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ON ASYMPTOTICS OF SOLUTION TO A BOUNDARY VALUE PROBLEM FOR A QUASILINEAR HYPERBOLIC EQUATION IN AN INFINITE STRIP

Abstract

In an infinite strip we consider a boundary value problem for a singularly-perturbed quasilinear equation degenerating to a parabolic equation. Asymptotic expansion of generalized solution of the considered problem is constructed with any accuracy and residual term is estimated.

Many researchers are interested in partial differential equations containing a small parameter for higher derivatives [1]-[6].

There are less papers on singularly-perturbed hyperbolic equations than the papers devoted to elliptic and parabolic equations. Such equations very often arise in studying different real phenomena with non-uniform transitions from one physical characteristics to another ones.

We cite some papers devoted to investigation of boundary value problems for singularly-perturbed hyperbolic equations.

In the paper [7], Sui-Yuh-Chen constructed the asymptotics of the solution to the following mixed problem

$$\varepsilon\left(\frac{\partial^2 U}{\partial t^2}-\frac{\partial^2 U}{\partial x^2}\right)-a\left(t,x,U\right)\frac{\partial U}{\partial t}+b\left(t,x,U\right)=0,$$

$$U|_{t=0}=\varphi\left(x\right), \quad \left.\frac{\partial U}{\partial t}\right|_{t=0}=\psi\left(x\right), \ U|_{x=0}=U|_{x=1}=0,$$

where $\varepsilon>0$ here and in the sequel is a small parameter.

In [8], M.G. Javadov investigated the Cauchy problem for a general $(m+1)$-th order linear hyperbolic equation degenerated into a linear hyperbolic equation of m -th order.

In [9], V.F. Butuzov constructed the asymptotics of the solution to the mixed problem for the following equation

$$\varepsilon^2\left(\frac{\partial^2 U}{\partial t^2}-\frac{\partial^2 U}{\partial x^2}\right)+\varepsilon^k a\left(t,x\right)\frac{\partial U}{\partial t}+b\left(t,x\right)U=f\left(t,x\right),$$

for different values of k.

In [10], the asymptotic solution to a nonlinear reduced wave equation is constructed by means of the theory of boundary layer.

The asymptotic representation of the solution to the mixed problem for the equation

$$\varepsilon U_{tt} - b(t,x) U_{xx} + a(t,x) U_t = f(t,x,U), \quad 0 < x < 1, \quad 0 < t < T$$

is obtained by means of the energetic method in the paper [11].

The convergence of the solution of the Cauchy problem for the equation

$$\varepsilon (U_{xx} - U_{tt}) = U_t + A(u) U_x$$

to the solution of the equation $W_t + A(W) W_x = 0$ as $\varepsilon \to 0$ is established in [12].

It should be noted that in references, the asymptotics of the solution to mixed problems for hyperbolic equations was studied only in finite (bounded) domains.

We also note that the in the known papers related to singularly perturbed hyperbolic equations, the derivatives of the desired function with respect to t enter into the equation in a linear way.

When constructing the asymptotics of the solution to the boundary value problems in infinite domains all the members of the asymptotic expansion should vanish under infinite increase of the appropriate variable. In the present paper, in $\Pi = \{(t,x) | \, 0 \le t \le T, \ -\infty < x < +\infty\}$ we consider the following boundary value problem

$$L_\varepsilon U \equiv \varepsilon \frac{\partial^2 U}{\partial t^2} + \varepsilon^{p-1} \left(\frac{\partial U}{\partial t} \right)^p + \frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} + aU - f(t,x) = 0, \tag{1}$$

$$U|_{t=0} = 0, \quad \left. \frac{\partial U}{\partial t} \right|_{t=0} = 0, \ (-\infty < x < +\infty) \tag{2}$$

$$\lim_{|x| \to +\infty} U = 0, \quad (0 \le t \le T) \tag{3}$$

where $p = 2k+1$, k is an arbitrary natural number, $a > 0$ is a constant, $f(t,x)$ is a given smooth function.

Our goal is to construct complete asymptotic expansion of the generalized solution of problem (1)-(3) by a small parameter ε.

