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ESTIMATIONS OF THE SMOOTHNESS MODULES OF DERIVATIVES OF CONVOLUTION OF TWO PERIODIC FUNCTIONS BY MEANS OF THEIR BEST APPROXIMATIONS IN $L_p(\mathbb{T})$

Abstract

In the paper the upper estimations of smoothness modules $\omega_k\left(h^{(s)};\delta\right)_r$ of derivative $h^{(s)}$ of order s of the convolution $h=f*g$ of two 2π periodic functions $f\in L_p(\mathbb{T})$ and $g\in L_q(\mathbb{T})$ are obtained by means of expression containing the product $E_{n-1}(f)_pE_{n-1}(g)_q$ of the best approximations of these functions in the metrics of $L_p(\mathbb{T})$ and $L_q(\mathbb{T})$ respectively, where $k,s\in\mathbb{N},p,q\in[1,\infty]$, $1/r=1/p+1/q-1\geq 0$, $\mathbb{T}=(-\pi,\pi]$. It is proved in the case $p,q\in(1,\infty)$ that the obtained estimations are exact in the sense of order on classes of convolutions with given majorants of sequences of the best approximations of f and g under some regularity of these majorants.

In what follows we use the following notation.

• $L_p(\mathbb{T})$, $1\leq p<\infty$, is the space of all measurable 2π periodic functions $f:\mathbb{R}\to\mathbb{C}$ with finite L_p-norm $\|f\|_p=\left((2\pi)^{-1}\int_{\mathbb{T}}|f(x)|^p\,dx\right)^{1/p}<\infty$.

• $C(\mathbb{T})\equiv L_\infty(\mathbb{T})$ is the space of all continuous 2π periodic functions with uniform norm $\|f\|_\infty\equiv\max\{|f(x)|:x\in\mathbb{T}\}$.

• $W_p^s(\mathbb{T}),s\in\mathbb{N},p\in[1,\infty)$, is the class of functions $f\in L_p(\mathbb{T})$ having an absolutely continuous derivative of order $s-1$ and $f^{(s)}\in L_p(\mathbb{T})$.

• $C^s(\mathbb{T})\equiv W_\infty^s(\mathbb{T}),s\in\mathbb{N}$, is the class of functions $f\in C(\mathbb{T})$ having an ordinary derivative $f^{(s)}\in C(\mathbb{T})$.

• $E_n(f)_p$ is the best approximation of a function f in the metric of $L_p(\mathbb{T})$ by the trigonometric polynomials of order $\leq n\in\mathbb{Z}_+$.

• $S_n(f;\cdot)$ is the partial sum of order $n\in\mathbb{Z}_+$ of the Fourier-Lebesque series of a function $f\in L_1(\mathbb{T}):S_n(f;x)=\sum\limits_{|\nu|=0}^{n}c_\nu(f)e^{i\nu x},\quad x\in\mathbb{T}$.

• $\omega_k(f;\delta)_p$ is the smoothness module of order k of a function $f\in L_p(\mathbb{T})$: $\omega_k(f;\delta)_p=\sup\left\{\left\|\Delta_t^kf\right\|_p:t\in\mathbb{R},|t|\leq\delta\right\},\ k\in\mathbb{N},\delta\in[0,\infty)$, where $\Delta_t^kf(x)=\sum\limits_{\nu=0}^{n}(-1)^{k-\nu}\begin{pmatrix}k\\ \nu\end{pmatrix}f(x+\nu t),\ x\in\mathbb{R}$.

• M_0 is the class of all sequences $\lambda=\{\lambda_n\}_{n=1}^\infty\subset\mathbb{R}$ such that $0<\lambda_n\downarrow 0$ $(n\uparrow\infty)$.

• $E_p[\lambda]=\{f\in L_p(\mathbb{T}):E_{n-1}(f)_p\leq\lambda_n,\ n\in\mathbb{N}\}$ for $p\in[1,\infty]$ and $\lambda\in M_0$.

The convolution $h=f*g$ of $f\in L_1(\mathbb{T})$ and $g\in L_1(\mathbb{T})$ is defined by the formula: $h(x)=(f*g)(x)=1/(2\pi)\int_{\mathbb{T}}f(x-y)g(y)dy$; it is known (see f.e. [1], v.1, §2.1, pp.64-65; [2], v.1, §3.1, pp.65-66) that the function h is defined almost everywhere, 2π periodic, measurable and $\|h\|_1\leq\|f\|_1\|g\|_1$ (whence it follows in particular that $h=f*g\in L_1(\mathbb{T})$). The last statement is a particular case of the following result known as the W.Young's inequality (see f.e. [1], v. 1, Theorem (1.15), pp. 67-68; [2], v.2, Theorem 13.6.1, pp. 176-177; [2], v.1, Theorem 3.1.4, p. 70, Theorem 3.1.6, p.72). Given $p\in[1,\infty]$, let $p'=p/(p-1)$ be the exponent conjugate to p. As usual, we assume that $p'=1$ for $p=\infty$ and $p'=\infty$ for $p=1$. If $p,q\in[1,\infty]$ and

$1/r = 1/p + 1/q - 1 \geq 0$, then $r = pq/(p+q-pq)$ and $r \in [1,\infty)$ for $1/r > 0$ and $r = \infty$ for $1/r = 0$ (in this case $1/p + 1/q = 1$, so that $q = p'$).

Theorem A. *Let* $p, q \in [1,\infty]$, $f \in L_p(\mathbb{T})$ *and* $g \in L_q(\mathbb{T})$, $h = f * g$, $1/r = 1/p + 1/q - 1 \geq 0$. *Then*

- *If* $1/r > 0$ *then* h *belongs to* $L_r(\mathbb{T})$ *and* $\|h\|_r \leq \|f\|_p \|g\|_q$.
- *If* $1/r = 0$ *then* h *belongs to* $C(\mathbb{T}) \equiv L_\infty(\mathbb{T})$ *and* $\|h\|_\infty \leq \|f\|_p \|g\|_{p'}$.

Recall that the Fourier coefficients $c_n(h)$ of $h = f * g$ of two arbitrary functions $f \in L_1(\mathbb{T})$ and $g \in L_1(\mathbb{T})$ are calculated by the formula (see [1], v.1, Theorem (1.5), p.64; [2], v.1, p.66, formula (3.1.5)) $c_n(h) = c_n(f * g) = c_n(f) \cdot c_n(g)$ for every $n \in \mathbb{Z}$.

We use also the following obvious inequalities (see f.e. [3], Lemma 1, pp. 18-19): let $f \in L_p(\mathbb{T}), p \in [1,\infty], k \in \mathbb{N}$ and $f = \operatorname{Re} f + i \operatorname{Im} f$; then

(i) $\max\{E_n(\operatorname{Re} f)_p, E_n(\operatorname{Im} f)_p\} \leq E_n(f)_p \leq$
$\leq E_n(\operatorname{Re} f)_p + E_n(\operatorname{Im} f)_p \leq 2E_n(f)_p,\ n \in \mathbb{Z}_+$.

(ii) $\max\{\omega_k(\operatorname{Re} f;\delta)_p, \omega_k(\operatorname{Im} f;\delta)_p\} \leq \omega_k(f;\delta)_p \leq$
$\leq \omega_k(\operatorname{Re} f;\delta)_p + \omega_k(\operatorname{Im} f;\delta)_p \leq 2\omega_k(f;\delta)_p,\ \delta \in [0,\infty)$.

The following statement be so called the inverse theorem "with derivatives" of the approximation theory of periodic functions in $L_p(\mathbb{T})$.

Theorem B. *Let* $p \in [1,\infty]$, $f \in L_p(\mathbb{T}), \theta = \theta(p) = \min\{2,p\}$ *for* $p \in [1,\infty)$ *and* $\theta(\infty) = 1$, $s \in \mathbb{N}$, $k \in \mathbb{N}$ *and*

$$\sum_{n=1}^{\infty} n^{\theta s - 1} E_{n-1}^{\theta}(f)_p < \infty. \tag{1}$$

Then $f \in W_p^s(\mathbb{T})$ *(more precisely,* f *almost everywhere equal to some function from* $W_p^s(\mathbb{T})$ *for* $p < \infty$ *and* $f \in C^s(\mathbb{T})$ *for* $p = \infty$*) and the following estimation holds:*

$$\omega_k\left(f^{(s)};\pi/n\right)_p \leq C_1(k,s,p)\left\{\left(\sum_{\nu=n+1}^{\infty} \nu^{\theta s-1} E_{\nu-1}^{\theta}(f)_p\right)^{1/\theta} + \right.$$
$$\left. + n^{-k}\left(\sum_{\nu=1}^{n} \nu^{\theta(k+s)-1} E_{\nu-1}^{\theta}(f)_p\right)^{1/\theta}\right\}, \quad n \in \mathbb{N}, \tag{2}$$

where $C_1(k,s,p)$ *is a positive constant depending only on parameters* k, s *and* p.

