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ON STRONG SOLVABILITY OF DIRICHLET PROBLEM FOR NON-UNIFORMLY DEGENERATE ELLIPTIC EQUATIONS OF SECOND ORDER

Abstract

The Dirichlet problem is considered for non-uniformly degenerate second order elliptic equations of divergent structure. The basic coercive inequality is proved and the conditions under which this problem is strongly solvable in Sobolevs anisotropic weight space are found.

Introduction. Let D be a bounded domain of n - dimensional Euclidean space E_n, $n \geq 3$, ∂D be its boundary, moreover $\partial D \subset C^2$ and $0 \in \overline{D}$. In D consider the Dirichlet problem

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x), \tag{1}$$

$$u|_{\partial D} = 0, \tag{2}$$

where $\|a_{ij}(x)\|$ is a real symmetric matrix with elements measurable in D, and for all $x \in D$, $\zeta \in E_n$ the condition

$$\mu \sum_{i=1}^{n} \lambda_i(x) \zeta_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \zeta_i \zeta_j \leq \mu^{-1} \sum_{i=1}^{n} \lambda_i(x) \zeta_i^2 \tag{3}$$

be fulfilled.

Here $\mu \in (0,1]$ is a constant, the functions $\lambda_i(x) = g_i(\rho(x))\ \rho(x) = \sum_{i=1}^{n} \omega_i(|x_i|)$, $g_i(t) \left(\frac{\omega_i^{-1}(t)}{t} \right)^2$, $i = 1, ..., n$. For the functions $\omega_i(t)$, $i = 1, ..., n$ we'll assume that the following conditions are satisfied: $\omega_i(t)$ are continuous and strongly monotonically increasing on $[0, d]$ functions ($diamD = d$), $\omega_i(0) = 0$, $\omega_i^{-1}(t)$ are the functions inverse to $\omega_i(t)$, and furthermore

$$C_1 \omega_i(t) \leq \omega_i(2t) \leq C_2 \omega_i(t); \quad t > 0 \tag{4}$$

for some $C_1 > 0$ and $C_2 > 0$, moreover, the constants C_1 and C_2 are independent of t. The function $\frac{\omega_i(t)}{t}$ decreases in $t > 0$. There exists the numbers $q > n$, $1 < p < \infty$ such that

$$\int_0^d \left(\frac{\omega_i(t)}{t} \right)^{\frac{2q}{q-n}} dt < \infty, \quad \int_0^d \left(\frac{\omega_i(t)}{t} \right)^{p} dt < \infty \tag{5}$$

For the coefficients of equations (1) we suppose the following condition

$$\widetilde{a}_{ij}(x) = \frac{a_{ij}(x)}{\sqrt{\lambda_i(x) \cdot \lambda_j(x)}} \in C\left(\overline{D}\right), \quad (i, j = 1, ..., n). \tag{6}$$

The goal of the paper is to find conditions on the functions $\lambda_i(x)$ and $f(x)$ $(i=1,...,n)$ under which problem $(1),(2)$ is uniquely strongly solvable in Sobolev's appropriate anisotropic weight space. The similar problem in the case of power form of degenerations was studied by I.T. Mamedov [1] and his followers [2]. As for solvability of divergent equations with degeneration (uniform) we indicate the monographs [3] and [4]. The first boundary value problem for a class of elliptic equations with non-uniform degeneration at the points are investigated in the papers $[5-6]$.

At first we give some denotation and determination. Let $W^p_{2,\widetilde{\omega}}(D)$ be a Banach space of functions $u(x)$ given on D with finite norm

$$\|u\|_{W^p_{2,\widetilde{\omega}}(D)}=$$

$$=\left(\int_D\left(|u(x)|^p+\sum_{i=1}^n\left(\sqrt{\lambda_i(x)}\left|\frac{\partial u}{\partial x_i}\right|\right)^p+\sum_{i,j=1}^n\left(\sqrt{\lambda_i(x)\lambda_j(x)}\left|\frac{\partial^2 u}{\partial x_i\partial x_j}\right|\right)^p\right)dx\right)^{1/p},$$

where $\omega=(\omega_1,...,\omega_n)$ $1\le p<\infty$. Let $W^p_{2,\widetilde{\omega}}(D)$ be a subspace of $W^p_{2,\widetilde{\omega}}(D)$, where in the dense set is the totality of all functions $u(x)\in C_0^\infty(D)$.