For constructing asymptotics we'll conduct iterative process. In the first iterative process we'll look for the approximate solution of equation (1) in the form

$$W = W_0(t,x) + \varepsilon W_1(t,x) + ... + \varepsilon^n W_n(t,x), \tag{4}$$

the functions W_i; $i = 0,1,...,n$ will be chosen so that

$$L_\varepsilon W = O\left(\varepsilon^{n+1}\right). \tag{5}$$

Substituting (4) into (5), expanding the nonlinear term in powers of ε and equating the terms with the same powers of ε, for determining W_i; $i = 0,1,...,n$ we get the following parabolic equations

$$L_0 W_0 \equiv \frac{\partial W_0}{\partial t} - \frac{\partial^2 W_0}{\partial x^2} + aW_0 = f(t,x), \tag{6}$$

$$L_0 W_i = f_i(t,x), \tag{7}$$

where $f_i(t,x)$ are the known functions, moreover $f_i(t,x) = -\frac{\partial^2 W_{i-1}}{\partial t^2}$ for $i = 1, 2, ..., 2k-1$, for $i = 2k,\ 2k+1, ..., n$ the functions f_i are expressed polynomially by the first and second order derivatives of the functions $W_0, W_1, ..., W_{i-1}$.

Find such solutions of equations (6), (7) that satisfy the boundary conditions

$$\lim_{|x|\to+\infty} W_i = 0; \quad i = 0, 1, ..., n. \tag{8}$$

Obviously, we can't use the both initial conditions (2) for equations (6), (7). The initial conditions for equation (6), (7) we'll be written below. Notice that we'll use the first condition from (2). Then the second condition will be lost. To compensate the initial condition, a boundary layer type function should be constructed near the boundary $S_0 = \{t = 0,\ 0 \le x \le 1\}$.

Write a new decomposition of the operator L_ε near the boundary S_0. For that we make change of variables: $t = \varepsilon\tau,\ x = x$. In the new coordinates, the operator L_ε is of the form

$$L_{\varepsilon,1}U \equiv \varepsilon^{-1}\left\{\frac{\partial^2 U}{\partial\tau^2} + \left(\frac{\partial U}{\partial\tau}\right)^p + \frac{\partial U}{\partial\tau} + \varepsilon\left[-\frac{\partial^2 U}{\partial x^2} + aU - f(\varepsilon\tau, x)\right]\right\}. \tag{9}$$

Introduce an auxiliary function

$$r = \sum_{j=0}^{n+1} \varepsilon^j r_j(\tau, x), \tag{10}$$

where $r_j(\tau,x)$ are some smooth functions. Substituting (10) into (9), after definite transformations we get the expansion of $L_{\varepsilon,1}r$ in powers of ε in the coordinates (τ,x) in the form

$$L_{\varepsilon,1}r \equiv \varepsilon^{-1}\left\{\frac{\partial^2 r_0}{\partial\tau^2} + \left(\frac{\partial r_0}{\partial\tau}\right)^p + \frac{\partial r_0}{\partial\tau} + \sum_{j=0}^{n+1}\varepsilon^j\left[\frac{\partial^2 r_j}{\partial\tau^2} + p\left(\frac{\partial r_0}{\partial\tau}\right)^{p-1}\frac{\partial r_j}{\partial\tau} + \frac{\partial r_j}{\partial\tau} + \right.\right.$$

$$\left.\left. + h_j(r_0, r_1, ..., r_{j-1})\right] + 0\left(\varepsilon^{n+2}\right)\right\}, \tag{11}$$

where $h_j(r_0, r_1, ..., r_{j-1})$ are the known functions dependent on $r_0, r_1, ..., r_{j-1}$ and their first and second derivatives.