The implication (1)$\Longrightarrow f \in W_p^s(\mathbb{T})$ for $p = \infty$ was proved by S.N.Bernstein [4], § 2.14 and § 2.17 (see also [5], Theorem 10 and Corollary 10.1, pp. 236-237). Theorem B independly was prowed by S.B.Stechkin [5], Theorem 11, p.238, for $p = \infty$, and by A.F.Timan [6] for $p = \infty$ and $p = 1$ (see also [7], § 1, p.490; [8], § 6.1.3, p.346-349). In the case $p \in (1,\infty)$ Theorem B in an equivalent form was obtained by O.V.Besov [9], Theorem 2, p.16, which amplify the corresponding result of M.F.Timan [10], Theorem 2, p.126 (see also [11], Theorem 3, p.109). With respect to Theorem B follow also to note the review of A.A.Andrienko [12], §3, p.220-224, and monograph of A.F.Timan [8], § 6.1.3-6.1.5, p.346-359. At last we denote that the first estimation is lake to (2) was obtained by Ch.J.-E. de la Vallée Poussin [13], § 39, for $k = 1, p = \infty$ and by E.S.Quade [14], Theorem 1, p.532, for $k = 1, 1 \leq p < \infty$ (see [14], pp.531-535).

Inequality (2) is exact in the sense of order on the class $E_p[\lambda]$ for all $p \in [1,\infty]$, namely

$$\sup\left\{\omega_k\left(f^{(s)};\pi/n\right)_p : f \in E_p[\lambda]\right\} \asymp$$

$$\asymp \left(\sum_{\nu=n+1}^{\infty} \nu^{\theta s-1}\lambda_\nu^\theta\right)^{1/\theta} + n^{-k}\left(\sum_{\nu=1}^{n} \nu^{\theta(k+s)-1}\lambda_\nu^\theta\right)^{1/\theta}, \quad n \in \mathbb{N}, \tag{3}$$

under condition that $\sum\limits_{n=1}^{\infty} n^{\theta s-1}\lambda_n^\theta < \infty$. Note that the convergence of the last series is necessary and sufficiently for validity of the imbedding $E_p[\lambda] \subset W_p^s(\mathbb{T})$. The sufficiency of denote condition follows from implication (1)$\Longrightarrow f \in W_p^s(\mathbb{T})$ (see Theorem B). The statement about necessity was anonced by the author [15], Theorem 2, point (2.2), p. 1302, and the proof was given in [16], Theorem 2, point (2.2), p. 133 (see also [17], p.39, the statement (2)).

The upper estimation in (3) immediately follows from inequality (2). The lower estimation in (3) is realized by means of individual functions in $E_p[\lambda]$; more precisely, for every $p \in [1,\infty]$ and for arbitrary $\lambda \in M_0$ there exists a function $f_0(\cdot;p;\lambda) \in L_p(\mathbb{T})$ with $E_{n-1}(f_0)_p \le \lambda_n$, $n \in \mathbb{N}$, such that

(i) $f_0 \in W_p^s(\mathbb{T}) \Longleftrightarrow \sum\limits_{n=1}^{\infty} n^{\theta s-1}\lambda_n^\theta < \infty$;

(ii) if the series in (i) converge, then $\omega_k\left(f_0^{(s)};\pi/n\right)_p \ge$

$$\ge C_2(k,s,p)\left\{\left(\sum_{\nu=n+1}^{\infty} \nu^{\theta s-1}\lambda_\nu^\theta\right)^{1/\theta} + n^{-k}\left(\sum_{\nu=1}^{n} \nu^{\theta(k+s)-1}\lambda_\nu^\theta\right)^{1/\theta}\right\}, \quad n \in \mathbb{N}.$$

The statement (i) and estimation (ii) was anonced by the author [15], Lemma 2, pp. 1302-1303, and the proof was given in [16], Lemma 3.7, p.75 (see also [17], Lemmas 3, 4 and 8; [18], Lemma 2, p.46).

Note also that the proof of ordering equality (3) in the case $p \in [1,\infty]$ was given by author in [17], p. 35. Later V.V.Geit [19], Theorem 3, p.25, by other method proved (3) in the case $p = \infty$.

Theorem 1. *Let* $p,q \in [1,\infty], 1/r = 1/p + 1/q - 1 \ge 0, \theta = \theta(r) = \min\{2,r\}$ *for* $r \in [1,\infty)$ *and* $\theta(\infty) = 1, f \in L_p(\mathbb{T}), g \in L_q(\mathbb{T}), h = f * g,\ s \in \mathbb{N},\ k \in \mathbb{N}$ *and*

$$\sum_{n=1}^{\infty} n^{\theta s-1} E_{n-1}^\theta(f)_p E_{n-1}^\theta(g)_q < \infty. \tag{4}$$

Then $h \in W_r^s(\mathbb{T})$ *and the following estimation holds:*

$$\omega_k\left(h^{(s)};\pi/n\right)_r \le C_3(k,s,r)\left\{\left(\sum_{\nu=n+1}^{\infty} \nu^{\theta s-1}E_{\nu-1}^\theta(f)_pE_{\nu-1}^\theta(g)_q\right)^{1/\theta} + \right.$$

$$\left. + n^{-k}\left(\sum_{\nu=1}^{n} \nu^{\theta(k+s)-1}E_{\nu-1}^\theta(f)_pE_{\nu-1}^\theta(g)_q\right)^{1/\theta}\right\}, \quad n \in \mathbb{N}. \tag{5}$$

Proof. Since $f \in L_p(\mathbb{T})$ and $g \in L_q(\mathbb{T})$ we have that $h \in L_r(\mathbb{T})$ for $1/r > 0$ and $h \in C(\mathbb{T}) \equiv L_\infty(\mathbb{T})$ for $1/r = 0$ by Theorem A. We need the following estimation (see [20], the inequality (2) in the proof of Theorem 1, p.41)

$$E_{n-1}(f * g)_r \le E_{n-1}(f)_p E_{n-1}(g)_q, \quad n \in \mathbb{N}, \quad r \in [1,\infty]. \tag{6}$$

Taking into account (4) and by inequality (6) we have that

$$\sum_{n=1}^{\infty} n^{\theta s-1} E_{n-1}^{\theta}(h)_r \leq \sum_{n=1}^{\infty} n^{\theta s-1} E_{n-1}^{\theta}(f)_p E_{n-1}^{\theta}(g)_q < \infty,$$

whence it follows that (1) hold for h. Therefore $h \in W_r^s(\mathbb{T})$ by Theorem B and applying the inequalities (2) for $h^{(s)} \in L_r(\mathbb{T})$ and (6), we obtain (5). Theorem 1 is proved.

Corollary. *Let under conditions of Theorem 1* $E_{n-1}(f)_p \leq n^{-\alpha}, E_{n-1}(g)_q \leq n^{-\beta}, n \in \mathbb{N}$, *where* $\alpha, \beta \in (0, \infty)$ *and* $\rho = \alpha + \beta - s > 0$. *Then* $h \in W_r^s(\mathbb{T})$ *and the estimations holds:*

$$(i) \quad \omega_k\left(h^{(s)}; \pi/n\right)_r \leq C_3(k,s,r) C_4(k,\rho,\theta) \begin{cases} n^{-\rho} & for \;\; \rho < k; \\ n^{-k}(\ln(en))^{1/\theta} & for \;\; \rho = k; \\ n^{-k} & for \;\; \rho > k. \end{cases}$$

$(ii) \quad \omega_{k+1}\left(h^{(s)}; \pi/n\right)_r \leq C_3(k+1,s,r) C_5(k,\theta) n^{-k}$ *for* $\rho = k$.

Proof. We have that

$$\sum_{n=1}^{\infty} n^{\theta s-1} E_{n-1}^{\theta}(f)_p E_{n-1}^{\theta}(g)_q \leq \sum_{n=1}^{\infty} n^{-\theta\rho-1} \leq 1 + (\theta\rho)^{-1},$$

whence $h \in W_r^s(\mathbb{T})$ by Theorem 1 and

$$(i) \; \omega_k\left(h^{(s)}; \pi/n\right)_r \leq C_3(k,s,r) \left\{ \left(\sum_{\nu=n+1}^{\infty} \nu^{-\theta\rho-1} \right)^{1/\theta} + n^{-k} \left(\sum_{\nu=1}^{n} \nu^{\theta(k-\rho)-1} \right)^{1/\theta} \right\} \leq$$

$$\leq C_3(k,s,r) \left((\theta\rho)^{-1/\theta} n^{-\rho} + n^{-k} C_6(k,\rho,\theta) \begin{cases} n^{k-\rho} & for \;\; \rho < k; \\ (\ln(en))^{1/\theta} & for \;\; \rho = k; \\ 1 & for \;\; \rho > k, \end{cases} \right)$$

$$\leq C_3(k,s,r) C_4(k,\rho,\theta) \begin{cases} n^{-\rho} & for \;\; \rho < k; \\ n^{-k}(\ln(en))^{1/\theta} & for \;\; \rho = k; \\ n^{-k} & for \;\; \rho > k, \end{cases}$$

where $C_4(k,\rho,\theta) = (\theta\rho)^{-1/\theta} + C_6(k,\rho,\theta)$, $C_6(k,\rho,\theta) = 1$ for $\rho = k$, $C_6(k,\rho,\theta) = \left(1 + (\theta(\rho-k))^{-1}\right)^{1/\theta}$ for $\rho > k$, $C_6(k,\rho,\theta) = 2^{k-\rho} (\theta(k-\rho))^{-1/\theta}$ for $\rho < k$ and $\theta(k-\rho) \geq 1$, $C_6(k,\rho,\theta) = (\theta(k-\rho))^{-1/\theta}$ for $\rho < k$ and $\theta(k-\rho) \leq 1$.