The function $u(x)\in W^p_{2,\widetilde{\omega}}(D)$ is called a strong (almost everywhere) solution of problem $(1),(2)$ if it satisfies equation (1) almost everywhere in D.

Everywhere in the sequel, the notation $C(.,.,.,.)$ means that the positive constant C depends only on the content of parenthesis.

Introduce the following denotation

$$E_R\left(x^0\right)=\left\{x\in E_n:\sum_{i=1}^n\frac{\left(x_i-x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2}\le 1\right\}\text{ assume }E_R=E_R(0).$$

1. Basic coercive inequality

Theorem 1. *Let $p>1$, the number $q>n$ from condition* (5), *conditions* $(1)-(6)$ *be fulfilled with respect to the coefficients of the operator L. Then for any function $u(x)\in W^p_{2,\widetilde{\omega}}(D)$ it is valid the estimate*

$$\|u\|_{W^p_{2,\widetilde{\omega}}(D)}\le\|Lu\|_{L_q(D)},$$

where the constant C depends on the constants in conditions $(1)-(5)$ *in domain D and on n,p,q.*

Proof. At first for an arbitrary function $u(x)\in W^p_{2,\widetilde{\omega}}(D)$ we prove the estimate

$$\|u\|_{W^p_{2,\widetilde{\omega}}(D)}\le C\left[\|Lu\|_{L^q(D)}+\|u\|_{L^q(D)}\right],\tag{7}$$

under some suppositions with respect to the functions $\{\omega_i(t)\}:i=1,2,...,n$; where the constant $C>0$ is independent of the function $u(x)$. Assume that $E_\sigma\subset\overline{D}$ for some $\sigma=2^{-\nu_0}$, $\nu_0\in E$; such ν_0 exists by virtue of the supposition $\partial D\in C^2$ at the beginning. For $\nu=\nu_0,\nu_0+1,\ \nu_0+2,...$ we denote the set $D_\nu=\left\{x\in D:x\in E_{R_\nu}\backslash E_{R_{\nu+1}}\right\}$, where $R_\nu=2^{-\nu}$. The boundary D_ν is piecewise smooth and consists of two parts; the part lying on ∂D and the part on the boundary $E_{R_\nu} \backslash E_{R_{\nu+1}}$. Apply a priori estimate [7]

$$\|u\|_{W^p_{2,\widetilde{\omega}}(G)} \leq C\left[\left\|\widetilde{L}u\right\|_{L^p(G)} + \|u\|_{L^p(G)}\right], \tag{8}$$

where $\widetilde{L}$ is a non-uniform elliptic operator: for any $x \in G$, $\zeta \in E_n$

$$\mu\,|\zeta|^2 \leq \sum_{i,j=1}^{n} \widetilde{a}_{ij}\,(x)\,\zeta_i\zeta_j \leq \mu^{-1}\,|\zeta|^2, \tag{9}$$

G is a domain with boundary of the class C^2, the constant $C > 0$ depends on n, p, G, μ.

For $x \in D_\nu$ we have the estimates $\rho\,(x) = \sum_{i=1}^{n} \omega_i\,(|x_i|) \leq nR_\nu$ i.e. $\omega_i^{-1}\,(\rho\,(x)) \leq \omega_i^{-1}\,(nR_\nu) \leq C\omega_i^{-1}\,(R_\nu)$, where C is independent of R_ν. For $x \in D_\nu$ we also have $\rho\,(x) \geq CR_{\nu+1}$, since i.e. $\exists i_1 \in \{1, 2, ..., n\}$ $|x_{i_1}| > \frac{1}{n}\omega_i^{-1}\,(R_{\nu+1})$, $\rho\,(x) = \sum_{i=1}^{n} \omega\,(|x_i|) \geq \omega_{i_1}\,(|x_{i_1}|) = \omega_{i_1}\left(\frac{1}{n}\omega_{i_1}^{-1}\,(R_{\nu+1})\right) \geq CR_{\nu+1}$, where C is independent of ν.