We'll look for the boundary layer type function S_0 near the boundary in the form

$$V = \varepsilon\left[V_0(\tau,x) + \varepsilon V_1(\tau,x) + ... + \varepsilon^n V_n(\tau,x)\right], \tag{12}$$

as a solution of the equation

$$L_{\varepsilon,1}(W+V) - L_{\varepsilon,1}W = 0\left(\varepsilon^{n+1}\right). \tag{13}$$

Assume that the functions $W_i\,(t,x);\ i=0,1,...,n$ have been already constructed. Then expanding each function $W_i\,(\varepsilon\tau,x)$ by the Taylor formula at the point $(0,x)$, we get a new expansion of W in powers of ε in the coordinates (τ,x) in the form

$$W=\sum_{j=0}^{n+1}\varepsilon^j\omega_j\,(\tau,x)+0\left(\varepsilon^{n+2}\right), \tag{14}$$

where $\omega_0=W_0\,(0,x)$ is independent of τ, the remaining functions ω_j, are determined by the formula

$$\omega_j=\sum_{s+r=j}\frac{1}{S!}\frac{\partial^s W_r\,(0,x)}{\partial t^s}\tau^s;\ \ j=1,2,...,n+1.$$

(12) and (14) yield that

$$W+V=\omega_0+\varepsilon\left(\omega_1+V_0\right)+\varepsilon\left(\omega_2+V_1\right)+...+\varepsilon^{n+1}\left(\omega_{n+1}+V_n\right)+0\left(\varepsilon^{n+2}\right). \tag{15}$$

Substituting expressions (14), (15) for the functions $W, W+V$ into (13) and considering (11), for determining $V_0,V_1,...,V_n$ we get the following equations:

$$\frac{\partial^2 V_0}{\partial\tau^2}+\frac{\partial V_0}{\partial\tau}=0, \tag{16}$$

$$\frac{\partial^2 V_j}{\partial\tau^2}+\frac{\partial V_j}{\partial\tau}=\frac{\partial^2 V_{j-1}}{\partial x^2}-aV_{j-1};\ \ j=1,2,...,n. \tag{17}$$

Now, write initial conditions for equations (6), (7) and boundary conditions for equations (16), (17). In order to find initial conditions for equations (6), (7) it is necessary to substitute the expressions of W from (4) and of V from (12) into the equality

$$\left.(W+V)\right|_{t=0}=0 \tag{18}$$

and compare the terms under the same powers of ε whose powers are less than $n+1$. Then we have

$$\left.W_0\right|_{t=0}=0, \tag{19}$$

$$\left.W_i\right|_{t=0}=\left.-V_{i-1}\right|_{\tau=0};\ \ \ i=1,2,...,n. \tag{20}$$

It should be noted that if the functions $W_i;\ i=0,1,...,n$ will satisfy the conditions (19), (20), the sum $W+v$ will satisfy condition (18) with ε^{n+1} -th accuracy, i.e.

$$\left.(W+V)\right|_{t=0}=\varepsilon^{n+1}\left.V_n\right|_{\tau=0}. \tag{21}$$

The boundary conditions for equations (16), (17) are found from the requirement

$$\left.\frac{\partial}{\partial t}\,(W+V)\right|_{t=0}=0 \tag{22}$$

and are of the form

$$\left.\frac{\partial V}{\partial \tau}\right|_{\tau=0}=-\left.\frac{\partial W_j}{\partial t}\right|_{t=0};\ j=0,1,...,n. \tag{23}$$

It should be noted that usually the first iterative process, then the second one that helps to construct boundary layer functions, are conducted. Here the functions W_i, V_i; $i=0,1,...,n$ should be constructed in turn, one after another: $W_0, V_0, W_1, V_1, ..., W_n, V_n$.

Now, construct the functions W_i, V_i; $i=0,1,...,n$. The function W_0 must be a solution of equation (6) satisfying boundary conditions (8) for $i=0$ and initial condition (19). Notice beforehand that vanishing of all consequent functions $V_0, W_1, V_1, ..., W_n, V_n$ as $|x|\to+\infty$ will be available at the expense of decrease of the function $W_0(t,x)$ and its derivatives as $|x|\to+\infty$. Decrease of $W_0(t,x)$ with respect to x will be attained by means of conditions that will be imposed on the function $f(t,x)$ at the right hand side of (6).

The bounded solution of equation (6) satisfying initial condition (19) may be written obviously. However, to provide the role of the function $W_0(t,x)$ we prove the following statement.