$(ii) \; \omega_{k+1}\left(h^{(s)}; \pi/n\right)_r \leq$

$$\leq C_3(k+1,s,r) \left\{ \left(\sum_{\nu=n+1}^{\infty} \nu^{-\theta k-1} \right)^{1/\theta} + n^{-(k+1)} \left(\sum_{\nu=1}^{n} \nu^{\theta-1} \right)^{1/\theta} \right\} \leq$$

$$\leq C_3 \left\{ (\theta k)^{-1/\theta} n^{-k} + n^{-(k+1)} n \right\} = C_3 \left((\theta k)^{-1/\theta} + 1 \right) n^{-k} = C_3 C_5(k,\theta) n^{-k}.$$

For further exposition we need preliminary lemmas.

Lemma 1. *Let* $1 < r \leq 2, s \in \mathbb{Z}_+, k \in \mathbb{N}, \psi \in W_r^s(\mathbb{T})$ *and have the Fourier series* $\psi(x) \sim \sum\limits_{n \in \mathbb{Z}} c_n(\psi) e^{inx}, x \in \mathbb{T}$. *Then*

$$(i) \quad n^{-k} \left(\sum_{\nu=1}^{n} \nu^{rk+r-2} \left|c_\nu(\psi)\right|^r \right)^{1/r} \leq C_7(k,r) \omega_k(\psi; \pi/n)_r, n \in \mathbb{N};$$

(ii) $\left(\sum\limits_{n=1}^{\infty} n^{rs+r-2}\left|c_n(\psi)\right|^r\right)^{1/r} \leq C_8(r)\left\|\psi^{(s)}\right\|_r$;

(iii) $\left(\sum\limits_{\nu=n+1}^{\infty} \nu^{rs+r-2}\left|c_\nu(\psi)\right|^r\right)^{1/r} \leq C_9(k,r)\omega_k\left(\psi^{(s)};\pi/n\right)_r, n\in\mathbb{N}$.

Proof. The inequality (i) was proved in [3], Lemma 2, pp.19-20. In the case $s=0$ the inequality (ii) immediately follows from the first part of Hardy-Littlewood Theorem (see [1], v. 2, Theorem 12.3.19, p.165; [2], v. 2, Theorem 13.11.1, p.215):

$$\left(\sum_{n=1}^{\infty} n^{r-2}\left|c_n(\psi)\right|^r\right)^{1/r} \leq \left(\sum_{|n|=0}^{\infty}(|n|+1)^{r-2}\left|c_n(\psi)\right|^r\right)^{1/r} \leq C_8(r)\left\|\psi\right\|_r.$$

Since in the case $s\in\mathbb{N}$ $\psi^{(s)}(x)\sim\sum\limits_{|n|=1}^{\infty}(in)^s c_n(\psi)e^{inx}, x\in\mathbb{T}$, then $c_n(\psi^{(s)})=(in)^s c_n(\psi)$, whence $\left|c_n(\psi^{(s)})\right| = n^s\left|c_n(\psi)\right|$ and therefore

$$\left(\sum_{n=1}^{\infty} n^{rs+r-2}\left|c_n(\psi)\right|^r\right)^{1/r} = \left(\sum_{n=1}^{\infty} n^{r-2}\left|c_n(\psi^{(s)})\right|^r\right)^{1/r} \leq C_8(r)\left\|\psi^{(s)}\right\|_r.$$

At last, applying the estimation (ii) to the difference $\psi^{(s)}(x)-S_n\left(\psi^{(s)};x\right)$ by the known M.Riesz inequality (see f.e. [8], Section 5.11, Inequality (6), p. 339; [21], Section 8.20, p. 594; [1], v. 1, Section 7.6, p.423, [2], v. 2, Section 12.10, p. 120):

$$\left\|f(\cdot)-S_n(f;\cdot)\right\|_r \leq C_{10}(r)E_n(f)_r, r\in(1,\infty), f\in L_r(\mathbb{T}), n\in\mathbb{Z}_+, \tag{7}$$

and by the L_r-analoque of known D.Jackson–S.B.Stechkin inequality (see [5], Theorem 1, p.226; [8], Section 5.11, p.338, Inequality (1), and references therein):

$$E_{n-1}(f)_r \leq C_{11}(k)\omega_k(f;\pi/n)_r, r\in[1,\infty], f\in L_r(\mathbb{T}), n\in\mathbb{N}, \tag{8}$$

we obtain that

$$\left(\sum_{\nu=n+1}^{\infty} \nu^{rs+r-2}\left|c_\nu(\psi)\right|^r\right)^{1/r} = \left(\sum_{\nu=n+1}^{\infty} \nu^{r-2}\left|c_\nu(\psi^{(s)})\right|^r\right)^{1/r} \leq$$

$$\leq C_8(r)\left\|\psi^{(s)}(\cdot)-S_n\left(\psi^{(s)};\cdot\right)\right\|_r \leq 2C_8(r)C_{10}(r)E_n(\psi^{(s)})_r \leq$$

$$\leq 4C_8(r)C_{10}(r)C_{11}(k)\omega_k\left(\psi^{(s)};\pi/(n+1)\right)_r \leq C_9(k,r)\omega_k\left(\psi^{(s)};\pi/n\right)_r,$$

whence it follows the estimation (iii) with constant $C_9(k,r)=4C_8(r)C_{10}(r)C_{11}(k)$. Lemma 1 is proved.

Lemma 2. *Let $s\in\mathbb{N}, k\in\mathbb{N}, \psi\in W_2^s(\mathbb{T})$ and have the Fourier series $\psi(x)\sim\sum\limits_{n=0}^{\infty} c_n(\psi)e^{inx}$, $x\in T$. Then*

(i) $n^{-k}\left(\sum\limits_{\nu=1}^{n}\nu^{2k-1}E_{\nu-1}^2(\psi)_2\right)^{1/2} \leq (2^{-k}+2C_{11}(k))\omega_k\left(\psi;\pi/n\right)_2$, $n\in\mathbb{N}$;

(ii) $\left(\sum\limits_{n=1}^{\infty} n^{2s-1}E_{n-1}^2(\psi)_2\right)^{1/2} \leq \left\|\psi^{(s)}\right\|_2$;

(iii) $\left(\sum\limits_{\nu=n+1}^{\infty}\nu^{2s-1}E^2_{\nu-1}(\psi)_2\right)^{1/2}\le E_n(\psi^{(s)})_2\le 2C_{11}(k)\omega_k\left(\psi^{(s)};\pi/n\right)_2, n\in\mathbb{N};$

(iv) $E_{n-1}(\psi)_2\le n^{-s}E_{n-1}(\psi^{(s)})_2\le 2C_{11}(k)n^{-s}\omega_k\left(\psi^{(s)};\pi/n\right)_2,\ \ n\in\mathbb{N}.$

Proof. The inequality (i) was proved in [3], Lemma 3, pp. 20-21. We have $E^2_{n-1}(\psi)_2=\|\psi(\cdot)-S_{n-1}(\psi;\cdot)\|^2_2=\sum\limits_{\nu=n}^{\infty}|c_\nu(\psi)|^2$ by the Parseval equality, whence

$$\sum_{n=1}^{\infty}n^{2s-1}E^2_{n-1}(\psi)_2=\sum_{n=1}^{\infty}n^{2s-1}\sum_{\nu=n}^{\infty}|c_\nu(\psi)|^2=\sum_{\nu=1}^{\infty}|c_\nu(\psi)|^2\sum_{n=1}^{\nu}n^{2s-1}\le$$

$$\le\sum_{\nu=1}^{\infty}\nu^{2s}\,|c_\nu(\psi)|^2=\sum_{\nu=1}^{\infty}\left|c_\nu(\psi^{(s)})\right|^2=\left\|\psi^{(s)}\right\|^2_2.$$