Therefore, $\sqrt{\lambda_i\,(x)} = \dfrac{\omega_i^{-1}\,(\rho\,(x))}{\rho\,(x)} \leq C\dfrac{\omega_i^{-1}\,(R_\nu)}{R_\nu}$ for $x \in D_\nu$, where C is independent of ν. Then

$$\sum_{i=1}^{n} \int_{D_\nu} \left(\sqrt{\lambda_i\,(x)}\left|\frac{\partial u}{\partial x_i}\right|\right)^p dx \leq C\sum_{i=1}^{n}\left(\frac{\omega_i^{-1}\,(R_\nu)}{R_\nu}\right)^p \cdot \int_{D_\nu}\left|\frac{\partial u}{\partial x_i}\right|^p dx. \tag{10}$$

Make change $x_i = \dfrac{\omega_i^{-1}\,(R_\nu)}{R_\nu}\zeta_i$; $i = 1..., n$ inside of the integral in the right hand side,then

$$\sum_{i=1}^{n} \int_{D_\nu} \left(\sqrt{\lambda_i\,(x)}\left|\frac{\partial u}{\partial x_i}\right|\right)^p dx \leq C\sum_{i=1}^{n}\left(\left|\frac{\partial u}{\partial \zeta_i}\right|^p d\zeta\right) \cdot \prod_{j=1}^{n}\left(\frac{\omega_i^{-1}\,(R_\nu)}{R_\nu}\right), \tag{11}$$

where $\widetilde{D_\nu}$ is an image of the set for the mapping $x \to \zeta$, the constant C is independent of ν and $u\,(x)$. Treating the following integral in the same way, we get

$$\sum_{i,j=1}^{n} \int_{D_\nu} \left(\sqrt{\lambda_i\,(x)\,\lambda_j\,(x)}\left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|\right)^p dx \leq$$

$$\leq C\sum_{i,j=1}^{n}\left(\frac{\omega_i^{-1}\,(R_\nu)\cdot\omega_j^{-1}\,(R_\nu)}{R_\nu^2}\right)^p \cdot \int_{D_\nu}\left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|^p dx.$$

After change of variables $x \to \zeta$ we get

$$\sum_{i,j=1}^{n} \int_{D_\nu} \left(\sqrt{\lambda_i\,(x)\,\lambda_j\,(x)}\left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|\right) dx \leq$$

$$\leq C\sum_{i,j=1}^{n}\prod_{j=1}^{n}\left(\frac{\omega_i^{-1}(R_\nu)}{R_\nu^2}\right)\cdot\int_{D_\nu}\left|\frac{\partial^2 u}{\partial\zeta_i\partial\zeta_j}\right|^p d\zeta. \qquad (12)$$

Then (11) and (12) yield

$$\|u\|_{W^p_{2,\widetilde{\omega}}(\widetilde{D}_\nu)}\leq$$

$$\leq C\cdot\prod_{j=1}^{n}\frac{\omega_i^{-1}(R_\nu)}{R_\nu}\left(\int\limits_{\widetilde{D}_\nu}\left(|\widetilde{u}(\zeta)|^p+\sum_{i=1}^{n}\left|\frac{\partial\widetilde{u}}{\partial\zeta_i}\right|^p+\sum_{i,j=1}^{n}\left|\frac{\partial^2\widetilde{u}}{\partial\zeta_i\partial\zeta_j}\right|^p\right)d\zeta\right), \qquad (13)$$

where $\widetilde{u}(\zeta)$ is the image of the function for the transform $x\to\zeta$, i.e. $\widetilde{u}(\zeta)=u(x(\zeta))$. Apply estimate (8) to the function $\widetilde{u}(\zeta)$ in the domain $\widetilde{D}_\nu$ to the operator

$$\widetilde{L}_\zeta=\sum_{i,j=1}^{n}a_{ij}(x(\zeta))\frac{R_\nu^2}{\omega_i^{-1}(R_\nu)\,\omega_j^{-1}(R_\nu)}\cdot\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}.$$

