and $\kappa = \lambda/n$ with $0 < \lambda < n$, then $M_{p,\kappa}(\mathbb{R}^n, w) = M_{p,\lambda}(\mathbb{R}^n)$, the classical Morrey spaces.

(2) Let $w \in \Delta_2$. If $\kappa = 0, M_{p,0}(\mathbb{R}^n, w) = L_{p,w}$. If $\kappa = 1, M_{p,1}(\mathbb{R}^n, w) = L_{\infty,w}(\mathbb{R}^n)$ by the Lebesgue differentiation theorem with respect to w (see [16]).

(3) In the one-dimensional case, let a weight $w(x) = |x|^\alpha$ for some $-1/2 < \alpha < 0$ and a function $f(x) = \chi_{(0,1)}(x)|x|^{-1/2}$. Then $f \in M_{1,\frac{\alpha+\frac{1}{2}}{\alpha+1}}(\mathbb{R}^n, w) \backslash L_{2(\alpha+1),w}(\mathbb{R}^n)$.

Lemma 2.1. *Let $0 < \kappa < 1$, $0 < p < \infty$, $q = p/(1-\kappa)$ and w weight functions. Then*

$$L_{q,w}(\mathbb{R}^n) \hookrightarrow M_{p,\kappa}(\mathbb{R}^n, w).$$

Proof. Let $t = \frac{q}{p} \geq 1$. Then $\frac{1}{t} + \frac{1}{t'} = 1$, $t = \frac{1}{1-\kappa}$ and $t' = \frac{1}{\kappa}$. Therefore

$$\begin{aligned}
\|f\|_{M_{p,w,\kappa}} &= \sup_Q \left(\frac{1}{w(Q)^\kappa} \int_Q |f(x)|^p w(x) dx \right)^{1/p} \\
&= \sup_Q w(Q)^{-\kappa/p} \left(\int_Q |f(x)|^p w(x)^{1/t} w(x)^{1/t'} dx \right)^{1/p} \\
&\leq \sup_Q w(Q)^{-\kappa/p} \left(\int_Q |f(x)|^{pt} w(x) dx \right)^{1/pt} \left(\int_Q w(x) dx \right)^{1/pt'} \\
&= \sup_Q w(Q)^{-\kappa/p} w(Q)^{1/pt'} \left(\int_Q |f(x)|^q w(x) dx \right)^{1/q} \\
&= \sup_Q \left(\int_Q |f(x)|^q w(x) dx \right)^{1/q} \\
&= \|f\|_{L_{q,w}}.
\end{aligned}$$

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The maximal operator M is defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)| dy,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$.

Let T be a singular integral Calderon-Zygmund operator, briefly a Calderon-Zygmund operator, i. e., a linear operator bounded from $L_2(\mathbb{R}^n)$ in $L_2(\mathbb{R}^n)$ taking all infinitely continuously differentiable functions f with compact support to the functions $Tf \in L_1^{\rm loc}(\mathbb{R}^n)$ represented by

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy \qquad \text{a. e.} \quad \text{on} \;\; \text{supp} f.$$

Here $K(x,y)$ is a continuous function away from the diagonal which satisfies the standard estimates: there exist $c_1 > 0$ and $0 < \varepsilon \le 1$ such that

$$|K(x,y)| \le c_1 |x-y|^{-n}$$

for all $x, y \in \mathbb{R}^n,\;\; x \ne y$, and

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \le c_1 \left(\frac{|x-x'|}{|x-y|} \right)^{\varepsilon} |x-y|^{-n},$$

whenever $2|x-x'| \le |x-y|$. Such operators were introduced in [6].

The operators M and T play an important role in real and harmonic analysis and applications (see, for example [18] and [19]).

F. Chiarenza and M. Frasca [5] studied the boundedness of the maximal operator M in these spaces. Their results can be summarized as follows:

Theorem 2.1. *Let $1 \le p < \infty$ and $0 \le \lambda < n$. Then for $p > 1$ the operator M is bounded in $M_{p,\lambda}(\mathbb{R}^n)$ and for $p = 1$ M is bounded from $L_{1,\lambda}(\mathbb{R}^n)$ to $WL_{1,\lambda}(\mathbb{R}^n)$.*

G.D.Fazio and M.A.Ragusa [11] studied the boundedness of the Calderón-Zygmund singular integral operators in Morrey spaces, and their results imply the following statement for Calderón-Zygmund operators T.