Furthermore, taking into account the inequality (8), we have that

$$\sum_{\nu=n+1}^{\infty}\nu^{2s-1}E^2_{\nu-1}(\psi)_2=\sum_{\nu=n+1}^{\infty}\nu^{2s-1}\sum_{\mu=\nu}^{\infty}|c_\mu(\psi)|^2=\sum_{\mu=n+1}^{\infty}|c_\mu(\psi)|^2\sum_{\nu=n+1}^{\mu}\nu^{2s-1}\le$$

$$\le\sum_{\mu=n+1}^{\infty}\mu^{2s}\,|c_\mu(\psi)|^2=\sum_{\mu=n+1}^{\infty}\left|c_\mu(\psi^{(s)})\right|^2=\left\|\psi^{(s)}(\cdot)-S_n\left(\psi^{(s)};\cdot\right)\right\|^2_2=$$

$$=E^2_n(\psi^{(s)})_2\le\left(2C_{11}(k)\omega_k\left(\psi^{(s)};\pi/(n+1)\right)_2\right)^2\le(2C_{11}(k))^2\,\omega^2_k\left(\psi^{(s)};\pi/n\right)_2.$$

At last by Parseval equality and by (8) we obtain that

$$E^2_{n-1}(\psi)_2=\sum_{\nu=n}^{\infty}|c_\nu(\psi)|^2=\sum_{\nu=n}^{\infty}\nu^{-2s}\nu^{2s}\,|c_\nu(\psi)|^2=\sum_{\nu=n}^{\infty}\nu^{-2s}\left|c_\nu(\psi^{(s)})\right|^2\le$$

$$\le n^{-2s}\sum_{\nu=n}^{\infty}\left|c_\nu(\psi^{(s)})\right|^2=n^{-2s}E^2_{n-1}\left(\psi^{(s)}\right)_2\le n^{-2s}\,(2C_{11}(k))^2\,\omega^2_k\left(\psi^{(s)};\pi/n\right)_2.$$

Lemma 2 is proved.

Lemma 3. *Let $s\in\mathbb{Z}_+,k\in\mathbb{N},\psi\in C^s(\mathbb{T})$ and have the Fourier series $\psi(x)\sim\sum\limits_{n=1}^{\infty}c_n(\psi)e^{inx},\ \ x\in\mathbb{T}$, with $c_n(\psi)\ge 0$ for every $n\in\mathbb{N}$. Then*

(i) $n^{-æ}\sum\limits_{\nu=1}^{n}\nu^{æ}c_\nu(\psi)\le 2^{-k}\omega_k\left(\operatorname{Re}\psi;\pi/n\right)_\infty,\ n\in\mathbb{N},$

where $æ=k+(1-(-1)^k)/2=\{k$ *for even* $k;\ k+1$ *for odd* $k\}$.

(ii) $n^{-æ}\sum\limits_{\nu=1}^{n}\nu^{æ}c_\nu(\psi)\le 2^{-(k+1)}\pi\omega_k\left(\operatorname{Im}\psi;\pi/n\right)_\infty,\ n\in\mathbb{N},$

where $æ=k+(1+(-1)^k)/2=\{k+1$ *for even* $k;\ k$ *for odd* $k\}$.

(iii) $$\sum_{n=1}^{\infty}n^s c_n(\psi)\le\begin{cases}\left\|\operatorname{Re}\psi^{(s)}\right\|_\infty & \textit{for}\ \ s=0,2,4,\cdots;\\ \left\|\operatorname{Im}\psi^{(s)}\right\|_\infty & \textit{for}\ \ s=1,3,\cdots.\end{cases}$$

(iv) $$\sum_{\nu=n+1}^{\infty}\nu^s c_\nu(\psi)\le 2^{k+2}C_{11}(k)\begin{cases}\omega_k\left(\operatorname{Re}\psi^{(s)};\pi/n\right)_\infty & \textit{for}\ \ s=0,2,4,\cdots;\\ \omega_k\left(\operatorname{Im}\psi^{(s)};\pi/n\right)_\infty & \textit{for}\ \ s=1,3,\cdots.\end{cases}$$

Proof. The inequalities (i) and (ii) was proved in [3], Lemma 4, pp.21-23. We proof now the inequalities (iii) and (iv). First we consider the case $s = 0$. It is clear that if ψ belongs to $C(\mathbb{T})$ then so do $\operatorname{Re}\psi$ and $\operatorname{Im}\psi$. Hence, since $c_n(\psi) \geq 0$ for every $n \in \mathbb{N}$, Fourier series of $\operatorname{Re}\psi$ and $\operatorname{Im}\psi$ (and ψ, respectively) uniformly converge everywhere on $\mathbb{T}$ by Paley's Theorem (see [21], Section 4.2, p.277), so that $\psi(x) = \sum\limits_{n=1}^{\infty} c_n(\psi)e^{inx} = \sum\limits_{n=1}^{\infty} c_n(\psi)\cos nx + i\sum\limits_{n=1}^{\infty} c_n(\psi)\sin nx = \operatorname{Re}\psi(x) + i\operatorname{Im}\psi(x)$, $x \in \mathbb{T}$, whence it follows that $\sum\limits_{n=1}^{\infty} c_n(\psi) = \operatorname{Re}\psi(0) \leq \|\operatorname{Re}\psi\|_\infty \leq \|\psi\|_\infty$. Further by virtue of N.K.Bari inequality ([22], see the proof of Theorem 4, p.293): $\sum\limits_{\nu=2n}^{\infty} c_\nu(f) \leq 4E_n(f)_\infty, n \in \mathbb{N}$, where $f \in C(\mathbb{T}), f(x) = \sum\limits_{n=1}^{\infty} c_n(f)\cos nx$ and $c_n(f) \geq 0$, $n \in \mathbb{N}$, and by inequality (8) we have that ($[t]$ −entire part of $t \in \mathbb{R}$)

$$\sum_{\nu=n+1}^{\infty} c_\nu(\psi) \leq \sum_{\nu=2[(n+1)/2]}^{\infty} c_\nu(\psi) \leq 4E_{[(n+1)/2]}\left(\operatorname{Re}\psi\right)_\infty \leq$$

$$\leq 4C_{11}(k)\omega_k\left(\operatorname{Re}\psi; \pi/\left([(n+1)/2]+1\right)\right)_\infty \leq 4C_{11}(k)\omega_k\left(\operatorname{Re}\psi; 2\pi/(n+1)\right)_\infty \leq$$
$$\leq 4C_{11}(k)2^k\omega_k\left(\operatorname{Re}\psi; \pi/(n+1)\right)_\infty \leq 2^{k+2}C_{11}(k)\omega_k\left(\operatorname{Re}\psi; \pi/n\right)_\infty.$$

Consider now the case $s > 0$. Since $\psi \in C^s(\mathbb{T})$, then $\operatorname{Re}\psi, \operatorname{Im}\psi \in C^s(\mathbb{T})$ and $\psi^{(s)} = (\operatorname{Re}\psi)^{(s)} + i\,(\operatorname{Im}\psi)^{(s)} = \operatorname{Re}\psi^{(s)} + i\operatorname{Im}\psi^{(s)}$.

For even s we have that $\psi^{(s)}(x) \sim \sum\limits_{n=1}^{\infty}(in)^s c_n(\psi)e^{inx} = (-1)^{s/2}\sum\limits_{n=1}^{\infty} n^s c_n(\psi)e^{inx}$, whence

$$\operatorname{Re}\psi^{(s)}(x) \sim (-1)^{s/2}\sum_{n=1}^{\infty} n^s c_n(\psi)\cos nx,\ \operatorname{Im}\psi^{(s)}(x) \sim (-1)^{s/2}\sum_{n=1}^{\infty} n^s c_n(\psi)\sin nx.$$

By Paley's Theorem above mentioned, Fourier series of $(-1)^{s/2}\operatorname{Re}\psi^{(s)}(x)$ and $(-1)^{s/2}\operatorname{Im}\psi^{(s)}(x)$ (and $(-1)^{s/2}\psi^{(s)}(x)$, respectively) uniformly converge everywhere on $\mathbb{T}$, whence it follows that

$$\psi^{(s)}(x) = (-1)^{s/2}\sum_{n=1}^{\infty} n^s c_n(\psi)e^{inx} = (-1)^{s/2}\sum_{n=1}^{\infty} n^s c_n(\psi)\cos nx +$$

$$+ i(-1)^{s/2}\sum_{n=1}^{\infty} n^s c_n(\psi)\sin nx = \operatorname{Re}\psi^{(s)}(x) + i\operatorname{Im}\psi^{(s)}(x),\quad x \in \mathbb{T},$$

and therefore we obtain that

$$\sum_{n=1}^{\infty} n^s c_n(\psi) = (-1)^{s/2}\operatorname{Re}\psi^{(s)}(0) \leq \left\|\operatorname{Re}\psi^{(s)}(\cdot)\right\|_\infty \leq \left\|\psi^{(s)}\right\|_\infty.$$

Further by virtue of Bari inequality and (8) we have that (see the proof (iv) for $s = 0$)

$$\sum_{\nu=n+1}^{\infty} \nu^s c_\nu(\psi) \leq \sum_{\nu=2[(n+1)/2]}^{\infty} \nu^s c_\nu(\psi) \leq 4E_{[(n+1)/2]}\left((-1)^{s/2}\operatorname{Re}\psi^{(s)}\right)_\infty \leq$$

$$\leq 4C_{11}(k)2^k\omega_k\left(\operatorname{Re}\psi^{(s)};\pi/(n+1)\right)_\infty \leq 2^{k+2}C_{11}(k)\omega_k\left(\operatorname{Re}\psi^{(s)};\pi/n\right)_\infty.$$