For that, at first we notice that the operator $\widetilde{L}$ is uniformly elliptic: for $\forall\zeta\in\widetilde{D}_\nu$, $\eta\in E_n$ we have

$$\sum_{i,j=1}^{n}\widetilde{a}_{ij}\eta_i\eta_j=\sum_{i,j=1}^{n}a_{ij}(x(\zeta))\frac{R_\nu^2}{\omega_i^{-1}(R_\nu)\,\omega_j^{-1}(R_\nu)}\eta_i\eta_j\asymp\sum_{i=1}^{n}\lambda_i(x)\left(\frac{\eta_i R_\nu}{\omega_i^{-1}(R_\nu)}\right)^2, \qquad (14)$$

where $\widetilde{a}_{ij}(\zeta)=a_{ij}(x(\zeta))\dfrac{R_\nu^2}{\omega_i^{-1}(R_\nu)\,\omega_j^{-1}(R_\nu)}$.

Hence by the fact that for $\zeta\in\widetilde{D}_\nu$ it holds $\lambda_i(x(\zeta))\leq\left(\dfrac{\omega_i^{-1}(R_\nu)}{R_\nu}\right)^2$, it follows from (14) that $\sum\limits_{i,j=1}^{n}\widetilde{a}_{ij}\eta_i\eta_j\asymp|\eta|^2$. Then

$$\|\widetilde{u}(\zeta)\|^p_{W^p_{2,\widetilde{\omega}}(D)}\leq C\left[\left\|\widetilde{L}_\zeta\widetilde{u}(\zeta)\right\|^p_{L^p(\widetilde{D}_\nu)}+\|u\|^p_{L^p(\widetilde{D}_\nu)}\right],$$

where the constant $C>0$ is independent of ν and $u(x)$. By the last estimate from (13) we get

$$\|u\|^p_{W^p_{2,\widetilde{\omega}}(D)}\leq C\prod_{j=1}^{n}\left(\frac{\omega_i^{-1}(R_\nu)}{R_\nu}\right)\cdot\left[\left\|\widetilde{L}_\zeta\widetilde{u}(\zeta)\right\|^p_{L^p(\widetilde{D}_\nu)}+\|\widetilde{u}(\zeta)\|^p_{L^p(\widetilde{D}_\nu)}\right]\leq$$

$$\leq C\left[\|Lu\|^p_{L^p(D_\nu)}+\|u\|^p_{L^p(D_\nu)}\right]$$

Summing up all the inequalities over $\nu=\nu_0,\nu_0+1,\,\nu_0+2,...$ we get

$$\|u\|^p_{W^p_{2,\widetilde{\omega}}(E_\sigma)}\leq C\left[\|Lu\|^p_{L^p(E_\sigma)}+\|u\|^p_{L^p(E_\sigma)}\right]. \qquad (15)$$

The same inequality holds in the domain $D\backslash E_\sigma$. Since the operator L has no degenerations in it , then by the Schauder inequality [7] up to the boundary and smoothness of the boundary of domain $D\backslash E_\sigma$, we have the estimate

$$\|u\|^p_{W^p_{2,\widetilde{\omega}}(D\backslash E_\sigma)}\leq C\left[\|Lu\|^p_{L^p(D\backslash E_\sigma)}+\|u\|^p_{L^p(D\backslash E_\sigma)}\right]. \qquad (15^1)$$

From (15) and $\left(15^1\right)$ we derive the estimate

$$\|u\|^p_{W^p_{2,\widetilde{\omega}}(D)} \leq C\left[\|Lu\|^p_{L^p(D)} + \|u\|^p_{L^p(D)}\right] \tag{16}$$

Here, the constant C depends only on n, p, operator L and domain D. In obtaining estimate (15)we used that

$$\prod_{j=1}^{n}\left(\frac{\omega_i^{-1}(R_\nu)}{R_\nu}\right)\left\|\widetilde{L}_\zeta\widetilde{u}(\zeta)\right\|^p_{L^p(D_\nu)} = \prod_{j=1}^{n}\left(\frac{\omega_j^{-1}(R_\nu)}{R_\nu}\right)\cdot$$

$$\cdot\int_{D_\nu}\left|\sum_{i,j=1}^{n}\frac{R_\nu^2}{\omega_i^{-1}(R_\nu)\,\omega_j^{-1}(R_\nu)}a_{ij}(x(\zeta))\frac{\partial^2 u}{\partial\zeta_i\partial\zeta_j}\right|^p d\zeta =$$

$$= \int_{D_\nu}\left|\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}\right|^p dx = \|Lu\|^p_{L^p(D_\nu)}.$$