Lemma 1. *Let $f(t,x)$ be a function given in Π and have continuous derivatives with respect to t up to the $(n+2)$-th order inclusively, and be infinitely differentiable with respect to the parameter x and satisfy the condition*

$$\sup_x\left(1+|x|^l\right)\left|\frac{\partial^k f(t,x)}{\partial t^{k_1}\partial x^{k_2}}\right|=C^{(1)}_{lk_1k_2}<+\infty, \tag{24}$$

moreover, l be a non-negative number, $k=k_1+k_2$, $k_1\le n+2$, k_2 be an arbitrary non-negative integer, $C^{(1)}_{lk_1k_2}$ be a positive number. Then the function $W_0(t,x)$ as a solution of problem (6), (8) for $i=0$, (19) has in Π continuous derivatives with respect to t up to the $(n+3)$-th order inclusively, is infinitely differentiable with respect to x and satisfies the condition

$$\sup_x\left(1+|x|^l\right)\left|\frac{\partial^k W_0(t,x)}{\partial t^{k_1}\partial x^{k_2}}\right|=C^{(2)}_{lk_1k_2}<+\infty, \tag{25}$$

where $k_1\le n+3$, $C^{(2)}_{lk_1k_2}>0$.

Proof. Using the Fourier transformation with respect to the variable x we reduce problem (6), (8) for $i=0$ (19) to the following problem

$$\frac{d\widetilde{W}}{dt}+\left(a+\lambda^2\right)\widetilde{W}_0=\widetilde{f}(t,\lambda),\ \left.\widetilde{W}\right|_{t=0}=0, \tag{26}$$

where $\widetilde{W}(t,\lambda)$, $\widetilde{f}(t,\lambda)$ are the Fourier transformations of the functions $W_0(t,x)$, $f(t,x)$. The solution of problem (26) is written in the form

$$\widetilde{W}_0(t,\lambda)=\int_0^t e^{-\left(a+\lambda^2\right)(t-\tau)}\ \widetilde{f}(\tau,\lambda)\,d\tau. \tag{27}$$

It follows from (24) that for each $\frac{\partial^{k_1}\widetilde{f}(t,\lambda)}{\partial t^{k_1}}$; $k_1 = 0,1,...,n+2$, the functions $t \in [0,T]$ belong to L. Schwarts space S_λ of rapidly decreasing functions as $|\lambda| \to +\infty$. Hence, it follows that $\widetilde{f}(t,\lambda)$ satisfies the condition

$$\sup_\lambda \left(1+|\lambda|^l\right)\left|\frac{\partial^k \widetilde{f}(t,\lambda)}{\partial t^{k_1}\partial\lambda^{k_2}}\right| = C^{(3)}_{lk_1k_2} < +\infty, \quad k_1 \le n+2. \tag{28}$$

Using formulae (28) and (27), we prove that $\frac{\partial^k \widetilde{W}_0(t,\lambda)}{\partial t^{k_1}} \in S_\lambda$; $k_1 = 0,1,...,\ n+3$, whence fulfillment of condition (25) will follow.

By the mathematical induction method we can prove the validity of the formulas:

$$\frac{\partial^k \widetilde{W}_0(t,\lambda)}{\partial\lambda^k} = \int_0^t \left[\sum_{j=0}^k a_j(\lambda,t-\tau)\frac{\partial^{k-j}\widetilde{f}(\tau,\lambda)}{\partial\lambda^{k-j}}\right] e^{-\left(a+\lambda^2\right)(t-\tau)}d\tau. \tag{29}$$

Here is $a_j(\lambda,t-\lambda)$ is a polynomial with respects to λ and

$$a_j(gl,t-\tau) = \sum_{r=0}^j C_r\lambda^r(t-\tau)^r,$$

the coefficients C_r are real numbers, $C_j \ne 0$ and among the remaining coefficients there may be the ones equal zero.