Theorem 2.2. *Let $1 \le p < \infty$, $0 < \lambda < n$. Then for $1 < p < \infty$ Calderón-Zygmund singular integral operator T is bounded in $M_{p,\lambda}(\mathbb{R}^n)$ and for $p = 1$ T is bounded from $L_{1,\lambda}(\mathbb{R}^n)$ to $WL_{1,\lambda}(\mathbb{R}^n)$.*

Note that in the case of the classical Calderón-Zygmund singular integral operators Theorem 2.2 was proved by J. Peetre [13]. If $\lambda = 0$, the statement of Theorem 2.2 reduces to the aforementioned result for $L_p(\mathbb{R}^n)$.

In the paper [12] was proved the boundedness of classical operators in harmonic analysis, that is, the Hardy-Littlewood maximal operator, a Calderon-Zygmund operator, the fractional integral operator, etc.

Theorem 2.3 ([12], Theorem 3.2]). *If $1 < p < \infty$,$0 < \kappa < 1$ and $w \in A_p$, then the Hardy-Littlewood maximal operator M is bounded on $M_{p,\kappa}(\mathbb{R}^n, w)$.*

If $p = 1$, $0 < \kappa < 1$ and $w \in A_1$, then for all $t > 0$ and any cube Q,

$$w(\{x \in Q : Mf(x) > t\}) \le \frac{C}{t} \|f\|_{M_{1,w,\kappa}} w(Q)^{\kappa}.$$

Theorem 2.4 ([[12], Theorem 3.3]). *If $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_p$, then a Calderon-Zygmund operator T is bounded on $M_{p,\kappa}(\mathbb{R}^n, w)$.*

If $p = 1$,$0 < \kappa < 1$ and $w \in A_1$, then for all $t > 0$ and any cube Q,

$$w(\{x \in Q : Tf(x) > t\}) \le \frac{C}{t} \|f\|_{M_{1,w,\kappa}} w(Q)^{\kappa}.$$

3. Weighted Morrey a priori estimates

We consider the Dirichlet problem (1.2) in bounded domains Ω. From now on we will assume that $\partial\Omega$ is of class C^2. The solution of this problem is given by

$$u(x) = \int_\Omega G(x,y)f(y)dy, \tag{3.1}$$

where $G(x,y)$ is the Green function, which can be written as

$$G(x,y) = \Gamma(x-y) + h(x,y)$$

with $h(x,y)$ satisfying, for each fixed $y \in \Omega$,

$$\begin{cases} \triangle_x h(x,y) = 0 & x \in \Omega, \\ h(x,y) = -\Gamma(x-y) & x \in \partial\Omega. \end{cases}$$

If $P(y,Q)$ is the Poisson kernel, $h(x,y)$ is given by

$$h(x,y) = -\frac{1}{(n-2)\omega_n}\int_{\partial\Omega}\frac{1}{|x-Q|^{n-2}}P(y,Q)dS(Q)$$

where dS denotes the surface measure on $\partial\Omega$.

In what follows the letter C will denote a generic constant, not necessarily the same at each occurrence. It is known that the Green function satisfies the following estimates (see [21]),

$$G(x,y) \le \begin{cases} C\log|x-y| & \text{if} \quad n=2, \\ C|x-y|^{2-n} & \text{if} \quad n \ge 3, \end{cases}$$

and

$$|D_{x_i}G(x,y)| \le C|x-y|^{1-n}.$$

Therefore

$$D_{x_i}u(x) = \int_\Omega D_{x_i}G(x,y)f(y)dy.$$

To obtain the second derivatives of u from the representation (3.1) we will use the following lemma. We denote with $d(x)$ the distance to the boundary, namely, $d(x) = \inf\limits_{Q\in\partial\Omega}|x-Q|$.

Lemma 3.2. *Given $\alpha \in \mathbb{Z}_+^n$ ($|\alpha| > 0$ if $n=2$) there exists a constant C depending only on n and α such that*

$$|D^\alpha h(x,y)| \le Cd(x)^{2-n-|\alpha|}.$$

It follows from this lemma that for each $x \in \Omega$, $D_{x_ix_j}h(x,y)$ is bounded uniformly in a neighborhood of x and so

$$D_{x_ix_j}\int_\Omega h(x,y)f(y)dy = \int_\Omega D_{x_ix_j}h(x,y)f(y)dy.$$

On the other hand, since $|D_{x_j}\Gamma(x)| \le C|x|^{1-n}$ we have

$$D_{x_j}\int_\Omega \Gamma(x-y)f(y)dy = \int_\Omega D_{x_j}\Gamma(x-y)f(y)dy.$$

However, $D_{x_i x_j}\Gamma$ is not an integrable function and we cannot interchange the order between second derivatives and integration. A known standard argument shows that

$$D_{x_i}\int_\Omega D_{x_j}\Gamma(x-y)f(y)dy = Kf(x) + c(x)f(x)$$

where c is a bounded function and

$$Kf(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} D_{x_i x_j}\Gamma(x-y)f(y)dy.$$