By means of the Alexandrov inequality [8] we have

$$\sup_{x\in D}|u(x)| \leq C(d)\left\|\frac{Lu}{\sqrt[n]{\det A(x)}}\right\|_{L^n(D)}, \tag{17}$$

where C depends on the operator L, n and d. Now, show that it holds the estimate

$$\|u\|_{\overset{\circ}{W}{}^p_{2,\widetilde{\omega}}(D)} \leq C\|Lu\|_{L^q(D)} \tag{18}$$

Indeed, $\det A(x) = \prod_{j=1}^{n}\lambda_j(x) = \prod_{j=1}^{n}\left(\frac{\omega_j^{-1}(\rho(x))}{\rho(x)}\right)^2.$

Then

$$\sup_{x\in D}|u(x)| \leq C\left(\int_D\frac{|Lu|^n}{\prod_{j=1}^{n}\lambda_j(x)}dx\right)^{1/n},$$

where C is independent of $u(x)$. Hence by the Holder inequality with the exponents $\frac{q}{n}$ and $\frac{q}{q-n}$ for $q>n$ we have

$$\sup_{x\in D}|u(x)| \leq C\left(\int_D\frac{dx}{\left(\prod_{j=1}^{n}\lambda_j(x)\right)^{\frac{q}{q-n}}}\right)^{\frac{q-n}{q^n}}\cdot\left(\int_D|Lu|^q\,dx\right)^{1/q}. \tag{19}$$

Show that the second multiplier in (19) is bounded. By the assumption on the function $\frac{\omega_i(t)}{t}$ it decreases with respect to t in $(0,\infty)$ for each $i \in \{1,2,...,n\}$. Therefore, the function $\frac{\omega_i^{-1}(t)}{t}$ will be an increasing function on $(0,\infty)$. Indeed, for any $t_2 > t_1$ we have

$$\frac{\omega_i^{-1}(t_2)}{t_2} < \frac{\omega_i^{-1}(t_1)}{t_1} \tag{20}$$

Let $\tau_2 > \tau_1 > 0$ be arbitrary numbers. Assume in inequality (20) $t_2 = \omega_i^{-1}(\tau_2)$, $t_1 = \omega_i^{-1}(\tau_1)$. Then for them we'll have $t_2 > t_1$. Therefore

$$\frac{\omega_i\left(\omega_i^{-1}(\tau_2)\right)}{\omega_i^{-1}(\tau_2)} < \frac{\omega_i\left(\omega_i^{-1}(\tau_1)\right)}{\omega_i^{-1}(\tau_1)}$$

or

$$\frac{\omega_i^{-1}(\tau_2)}{\tau_2} > \frac{\omega_i^{-1}(\tau_1)}{\tau_1}$$

i.e. the function $\frac{\omega_i^{-1}(t)}{t}$ increases. For $x \in D$ we have

$$\frac{1}{\left(\prod_{j=1}^{n}\lambda_j(x)\right)^{\frac{q}{q-n}}} = \frac{1}{\left(\prod_{j=1}^{n}\frac{\omega_i^{-1}(\rho(x))}{\rho(x)}\right)^{\frac{2q}{q-n}}}. \tag{21}$$

By the inequality $\rho(x) = \sum_{j=1}^{n}\omega_j(|x_j|) \geq \omega_i(|x_i|)$; $i = 1,2,...,n$ and that the functions $\frac{\omega_i^{-1}(t)}{t}$ increase, we have

$$\left(\prod_{j=1}^{n}\lambda_j(x)\right)^{\frac{q}{q-n}} \leq \left(\prod_{j=1}^{n}\left(\frac{\omega_j(|x_j|)}{|x_j|}\right)\right)^{\frac{2q}{q-n}}. \tag{22}$$