For odd s we have that

$$\psi^{(s)}(x)\sim\sum_{n=1}^{\infty}(in)^s c_n(\psi)e^{inx}=$$

$$=(-1)^{(s+1)/2}\sum_{n=1}^{\infty}n^s c_n(\psi)\sin nx+i(-1)^{(s+1)/2+1}\sum_{n=1}^{\infty}n^s c_n(\psi)\cos nx,$$

whence

$$\operatorname{Re}\psi^{(s)}(x)\sim(-1)^{(s+1)/2}\sum_{n=1}^{\infty}n^s c_n(\psi)\sin nx,$$

$$\operatorname{Im}\psi^{(s)}(x)\sim(-1)^{(s+1)/2+1}\sum_{n=1}^{\infty}n^s c_n(\psi)\cos nx.$$

The arguments using above in considered the case of even s give the following estimations

$$\sum_{n=1}^{\infty}n^s c_n(\psi)=(-1)^{(s+1)/2+1}\operatorname{Im}\psi^{(s)}(0)\leq\left\|\operatorname{Im}\psi^{(s)}(\cdot)\right\|_\infty\leq\left\|\psi^{(s)}\right\|_\infty,$$

$$\sum_{\nu=n+1}^{\infty}\nu^s c_\nu(\psi)\leq 4E_{[(n+1)/2]}\left((-1)^{(s+1)/2+1}\operatorname{Im}\psi^{(s)}\right)_\infty\leq$$

$$\leq 4C_{11}(k)2^k\omega_k\left(\operatorname{Im}\psi^{(s)};\pi/(n+1)\right)_\infty\leq 2^{k+2}C_{11}(k)\omega_k\left(\operatorname{Im}\psi^{(s)};\pi/n\right)_\infty.$$

Lemma 3 is proved.

Given $\alpha\in(0,\infty)$, let $M_0(\alpha)$ be the set of all sequences $\lambda=\{\lambda_n\}_{n=1}^{\infty}\in M_0$ such that $n^\alpha\lambda_n\downarrow(n\uparrow)$.

Lemma 4. *Let $p,q\in(1,\infty), r=pq/(p+q-pq)\in(1,\infty], \theta=\theta(r)=\min\{2,r\}$ for $r\in(1,\infty)$ and $\theta(\infty)=1, k\in\mathbb{N}, s\in\mathbb{N}, \lambda=\{\lambda_n\}_{n=1}^{\infty}\in M_0(\alpha)$ and $\varepsilon=\{\varepsilon_n\}_{n=1}^{\infty}\in M_0(\beta)$ for some $\alpha,\beta\in(0,\infty)$. Then there are functions $f_0(\cdot;p;\lambda)\in L_p(\mathbb{T})$ and $g_0(\cdot;q;\varepsilon)\in L_q(\mathbb{T})$ such that*

(i) $E_{n-1}(f_0)_p\leq C_{12}(p,\alpha)\lambda_n,\ E_{n-1}(g_0)_q\leq C_{12}(q,\beta)\varepsilon_n,\ n\in\mathbb{N};$

(ii) $h_0=f_0*g_0\in W_r^s(\mathbb{T})\Longleftrightarrow\sum\limits_{n=1}^{\infty}n^{\theta s-1}\lambda_n^\theta\varepsilon_n^\theta<\infty;$

(iii) if the series in (ii) converge, then

$$\left(\sum_{\nu=n+1}^{\infty}\nu^{\theta s-1}\lambda_\nu^\theta\varepsilon_\nu^\theta\right)^{1/\theta}+n^{-k}\left(\sum_{\nu=1}^{n}\nu^{\theta(k+s)-1}\lambda_\nu^\theta\varepsilon_\nu^\theta\right)^{1/\theta}\leq$$

$$\leq C_{13}(k,s,r)\omega_k\left(h_0^{(s)};\pi/n\right)_r,\ n\in\mathbb{N}.$$

Proof. First we consider the case $1<r\leq 2$. For $p,q\in(1,\infty)$ $\Big(p'=p/(p-1)$, $q'=q/(q-1)\Big)$, let

$$f_0(x;p;\lambda)=\sum_{n=1}^{\infty}n^{-1/p'}\lambda_n e^{inx},\quad g_0(x;q;\varepsilon)=\sum_{n=1}^{\infty}n^{-1/q'}\varepsilon_n e^{inx},\quad x\in\mathbb{T}.$$

Since $\lambda \in M_0(\alpha)$ and $\varepsilon \in M_0(\beta)$, in virtue of Lemma 1 [23] we have $f_0 \in L_p(\mathbb{T})$, $E_{n-1}(f_0)_p \leq C_{12}(p,\alpha)\lambda_n$ and $g_0 \in L_q(\mathbb{T})$, $E_{n-1}(g_0)_q \leq C_{12}(q,\beta)\varepsilon_n$, $n \in \mathbb{N}$.

If the series in (ii) converge, then by (i) we have that

$$\sum_{n=1}^{\infty} n^{rs-1} E_{n-1}^r(f_0)_p E_{n-1}^r(g_0)_q \leq (C_{12}(p,\alpha)C_{12}(q,\beta))^r \sum_{n=1}^{\infty} n^{rs-1}\lambda_n^s \varepsilon_n^s < \infty,$$

whence $h_0 = f_0 * g_0 \in W_r^s(\mathbb{T})$ by Theorem 1. On the other hand, if $h_0 \in W_r^s(\mathbb{T})$, then taking into account $c_n(h_0) = c_n(f_0) \cdot c_n(g_0) = n^{-(1/p'+1/q')}\lambda_n\varepsilon_n$ and $r - 1 - r(1/p' + 1/q') = 0$, we have by (ii) of Lemma 1 that

$$\left(\sum_{n=1}^{\infty} n^{rs-1}\lambda_n^r\varepsilon_n^r\right)^{1/r} = \left(\sum_{n=1}^{\infty} n^{r-2} n^{rs-r(1/p'+1/q')}\lambda_n^r\varepsilon_n^r\right)^{1/r} =$$

$$= \left(\sum_{n=1}^{\infty} n^{rs+r-2}\left|c_n(h_0)\right|^r\right)^{1/r} \leq C_8(r)\left\|h_0^{(s)}\right\|_r < \infty.$$

Further applying the inequality from (iii) of Lemma 1 and taking into account the estimation from (ii) of Lemma 1 [23] (for estimation of the second summand) we obtain that

$$\left(\sum_{\nu=n+1}^{\infty} \nu^{rs-1}\lambda_\nu^r\varepsilon_\nu^r\right)^{1/r} + n^{-k}\left(\sum_{\nu=1}^{n} \nu^{r(k+s)-1}\lambda_\nu^r\varepsilon_\nu^r\right)^{1/r} \leq C_9(k,r)\omega_k\left(h_0^{(s)};\pi/n\right)_r +$$

$$+C_{14}(k+s,r)n^s\omega_{k+s}\left(h_0;\pi/n\right)_r \leq \left(C_9(k,r) + \pi^s C_{14}(k+s,r)\right)\omega_k\left(h_0^{(s)};\pi/n\right)_r,$$

whence the estimation (iii) follows in the case $1 < r \leq 2$.

Consider now the case $2 < r < \infty$. Put

$$f_0(x;\lambda) = \sum_{\nu=0}^{\infty} \lambda_{2^\nu} e^{i2^\nu x},\ g_0(x;\varepsilon) = \sum_{\nu=0}^{\infty} \varepsilon_{2^\nu} e^{i2^\nu x},\ x \in \mathbb{T}.$$

Since $\lambda \in M_0(\alpha)$ and $\varepsilon \in M_0(\beta)$, then by Lemma 1 [23] (see the case $2 < r < \infty$) we have that $f_0 \in L_p(\mathbb{T}), E_{n-1}(f_0)_p \leq C_{12}(p,\alpha)\lambda_n$ and $g_0 \in L_q(\mathbb{T}), E_{n-1}(g_0)_q \leq C_{12}(q,\beta)\varepsilon_n, n \in \mathbb{N}$, for every $p, q \in (1,\infty)$, whence it follows that $h_0 = f_0 * g_0 \in L_r(\mathbb{T})$ for all $r \in (1,\infty]$ by Theorem A.