By (28) and (29) we get

$$\sup_\lambda\left(1+|\lambda|^l\right)\left|\frac{\partial^k \widetilde{W}(t,\lambda)}{\partial\lambda^k}\right| \le$$

$$\le \sup_\lambda\left(1+|\lambda|^l\right)\int_0^t\left[\left|\sum_{j=0}^k a_j(\lambda,t-\tau)\right| e^{-\lambda(t-\tau)}\left|\frac{\partial^{k-j}\widetilde{f}(\tau,\lambda)}{\partial\lambda^{k-j}}\right|\right]d\tau \le$$

$$\le \sum_{j=0}^k b_j\int_0^t\left[\sup_\lambda\left(1+|\lambda|^l\right)\left|\frac{\partial^{k-j}\widetilde{f}(\tau,\lambda)}{\partial\lambda^{k-j}}\right|\right]d\tau \le \sum_{j=0}^k Tb_jC^{(3)}_{l0k-j}.$$

Denoting $T\sum_{j=0}^k Tb_jC_{l0k-j} = C$, we get $\widetilde{W}_0(t,\lambda) \in S_\lambda$. In transformation above we used that $\left|a_j(\lambda,\sigma)e^{-\sigma\lambda^2}\right| \le b_j$; where $\sigma \ge 0$, $j = 0,1,...,k$, $b_j > 0$ are some numbers.

Now, prove $\frac{\partial^{k_1}\widetilde{W}_0(t,\lambda)}{\partial t^{k_1}} \in S_\lambda$. It is easy to show that the derivatives with respect to t of any order of the function $\widetilde{W}_0(t,\lambda)$ being the solution of problem (26) are expressed by the formula

$$\frac{\partial^{k_1}\widetilde{W}_0(t,\lambda)}{\partial t^{k_1}} = \left[-a\left(a+\lambda^2\right)\right]^{k_1}\widetilde{W}_0 + \sum_{j=0}^{k_1-1}\left[-\left(a+\lambda^2\right)\right]^{k_1-j-1}\frac{\partial^j\widetilde{f}(t,\lambda)}{\partial t^j}. \tag{30}$$

Each term contained in the right hand side of (30) is a product of such two functions. One of them has a polynomial growth, the another one is contained in space S_λ. Therefore, the relation

$$\frac{\partial^{k_1}\widetilde{W}_0(t,\lambda)}{\partial t^{k_1}} \in S_\lambda;\ k=1,2,...,2n+2.$$

is valid.

Lemma 1 is proved.

Since the function $W_0(t,x)$ is known, we can determine the function V_0 as a solution of the boundary layer type equation (16) satisfying condition (23) for $j = 0:\ \left.\frac{\partial V_0}{\partial \tau}\right|_{\tau=0} = -\left.\frac{\partial W_0}{\partial t}\right|_{t=0}$. Obviously, V_0 is determined by the formula

$$V_0(\tau,x) = \left.\frac{\partial W_0}{\partial t}\right|_{t=0} e^{-\tau}. \tag{31}$$

By lemma 1, from (31) we have that the function $V_0(\tau,x)$ is infinitely differentiable with respect to both variables and for any value of τ from $[0,+\infty)$ $V_0(\tau,x) \in S_x$, whence fulfillment of the condition $\lim\limits_{|x|\to+\infty} V_0 = 0$ follows.

For $i=1$, it follows from (7), (8) and (20) that the function W_1 contained in expansion (4) is a solution of the following problem:

$$\frac{\partial W_1}{\partial t} - \frac{\partial^2 W_1}{\partial x^2} + aW_1 = -\frac{\partial^2 W_0}{\partial t^2};$$

$$W_1|_{t=0} = -\left.V_0\right|_{\tau=0},\ \lim_{|x|\to+\infty} W_1 = 0. \tag{32}$$