Here and in what follows we consider f defined in $\mathbb{R}^n$ extending the original f by zero.

The operator K is a Calderón-Zygmund singular integral operator. Indeed, since $D_{x_j}\Gamma \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and it is a homogeneous function of degree $1-n$, it follows that $D_{x_i x_j}\Gamma(x-y)$ is homogeneous of degree $-n$ and has vanishing average on the unit sphere (see Lemma 11.1 in [2], page 152). Then, it follows from the general theory given in [5] that K is a bounded operator in $L_p(\mathbb{R}^n)$ for $1 < p < \infty$.

Moreover, the maximal operator

$$\widetilde{K}f(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} D_{x_i x_j}\Gamma(x-y)f(y)dy\right|$$

is also bounded in $L_p(\mathbb{R}^n)$ for $1 < p < \infty$.

We can now state and prove our main result.

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain. If $1 < p < \infty$, $0 \le \kappa < 1$, $\omega \in A_p$, $f \in M_{p,\kappa}(\Omega, w)$ and u is the solution of problem (1.2), then there exists a constant C depending only on n, ω and Ω such that*

$$\|u\|_{W^2_{p,\kappa}(\Omega,w)} \le C\|f\|_{M_{p,\kappa}(\Omega,w)}. \tag{3.2}$$

Proof. We will need the following estimate for the Green function. This estimate has been proved by A. Dall'Acqua and G. Sweers in [7], however they assume that the domain is more regular than C^2.

Let Ω be a bounded C^2 domain and $G(x,y)$ be the Green function of problem (1.2) in Ω. There exists a constant C depending only on n and Ω such that for $(x,y) \in \Omega\times\Omega$

$$|D_{x_i x_j}G(x,y)| \le C\frac{d(x)}{|x-y|^{n+1}}.$$

Our result follows from the following inequalities (see [8]).

There exists a constant C depending only on n and Ω such that, for any $x \in \Omega$,

$$|u(x)| + |D_{x_i}u(x)| \le CMf(x), \tag{3.3}$$

$$|D_{x_i x_j}u(x)| \le C\left(\widetilde{K}f(x) + Mf(x) + |f(x)|\right). \tag{3.4}$$

By Theorems 2.3 and 2.4 implies that the operators M and $\widetilde{K}$ are bounded in $M_{p,\kappa}(\Omega, w)$. Therefore (3.2) follows immediately from inequalities (3.3) and (3.4).

4. Application to weights of the form $d(x)^\beta$

In this section we show how the weighted Morrey estimate proved in the previous section can be used to obtain some of the a priori estimates given in [20]. Moreover, our arguments allows us to prove a new estimate which was not contained in the results in [20].

We will also make use of some imbedding theorems for weighted Sobolev spaces which, as we will show, can be proved in a simple way by using an argument of Buckley and Koskela [4].

Theorem 4.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain, $f \in L_{p,\kappa}(\Omega, d^\gamma)$ and u be the solution of problem (1.2). If $0 \le \kappa < 1$ and $-1 < \gamma < p-1$, then there exists a constant C depending only on κ, γ, p, n and Ω such that*

$$\|u\|_{W^2_{p,\kappa}(\Omega,d^\gamma)} \le C\|f\|_{L_{p,\kappa}(\Omega,d^\gamma)}. \tag{4.1}$$

Proof. In [8] proven that if $\Omega \subset \mathbb{R}^n$ be a bounded C_2 domain and $d(x)$ the distance from x to $\partial\Omega$, then, $d(x)^\beta \in A_p$ for $-1 < \beta < p-1$. For the particular case of Ω being a ball, it was shown in [14]. Therefore from Theorem 3.5 we get (4.1).

Acknowledgements. Author want to express his thanks to supervisor prof. V.S.Guliyev for the statement of the problem and many helpful conversations.

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Received March 15, 2010; Revised June 03, 2010.