If the series in (ii) converge, then by (i) we have that

$$\sum_{n=1}^{\infty} n^{2s-1} E_{n-1}^2(f_0)_p E_{n-1}^2(g_0)_q \leq (C_{12}(p,\alpha)C_{12}(q,\beta))^2 \sum_{n=1}^{\infty} n^{2s-1}\lambda_n^2 \varepsilon_n^2 < \infty,$$

whence by Theorem 1 we obtain that $h_0 = f_0 * g_0 \in W_r^s(\mathbb{T})$ for all $r \in (1,\infty]$ and in the sense of convergence in $L_r(\mathbb{T})$

$$h_0^{(s)}(x) = (f_0 * g_0)^{(s)}(x) = \sum_{\nu=0}^{\infty} (i2^\nu)^s \lambda_{2^\nu}\varepsilon_{2^\nu} e^{i2^\nu x},\ \ x \in \mathbb{T}.$$

On the other hand if $h_0 = f_0 * g_0 \in W_r^s(\mathbb{T})$ for $r \in (1,\infty]$ and since $2 < r < \infty$ thereafter assumption, then $h_0 \in W_2^s(\mathbb{T})$, and therefore $h_0^{(s)} \in L_2(\mathbb{T})$. Clearly we have that $E_0^2(h_0)_2 = \sum\limits_{\nu=0}^{\infty} \lambda_{2^\nu}^2 \varepsilon_{2^\nu}^2 \geq \lambda_1^2 \varepsilon_1^2$, $E_{2^j}^2(h_0)_2 = \sum\limits_{\nu=j+1}^{\infty} \lambda_{2^\nu}^2 \varepsilon_{2^\nu}^2 \geq \lambda_{2^{j+1}}^2 \varepsilon_{2^{j+1}}^2$ for $j \in \mathbb{Z}_+$.

Taking into account these estimations, we obtain that $\left(C_{15}(s) = (2s)^{-1}\left(2^{2s}-1\right)\right)$

$$\sum_{n=1}^{\infty} n^{2s-1}\lambda_n^2\varepsilon_n^2 = \sum_{j=0}^{\infty}\sum_{n=2^j}^{2^{j+1}-1} n^{2s-1}\lambda_n^2\varepsilon_n^2 \leq C_{15}(s)\sum_{j=0}^{\infty} 2^{2sj}\lambda_{2^j}^2\varepsilon_{2^j}^2 =$$

$$= C_{15}(s)\left\{\lambda_1^2\varepsilon_1^2 + 2^{2s}\lambda_2^2\varepsilon_2^2 + \sum_{j=1}^{\infty} 2^{2s(j+1)}\lambda_{2^{j+1}}^2\varepsilon_{2^{j+1}}^2\right\} \leq$$

$$\leq C_{15}(s)\left\{E_0^2\left(h_0\right)_2 + 2^{2s}E_1^2\left(h_0\right)_2 + \sum_{j=1}^{\infty} 2^{2s(j+1)}E_{2^j}^2\left(h_0\right)_2\right\} \leq$$

$$\leq C_{15}(s)\left\{E_0^2\left(h_0\right)_2 + 2^{2s}E_1^2\left(h_0\right)_2 + \left(C_{15}(s)\right)^{-1}2^{4s}\sum_{j=1}^{\infty}\sum_{\nu=2^{j-1}+1}^{2^j}\nu^{2s-1}E_\nu^2\left(h_0\right)_2\right\} =$$

$$= C_{15}(s)\left\{E_0^2\left(h_0\right)_2 + 2^{2s}E_1^2\left(h_0\right)_2 + \left(C_{15}(s)\right)^{-1}2^{4s}\sum_{\nu=2}^{\infty}\nu^{2s-1}E_\nu^2\left(h_0\right)_2\right\} \leq$$

$$\leq C_{16}(s)\sum_{\nu=1}^{\infty}\nu^{2s-1}E_{\nu-1}^2\left(h_0\right)_2,$$

whence we have by (ii) of Lemma 2 and for $r \in (2,\infty)$ that

$$\left(\sum_{n=1}^{\infty} n^{2s-1}\lambda_n^2\varepsilon_n^2\right)^{1/2} \leq \left(C_{16}(s)\right)^{1/2}\left(\sum_{\nu=1}^{\infty}\nu^{2s-1}E_{\nu-1}^2\left(h_0\right)_2\right)^{1/2} \leq$$

$$\leq \left(C_{16}(s)\right)^{1/2}\left\|h_0^{(s)}\right\|_2 \leq \left(C_{16}(s)\right)^{1/2}\left\|h_0^{(s)}\right\|_r < \infty.$$

It follows from this estimation that (ii) holds for $r \in (2,\infty)$.

We proof now the estimation in point (iii). We have that

$$\sum_{\nu=n+1}^{\infty}\nu^{2s-1}\lambda_\nu^2\varepsilon_\nu^2 = \sum_{\nu=n+1}^{4n-1}\nu^{2s-1}\lambda_\nu^2\varepsilon_\nu^2 + \sum_{\nu=4n}^{\infty}\nu^{2s-1}\lambda_\nu^2\varepsilon_\nu^2 = \sigma_1 + \sigma_2.$$

For σ_1 we obtain that

$$\sigma_1 \leq \lambda_{n+1}^2\varepsilon_{n+1}^2\sum_{\nu=n+1}^{4n-1}\nu^{2s-1} \leq (2s)^{-1}\left(4^{2s}-1\right)n^{2s}\lambda_{n+1}^2\varepsilon_{n+1}^2.$$

Since for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $2^{m-1} \leq n < 2^m$, we have that (see above the proof of necessity in point (ii))

$$\sigma_2 \leq \sum_{\nu=2^{m+1}}^{\infty}\nu^{2s-1}\lambda_\nu^2\varepsilon_\nu^2 = \sum_{j=m+1}^{\infty}\sum_{\nu=2^j}^{2^{j+1}-1}\nu^{2s-1}\lambda_\nu^2\varepsilon_\nu^2 \leq C_{15}(s)\sum_{j=m+1}^{\infty}2^{2sj}\lambda_{2^j}^2\varepsilon_{2^j}^2 =$$

$$= C_{15}(s)\left\{2^{2s(m+1)}\lambda^2_{2^{m+1}}\varepsilon^2_{2^{m+1}} + \sum_{j=m+1}^{\infty} 2^{2s(j+1)}\lambda^2_{2^{j+1}}\varepsilon^2_{2^{j+1}}\right\} \le$$

$$\le C_{15}(s)\left\{2^{2s(m+1)}E^2_{2^m}(h_0)_2 + \sum_{j=m+1}^{\infty} 2^{2s(j+1)}E^2_{2^j}(h_0)_2\right\} \le$$

$$\le C_{15}(s)\left\{2^{2s(m+1)}E^2_{2^m}(h_0)_2 + (C_{15}(s))^{-1}\, 2^{4s}\sum_{\nu=2^m+1}^{\infty} \nu^{2s-1}E^2_{\nu}(h_0)_2\right\} \le$$

$$\le C_{15}(s)2^{4s}\left\{n^{2s}E^2_n(h_0)_2 + (C_{15}(s))^{-1}\sum_{\nu=n+1}^{\infty} \nu^{2s-1}E^2_{\nu}(h_0)_2\right\}.$$

Taking into account the estimations for σ_1 and σ_2, the inequalities in (iii) and (iv) of Lemma 2 and (8) we have that

$$\left(\sum_{\nu=n+1}^{\infty}\nu^{2s-1}\lambda^2_{\nu}\varepsilon^2_{\nu}\right)^{1/2} \le (2s)^{-1/2}\left(4^{2s}-1\right)^{1/2} n^s\lambda_{n+1}\varepsilon_{n+1}+$$

$$+2^{2s}\left(C_{15}(s)\right)^{1/2}\left\{n^sE_{n-1}(h_0)_2 + (C_{15}(s))^{-1/2}\left(\sum_{\nu=n+1}^{\infty}\nu^{2s-1}E^2_{\nu-1}(h_0)_2\right)^{1/2}\right\} \le$$

$$\le (2s)^{-1/2}\left(4^{2s}-1\right)^{1/2} n^s\lambda_{n+1}\varepsilon_{n+1} + 2^{2s}\left(C_{15}(s)\right)^{1/2}E_{n-1}\left(h_0^{(s)}\right)_2 + 2^{2s}E_n\left(h_0^{(s)}\right)_2 \le$$

$$\le C_{17}(k,s)\omega_k\left(h_0^{(s)};\pi/n\right)_2 + C_{18}(s)n^s\lambda_{n+1}\varepsilon_{n+1},$$

where $C_{17}(k,s) = 2^{2s+1}C_{11}(k)\left(1+(C_{15}(s))^{1/2}\right), C_{18}(s) = (2s)^{-1/2}\left(4^{2s}-1\right)^{1/2}.$

In virtue of estimation in (ii) of Lemma 1 [23] (the case $2 < r < \infty$) we have the estimation for second summand in right part of the last inequality:

$$n^s\lambda_{n+1}\varepsilon_{n+1} \le n^s\lambda_n\varepsilon_n \le (2(k+s))^{1/2}\, n^{-k}\left(\sum_{\nu=1}^{n}\nu^{2(k+s)-1}\lambda^2_{\nu}\varepsilon^2_{\nu}\right)^{1/2} \le$$

$$\le (2(k+s))^{1/2}\, C_{14}(k+s,2)\, n^s\omega_{k+s}(h_0;\pi/n)_r \le$$

$$\le (2(k+s))^{1/2}\, C_{14}(k+s,2)\, \pi^s\omega_k\left(h_0^{(s)};\pi/n\right)_r,$$

and by this we obtain that

$$\left(\sum_{\nu=n+1}^{\infty}\nu^{2s-1}\lambda^2_{\nu}\varepsilon^2_{\nu}\right)^{1/2} \le C_{19}(k,s)\omega_k\left(h_0^{(s)};\pi/n\right)_r,$$

where $C_{19}(k,s) = C_{17}(k,s) + C_{18}(s)\left(2(k+s)\right)^{1/2}C_{14}(k+s,2)\,\pi^s$.