The solution of problem (32) may be found in the form: $W_1 = W_1^{(1)} + W_1^{(2)}$, where $W_1^{(1)}$, $W_1^{(2)}$ are the solutions of the following problems:

$$\frac{\partial W_1^{(1)}}{\partial t} - \frac{\partial W_1^{(1)}}{\partial x^2} + aW_1^{(1)} = -\frac{\partial^2 W_0}{\partial t^2};$$

$$\left.W_1^{(1)}\right|_{t=0} = 0,\ \lim_{|x|\to+\infty} W_1^{(1)} = 0, \tag{33}$$

$$\frac{\partial W_1^{(2)}}{\partial t} - \frac{\partial^2 W_1^{(2)}}{\partial x^2} + aW_1^{(2)} = 0,$$

$$\left.W_1^{(2)}\right|_{t=0} = -\left.V_0\right|_{\tau=0},\ \lim_{|x|\to+\infty} W_1^{(2)} = 0. \tag{34}$$

By lemma 1, the solution of problem (33) satisfies condition (25) for $k_1 \le n+2$. By the Fourier transformation the problem (34) is reduced to the problem $\frac{\partial \widetilde{W}_1^{(2)}}{\partial t} + \left(a+\lambda^2\right)\widetilde{W}_1^{(2)} = 0,\ \left.\widetilde{W}_1\right|_{t=0} = \widetilde{\varphi}_1(\lambda)$ whose solution is of the form

$$\widetilde{W}_1^{(2)}(t,\lambda) = \widetilde{\varphi}_1(\lambda)\, e^{-\left(a+\lambda^2\right)t}. \tag{35}$$

By $\widetilde{\varphi}_1(\lambda)$, $\widetilde{W}_1^{(2)}(t,\lambda)$ the Fourier transformations of the functions $-V_0|_{\tau=0} = -\frac{\partial W_0(0,x)}{\partial t}$, $W_1^{(2)}(t,x)$ are denoted.

By lemma 1 $\frac{\partial W_0(0,x)}{\partial t} \in S_x$, whence it follows that $\widetilde{\varphi}_1(\lambda) \in S_\lambda$. The function $\widetilde{W}_1^{(2)}(t,\lambda)$ determined by formula (35) is infinitely differentiable with respect to both arguments λ and t. Any order derivatives of the function $\widetilde{W}_1^{(2)}(t,\lambda)$ with respect to t is expressed by the equality $\frac{\partial^{k_1}\widetilde{W}_1^{(2)}(t,\lambda)}{\partial t^{k_1}} = \left[-\left(a+\lambda^2\right)\right]^{k_1}\varphi_1(\lambda)\,e^{-\left(a+\lambda^2\right)t}$. Arbitrary order derivatives of the function $\widetilde{W}_1^{(2)}(t,\lambda)$ with respect to λ consist of a finite sum. Each addend of this sum is a product of three functions: one of them is exp $-\exp\left[-\left(a+\lambda^2\right)t\right]$. The second one is $\widetilde{\varphi}_1(\lambda)$ or its derivatives. The third one is a polynomial with respect to λ and t. Therefore, for any value of t from $[0,T]$ and for arbitrary non-negative integer k_1, the relation $\frac{\partial^{k_1}\widetilde{W}_1^{(2)}(t,\lambda)}{\partial t^{k_1}} \in S_\lambda$ is valid. Whence it follows that $\frac{\partial^{k_1}W_1^{(2)}(t,x)}{\partial t^{k_1}} \in S_x$. Consequently, for the function $W_1(t,x)$ represented by the sum of $W_1^{(1)}(t,x)$ and $W_1^{(2)}(t,x)$ for each $t \in [0,T]$ we have $\frac{\partial^{k_1}W_1(t,x)}{\partial t^{k_1}} \in S_x$, $\mathrm{k}_1 = 0,1,...,n+2$.

For $j=1$ it follows from (17), (23) and (31) that the function V_1 is a solution of a boundary layer type problem:

$$\frac{\partial^2 V_1}{\partial\tau^2} + \frac{\partial V_1}{\partial\tau} = \Psi_1(x)\,e^{-\tau}, \tag{36}$$

$$\left.\frac{\partial V_1}{\partial\tau}\right|_{\tau=0} = \left.\frac{\partial W_1}{\partial t}\right|_{t=0}, \tag{37}$$

where $\Psi_1(x) = \frac{\partial^3 W_0(0,x)}{\partial t\partial x^2} - a\frac{\partial W_0(0,x)}{\partial t}$. It is easy to show that the function

$$V_1^{(1)} = -\Psi_1(x)\,\tau e^{-\tau} \tag{38}$$

is a special solution of equation (36). Represent V_1 in the form $V_1 = V_1^{(1)} + V_1^{(2)}$. Then $V_1^{(2)}$ will be a solution of a boundary layer type problem

$$\frac{\partial^2 V_1^{(2)}}{\partial\tau^2} + \frac{\partial V_1^{(2)}}{\partial\tau^2} = 0, \quad \left.\frac{\partial V_1^{(2)}}{\partial\tau}\right|_{\tau=0} = -\frac{\partial W_1(0,x)}{\partial t} + \Psi_1(x).$$

Obviously, the solution of the last problem is of the form

$$V_1^{(2)} = \left[\frac{\partial W_1(0,x)}{\partial t} - \psi_1(x)\right]e^{-\tau}. \tag{39}$$

It follows from (38) and (39) that the function V_1 as a sum of $V_1^{(1)}$ and $V_1^{(2)}$ is determined by the formula

$$V_1 = \left[b_{10}(x) + b_{11}(x)\,\tau\right]e^{-\tau}, \tag{40}$$

and the functions $b_{10}(x), b_{11}(x)$ are determined by the formulae

$$b_{10}(x) = \frac{\partial W_1(0,x)}{\partial t} - \frac{\partial^3 W_0(0,x)}{\partial t\partial x^2} + a\frac{\partial W_0(0,x)}{\partial t}, \tag{41}$$

$$b_{11}(x) = a\frac{\partial W_0(0,x)}{\partial t} - \frac{\partial^3 W_0(0,x)}{\partial t \partial x^2}. \tag{42}$$

It follows from the above mentioned properties of the functions W_0, W_1 and from (40)-(42) that the function $V_1(\tau, x)$ is infinitely differentiable with respect to both variables and for any value of τ from $[0, +\infty)$ $\eta_1(\tau, x) \in S_x$. Therefore, the function $V_1(\tau, x)$ satisfied also the following condition: $\lim\limits_{|x|\to+\infty} V_1 = 0$.

Continuing the process, we successively determine all the functions W_2, V_2, W_3, $V_3, ..., W_n, V_n$. The functions W_i are determined from problems(7), (8), (20) for $i = 2, 3, ..., n$ that are the problems of such types as (32) for the function W_1. The functions V_j are found as boundary layer type solutions of problem (17), (23) for $j = 2, 3, ..., n$ and are of the form

$V_j = \left[b_{j0}(x) + b_{j1}(x)\tau + b_{j2}(x)\tau^2 + ... + b_{jj}(x)\tau^j\right]e^{-\tau}$. Moreover, the functions $b_{js}(x)$; $s = 0, 1, ..., j$ are determined by the functions $\frac{\partial^{2k+1} W_s(0,x)}{\partial t \partial x^{2k}}$, $s + k \le j$ contained in the space S_x. Hence, it follows that all the functions V_j satisfy the condition

$$\lim_{|x|\to+\infty} V_j = 0; \quad j = 0, 1, ..., n. \tag{43}$$

Multiply the functions V_j by the smoothing functions and leave the previous denotation V_j; $j = 0, 1, ..., n$ for the obtained new functions.

It follows from (8) and (43) that the constructed sum $\widetilde{U} = W + V$ in addition to initial conditions (21), (22) satisfies the boundary condition

$$\lim_{|x|\to+\infty} (W + V) = 0. \tag{44}$$

Having denoted $U - \widetilde{U} = z$, we get the following asymptotic expansion by small parameter of the solution of problem (1)-(3):

$$U = \sum_{i=0}^{n} \varepsilon^i W_i + \sum_{i=0}^{n} \varepsilon^{1+i} V_i + z, \tag{45}$$

where z is a residual term.

Now, estimate the residual term.

The following lemma is valid.