By last estimation and estimation in (ii) of Lemma 1 [23] (the estimation of the second summand for $2 < r < \infty$) we have that

$$\left(\sum_{\nu=n+1}^{\infty}\nu^{2s-1}\lambda^2_{\nu}\varepsilon^2_{\nu}\right)^{1/2} + n^{-k}\left(\sum_{\nu=1}^{n}\nu^{2(k+s)-1}\lambda^2_{\nu}\varepsilon^2_{\nu}\right)^{1/2} \le$$

$$\leq C_{19}(k,s)\omega_k\left(h_0^{(s)};\pi/n\right)_r + C_{14}\left(k+s,2\right)n^s\omega_{k+s}\left(h_0;\pi/n\right)_r \leq$$

$$\leq \left\{C_{19}(k,s) + C_{14}\left(k+s,2\right)\pi^s\right\}\omega_k\left(h_0^{(s)};\pi/n\right)_r,$$

whence the estimation (iii) follows in the case $2 < r < \infty$.

At last we consider the case $r = \infty$. In this case $1/p + 1/q = 1$, that is $q = p'$, and therefore $1/p' + 1/q' = 1$. Let $f_0(\cdot;p;\lambda)$ and $g_0(\cdot;q;\varepsilon)$ be functions such as in the case $1 < r \leq 2$, and $h_0 = f_0 * g_0$. If the series in (ii) converge, then by (i) we have that

$$\sum_{n=1}^{\infty} n^{s-1}E_{n-1}\left(f_0\right)_p E_{n-1}\left(g_0\right)_q \leq C_{12}(p,\alpha)C_{12}(q,\beta)\sum_{n=1}^{\infty} n^{s-1}\lambda_n\varepsilon_n < \infty,$$

whence $h_0 \in W^s_\infty(\mathbb{T}) \equiv C^s(\mathbb{T})$ by Theorem 1. On the other hand, if $h_0 \in C^s(\mathbb{T})$, then by inequality in (iii) of Lemma 3 we have that $(1/p' + 1/q' = 1)$

$$\sum_{n=1}^{\infty} n^{s-1}\lambda_n\varepsilon_n = \sum_{n=1}^{\infty} n^s n^{-(1/p'+1/q')}\lambda_n\varepsilon_n = \sum_{n=1}^{\infty} n^s c_n(h_0) \leq \left\|h_0^{(s)}\right\|_\infty.$$

Further, applying the inequality (iv) of Lemma 3 and taking into account the estimation in (ii) of Lemma 1 [23] (the estimation of the second summand in the case $r = \infty$) we obtain that

$$\sum_{\nu=n+1}^{\infty} \nu^{s-1}\lambda_\nu\varepsilon_\nu + n^{-k}\sum_{\nu=1}^{n} \nu^{k+s-1}\lambda_\nu\varepsilon_\nu \leq$$

$$\leq 2^{k+2}C_{11}(k)\omega_k\left(h_0^{(s)};\pi/n\right)_\infty + C_{14}\left(k+s,\infty\right)n^s\omega_{k+s}\left(h_0;\pi/n\right)_\infty \leq$$

$$\leq \left\{2^{k+2}C_{11}(k) + \pi^s C_{14}\left(k+s,\infty\right)\right\}\omega_k\left(h_0^{(s)};\pi/n\right)_\infty,$$

whence the estimation (iii) follows in the case $r = \infty$.

Lemma 4 is proved.

Given $p, q \in [1,\infty]$ and $\lambda, \varepsilon \in M_0$, put

$$E_p[\lambda] * E_q[\varepsilon] = \left\{h = f * g : f \in E_p[\lambda], g \in E_q[\varepsilon]\right\}.$$

The following theorem shows that estimation (5) of Theorem 1 is exact in the sense of order on classes $E_p[\lambda] * E_q[\varepsilon]$ in the case $p, q \in (1,\infty)$ under conditions that $\lambda \in M_0(\alpha)$ and $\varepsilon \in M_0(\beta)$, for some $\alpha, \beta \in (0,\infty)$.

Theorem 2. *Let $p, q \in (1,\infty)$, $r = pq/(p+q-pq) \in (1,\infty]$, $\theta = \theta(r) = \min\{2, r\}$ for $r \in (1,\infty)$ and $\theta(\infty) = 1$, $k \in \mathbb{N}$, $s \in \mathbb{N}$, $\lambda = \{\lambda_n\} \in M_0(\alpha)$, $\varepsilon = \{\varepsilon_n\} \in M_0(\beta)$ for some $\alpha, \beta \in (0,\infty)$, and*

$$\sum_{n=1}^{\infty} n^{\theta s-1}\lambda_n^\theta\varepsilon_n^\theta < \infty. \tag{9}$$

Then

$$\sup\left\{\omega_k\left(h^{(s)};\pi/n\right)_r : h \in E_p[\lambda] * E_q[\varepsilon]\right\} \asymp$$

$$\asymp \left(\sum_{\nu=n+1}^{\infty} \nu^{\theta s-1} \lambda_\nu^\theta \varepsilon_\nu^\theta \right)^{1/\theta} + n^{-k} \left(\sum_{\nu=1}^{n} \nu^{\theta(k+s)-1} \lambda_\nu^\theta \varepsilon_\nu^\theta \right)^{1/\theta}, \quad n \in \mathbb{N}.$$

Proof. Indeed, the upper estimation for every $p, q \in [1, \infty]$ and for arbitrary $\lambda, \varepsilon \in M_0$ immediately follows by inequality (5) of Theorem 1. The lower estimation is realized by function

$$h_0(\cdot; p, q; \lambda, \varepsilon) = (C_{12}(p, \alpha))^{-1} f_0(\cdot; p; \alpha) * (C_{12}(q, \beta))^{-1} g_0(\cdot; q; \varepsilon) \in E_p[\lambda] * E_q[\varepsilon]$$

in virtue of (iii) of Lemma 4.

Remark. The condition convergence of the series (9) it is necessary and sufficiently for imbedding $E_p[\lambda] * E_q[\varepsilon] \subset W_r^s(\mathbb{T})$. The sufficiency for arbitrary $\lambda, \varepsilon \in M_0$ immediately follows from the first part of the statement of Theorem 1. The necessity under conditions $\lambda \in M_0(\alpha)$ and $\varepsilon \in M_0(\beta)$ follows from the statement (ii) of Lemma 4.

Given $p, q \in [1, \infty]$ and $\alpha, \beta \in (0, \infty)$ we denote

$$E_{p,\alpha} = E_p\left[\left\{n^{-\alpha}\right\}_{n=1}^{\infty}\right], \quad E_{q,\beta} = E_q\left[\left\{n^{-\beta}\right\}_{n=1}^{\infty}\right].$$

Theorem 3. *Let* $p, q \in (1, \infty), r = pq/(p+q-pq) \in (1, \infty], \theta = \theta(r) = \min\{2, r\}$ *for* $r \in (1, \infty)$ *and* $\theta(\infty) = 1, k \in \mathbb{N}, s \in \mathbb{N}, \alpha, \beta \in (0, \infty), \rho = \alpha + \beta - s > 0$. *Then for* $\delta \in (0, \pi]$

(i) $\sup\left\{\omega_k\left(h^{(s)}; \delta\right)_r : h \in E_{p,\alpha} * E_{q,\beta}\right\} \asymp$
$\asymp \left\{\delta^\rho \text{ for } \rho < k;\ \ \delta^k \left(\ln(\pi e/\delta)\right)^{1/\theta} \text{ for } \rho = k; \delta^k \text{ for } \rho > k\right\}.$

(ii) $\sup\left\{\omega_{k+1}\left(h^{(s)}; \delta\right)_r : h \in E_{p,\alpha} * E_{q,\beta}\right\} \asymp \delta^k$ *for* $\rho = k$.

Proof. First note the following. For every $\delta \in (0, \pi]$ there exists an $n \in \mathbb{N}$ such that $\pi/(n+1) < \delta \le \pi/n$, whence we have the following estimations:

$$2^{-k}\omega_k\left(h^{(s)}; \pi/n\right)_r \le \omega_k\left(h^{(s)}; \delta\right)_r \le \omega_k\left(h^{(s)}; \pi/n\right)_r;$$

$$2^{-\rho}(\pi/n)^\rho < \delta^\rho \le (\pi/n)^\rho \quad \text{for every} \quad \rho \in (0, \infty);$$

$$\delta^k (\ln(\pi e/\delta))^{1/\theta} \le (\pi/n)^k (\ln(e(n+1)))^{1/\theta} =$$

$$= \pi^k n^{-k} (1 + \ln(n+1))^{1/\theta} \le 3^{1/\theta} \pi^k n^{-k} (\ln(n+1))^{1/\theta};$$

$$n^{-k} (\ln(en))^{1/\theta} \le (2/\pi)^k (\pi/(n+1))^k (\ln(\pi e/\delta))^{1/\theta} < (2/\pi)^k \delta^k (\ln(\pi e/\delta))^{1/\theta}.$$

Upper estimations. For every function $h \in E_{p,\alpha} * E_{q,\beta}$ we have that $h = f * g$ for some $f \in L_p(\mathbb{T})$ and $g \in L_q(\mathbb{T})$ with $E_{n-1}(f)_p \le n^{-\alpha}$ and $E_{n-1}(g)_q \le n^{-\beta}$, for every $n \in \mathbb{N}$. Hence we obtain by Corollary that $\Big(C_{20}(k, s, r, \rho, \theta) = C_3(k, s, r) C_4(k, \rho, \theta),$ $C_{21}(k, s, r, \theta) = C_3(k+1, s, r) C_5(k, \theta)\Big)$

$$C_{20}^{-1}\omega_k\left(h^{(s)}; \delta\right)_r \le C_{20}^{-1}\omega_k\left(h^{(s)}; \pi/n\right)_r \le n^{-\rho} < (2/\pi)^\rho \delta^\rho \quad \text{for} \quad \rho < k,$$

$$C_{20}^{-1}\omega_k\left(h^{(s)}; \delta\right)_r \le C_{20}^{-1}\omega_k\left(h^{(s)}; \pi/n\right)_r \le n^{-k} (\ln(en))^{1/\theta} < (2/\pi)^k \delta^k (\ln(\pi e/\delta))^{1/\theta}$$

for $\rho = k$;

$$C_{20}^{-1}\omega_k\left(h^{(s)}; \delta\right)_r \le C_{20}^{-1}\omega_k\left(h^{(s)}; \pi/n\right)_r \le n^{-k} \le (2/\pi)^k \delta^k \quad \text{for } \rho > k;$$

$$C_{21}^{-1}\omega_{k+1}\left(h^{(s)};\delta\right)_r \le C_{21}^{-1}\omega_{k+1}\left(h^{(s)};\pi/n\right)_r \le n^{-k} \le (2/\pi)^k\delta^k \quad \text{for } \rho = k.$$

It follows from these inequalities that the upper estimations in (i) and (ii) of Theorem 3 hold.

Lower estimations. We have by (i) of Lemma 4 that

$$(C_{12}(p,\alpha))^{-1} f_0(\cdot;p;\lambda) \in E_{p,\alpha} \text{ and } (C_{12}(q,\beta))^{-1} g_0(\cdot;q;\varepsilon) \in E_{q,\beta}$$

for $\lambda = \{n^{-\alpha}\}_{n=1}^{\infty}$ and $\varepsilon = \{n^{-\beta}\}_{n=1}^{\infty}$, whence

$$h_0 = (C_{12}(p,\alpha))^{-1} f_0 * (C_{12}(q,\beta))^{-1} g_0 \in E_{p,\alpha} * E_{q,\beta}.$$

So, we have by (iii) of Lemma 4 that $(\rho = \alpha + \beta - s > 0)$

$$C_{13}(k,r,s)\cdot C_{12}(p,\alpha)\cdot C_{12}(q,\beta)\omega_k\left(h_0^{(s)};\pi/n\right)_r = C_{13}(k,r,s)\omega_k\left((f_0*g_0)^{(s)};\pi/n\right)_r \ge$$

$$\ge \left(\sum_{\nu=n+1}^{\infty}\nu^{-\theta\rho-1}\right)^{1/\theta} + n^{-k}\left(\sum_{\nu=1}^{n}\nu^{\theta(k-\rho)-1}\right)^{1/\theta}.$$

Taking into account the following inequalities

$$\left(\sum_{\nu=n+1}^{\infty}\nu^{-\theta\rho-1}\right)^{1/\theta} \ge (\theta\rho)^{-1/\theta}(n+1)^{-\rho} \ge (\theta\rho)^{-1/\theta}2^{-\rho}n^{-\rho};$$

$$n^{-k}\left(\sum_{\nu=1}^{n}\nu^{\theta(k-\rho)-1}\right)^{1/\theta} \ge \varphi_n(k-\rho;\theta), \quad \text{where } \varphi_n(k-\rho;\theta) =$$

$$= (\theta(k-\rho))^{-1/\theta} n^{-\rho} \quad \text{for } \rho < k, \theta(k-\rho) \ge 1, \varphi_n(k-\rho;\theta) =$$

$$= 2^{k-\rho-1/\theta}n^{-\rho} \quad \text{for } \rho < k, \theta(k-\rho) \le 1, \varphi_n(k-\rho;\theta) = n^{-k}\left(\ln(n+1)\right)^{1/\theta}$$

for $\rho = k$ and $\varphi_n(k-\rho;\theta) = n^{-k}$ for $\rho > k$, we obtain that $(C_{22} = C_{13}(k,r,s)\times$ $\times C_{12}(p,\alpha)C_{12}(q,\beta))$

$$2^k C_{22}\omega_k\left(h_0^{(s)};\delta\right)_r \ge C_{22}\omega_k\left(h_0^{(s)};\pi/n\right)_r \ge \left\{(\theta\rho)^{-1/\theta}2^{-\rho} + (\theta(k-\rho))^{-1/\theta}\right\}n^{-\rho} \ge$$

$$\ge \left\{(\theta\rho)^{-1/\theta}2^{-\rho} + (\theta(k-\rho))^{-1/\theta}\right\}\pi^{-\rho}\delta^{\rho} \quad \text{for } \rho < k, \theta(k-\rho) \ge 1;$$

$$2^k C_{22}\omega_k\left(h_0^{(s)};\delta\right)_r \ge C_{22}\omega_k\left(h_0^{(s)};\pi/n\right)_r \ge \left\{(\theta\rho)^{-1/\theta}2^{-\rho} + 2^{k-\rho-1/\theta}\right\}n^{-\rho} \ge$$

$$\ge \left\{(\theta\rho)^{-1/\theta}2^{-\rho} + 2^{k-\rho-1/\theta}\right\}\pi^{-\rho}\delta^{\rho} \quad \text{for } \rho < k, \theta(k-\rho) \le 1;$$

$$2^k C_{22}\omega_k\left(h_0^{(s)};\delta\right)_r \ge C_{22}\omega_k\left(h_0^{(s)};\pi/n\right)_r \ge (\theta\rho)^{-1/\theta}2^{-\rho}n^{-\rho} + n^{-k}\left(\ln(n+1)\right)^{1/\theta} =$$

$$= \left\{(\theta k)^{-1/\theta}2^{-k} + (\ln(n+1))^{1/\theta}\right\}n^{-k} \ge n^{-k}(\ln(n+1))^{1/\theta} \ge$$

$$\ge 3^{-1/\theta}\pi^{-k}\delta^k\left(\ln(\pi e/\delta)\right)^{1/\theta} \quad \text{for } \rho = k,$$

$$2^k C_{22}\omega_k\left(h_0^{(s)};\delta\right)_r \ge C_{22}\omega_k\left(h_0^{(s)};\pi/n\right)_r \ge (\theta\rho)^{-1/\theta}2^{-\rho}n^{-\rho} + n^{-k} \ge$$

$$\geq n^{-k} \geq \pi^{-k}\delta^k \quad \text{for} \quad \rho > k.$$

At last, by (iii) of Lemma 4 we have that (the case $\rho = k$)

$$2^{k+1}C_{22}\omega_{k+1}\left(h_0^{(s)};\delta\right)_r \geq C_{22}\omega_{k+1}\left(h_0^{(s)};\pi/n\right)_r =$$

$$= C_{13}\left(k+1,r,s\right)\omega_{k+1}\left(\left(f_0 * g_0\right)^{(s)};\pi/n\right)_r \geq \left(\sum_{\nu=n+1}^{\infty}\nu^{-\theta k-1}\right)^{1/\theta} +$$

$$+n^{-(k+1)}\left(\sum_{\nu=1}^{n}\nu^{\theta-1}\right)^{1/\theta} \geq (\theta k)^{-1/\theta}2^{-k}n^{-k} + n^{-(k+1)}\theta^{-1/\theta}n =$$

$$= \left\{(\theta k)^{-1/\theta}2^{-k} + \theta^{-1/\theta}\right\}n^{-k} \geq \left\{(\theta k)^{-1/\theta}2^{-k} + \theta^{-1/\theta}\right\}\pi^{-k}\delta^k.$$

It follows from these inequalities that the lower estimations in (i) and (ii) of Theorem 3 hold.

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