Lemma 2. *The following estimation is valid for the residual term z in (45)*

$$\varepsilon \int_{-\infty}^{+\infty} \left(\left.\frac{\partial z}{\partial t}\right|_{t=T} \right)^2 dx + \varepsilon^{2k} \iint_{\Pi} \left(\frac{\partial z}{\partial t} \right)^{2k+2} dt dx + \iint_{\Pi} \left(\frac{\partial z}{\partial t} \right)^2 dt dx +$$

$$+ \int_{-\infty}^{+\infty} \left(\left.\frac{\partial z}{\partial x}\right|_{t=T} \right)^2 dx + C_1 \int_{-\infty}^{+\infty} \left(z|_{t=T} \right)^2 dx \le C_2 \varepsilon^{2(n+1)}, \tag{46}$$

where $C_1 > 0$, $C_2 > 0$ are the constants independent of ε.

Proof. Summing up (5) and (13), notice that $\widetilde{U}$ satisfies the equation

$$L_\varepsilon \widetilde{U} = 0\left(\varepsilon^{n+1}\right) \tag{47}$$

Subtracting (47) from (1), we have

$$\varepsilon \frac{\partial^2 z}{\partial t^2} + \varepsilon^{2k}\left[\left(\frac{\partial U}{\partial t}\right)^{2k+1} - \left(\frac{\partial \widetilde{U}}{\partial t}\right)^{2k+1}\right] + \frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} + az = 0\left(\varepsilon^{n+1}\right). \tag{48}$$

It follows from (2), (3) and (21), (22), (44), that z satisfies the following initial and boundary conditions:

$$z|_{t=0} = -\varepsilon^{m+1}\, V_n|_{\tau=0}\,, \quad \left.\frac{\partial z}{\partial t}\right|_{t=0} = 0, \quad \lim_{|x|\to+\infty} z = 0. \tag{49}$$

In obtaining a uniform estimation for z, the inhomogeneity of the first initial condition in (49) makes some difficulty. In this connection consider the auxiliary function

$$z_1 = \varepsilon^{n+1}\left(t^2 e^{-x^2} - V_n|_{\tau=0}\right), \tag{50}$$

that satisfies initial and boundary conditions (49). Representing the residual term z in the form

$$z = z_1 + z_2, \tag{51}$$

at first we get estimation for z_2 and then for z. Obviously, the function z_2 will satisfy the homogeneous boundary conditions:

$$z_2|_{t=0} = 0, \quad \left.\frac{\partial z_2}{\partial t}\right|_{t=0} = 0, \quad \lim_{|x|\to+\infty} z_2 = 0, \tag{52}$$

Substituting the expression of z from (51) into (48) and considering (50), after some transformations we get the equation

$$\varepsilon \frac{\partial^2 z_2}{\partial t^2} + \varepsilon^{2k}\left\{\left[\frac{\partial\left(\widetilde{U} + z_1 + z_2\right)}{\partial t}\right]^{2k+1} - \left[\frac{\partial\left(\widetilde{U} + z_1\right)}{\partial t}\right]^{2k+1}\right\} +$$

$$+\varepsilon^{2k}\left\{\left[\frac{\partial\left(\widetilde{U} + z_1\right)}{\partial t}\right]^{2k+1} - \left(\frac{\partial \widetilde{U}}{\partial t}\right)^{2k+1}\right\} + \frac{\partial z_2}{\partial t} - \frac{\partial^2 z_2}{\partial x^2} + az_2 = 0\left(\varepsilon^{n+1}\right). \tag{53}$$

Multiplying (53) by $\frac{\partial z_2}{\partial t}$ and integrating by parts the both of the obtained equality with regard to boundary conditions (52), after definite transformations we get validity of estimations (46) for z_2. From (50), (51) and from the estimation for z_2 it follows the validity of estimation (46) for z. Lemma 2 is proved.

Theorem. *Assume that the function $f(t,x)$ given in Π has continuous derivatives with respect up t to the $(n+2)$-th order inclusively, is finitely differentiable* *with respect to x and satisfies condition (24). Then the asymptotic representation (45) is true for the solution of problem (1)-(3). In this representation, the functions W_i are determined by the first iterative process, V_i are the boundary layer type functions near the boundary S_0 and are determined by the second iterative process, z is a remainder term and estimation (46) is valid for it.*

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