From (19) and (22) we find

$$\sup_{x\in D}|u(x)| \leq C\left(\int_D |Lu|^q\,dx\right)^{1/q}\left(\int_D\left(\prod_{j=1}^{n}\left(\frac{\omega_j(|x_j|)}{|x_j|}\right)\right)^{\frac{2q}{q-n}}dx\right)^{\frac{q-n}{qn}} \leq$$

$$\leq C\left[\prod_{j=1}^{n}\left(\int_D^{\omega_j^{-1}(d)}\left(\frac{\omega_j(t)}{t}\right)^{\frac{2q}{q-n}}dt\right)\right]\cdot\left(\int_D |Lu|^q\,dx\right)^{1/q}.$$

Hence, allowing for condition (5)

$$\sup_{x\in D}|u(x)| \leq C(d)\,\|Lu\|_{L^q(D)}, \tag{23}$$

where the constant $C(d)$ is independent of the arbitrary function $u(x)$.

Now, by estimates (16) and (23) we have

$$\|u\|_{W^p_{2,\tilde{\omega}}(D)} \leq C\left[\|Lu\|_{L^p(D)} + \|Lu\|_{L^q(D)}\right]$$

By the boundedness of the domain D and the Holder inequality

$$\|Lu\|_{L^p(D)} \leq (mes_n D)^{\frac{q-n}{p}} \|Lu\|_{L^q(D)}\,. \tag{24}$$

From (24) we get the inequality

$$\|u\|_{W^p_{2,\tilde{\omega}}(D)} \leq C\|Lu\|_{L^q(D)}\,, \tag{25}$$

where the constant C is independent of $u(x)$.

Theorem 1 is proved.

2. Strong solvability of the first boundary value problem

Theorem 2. *Let in the bounded domain $D \subset E_n$ with a boundary $\partial D \subset C^2$ the coefficients of the operator $L = \sum\limits_{i,j=1}^{n} a_{ij}(x)\dfrac{\partial^2}{\partial x_i \partial x_j}$ satisfying conditions $(1)-(6)$ be determined. Then boundary value problem $(1), (2)$ is uniquely solvable in the space $W^p_{2,\tilde{\omega}}(D)$; the number $q > n$ is determined by condition (5). For the solution the following estimate is true*

$$\|u\|_{W^p_{2,\tilde{\omega}}(D)} \leq C\|f\|_{L^q(D)}\,, \tag{26}$$

where the constant C depends on the constants in conditions $(1)-(6)$, on domain D and dimension of the space n, p, q.

Proof. For each $\sigma > 0$ we set $\Pi_\sigma = \{x : \rho(x) < \sigma\}$. Without loss of generality, we'll assume that $\overline{\Pi}_1 \subset D$. Let for the natural numbers m

$$\lambda_i^m = \begin{cases} \lambda_i(x) & if \quad x \in D\backslash\Pi_{1/m} \\ \left[\frac{\omega_i^{-1}(1/m)}{1/m}\right]^2 & if \quad x \in \Pi_{1/m} \end{cases}$$

Consider the operator L^m of the following form $L^m = \sum\limits_{i,j=1}^{n} a_{ij}^m(x)\dfrac{\partial^2}{\partial x_i \partial x_j}$, where $a_{ij}^m(x) = \sqrt{\lambda_i^m(x)\lambda_j^m(x)}\tilde{a}_{ij}(x)$, $i,j = 1,...,n$. From the uniform ellipticity and condition (3). for it we have

$$\mu\sum_{i=1}^{n}\lambda_i^m(x)\zeta_i^2 \leq \sum_{i,j=1}^{n} a_{ij}^m(x)\zeta_i\zeta_j \leq \mu^{-1}\sum_{i=1}^{n}\lambda_i^m(x)\zeta_i^2.$$

Introduce the spaces $W^p_{2,\tilde{\omega}^m}(D)$ and $\dot{W}^p_{2,\tilde{\omega}^m}(D)$ with change of the system of functions $\{\lambda_i(x)\}$ by $\{\lambda_i^m(x)\}$, $i = 1....,n$. By means of the reasonings similar to above mentioned ones we can show that the inequality

$$\|u\|_{W^p_{2,\tilde{\omega}^m}(D)} \leq C\|Lu\|_{L^q(D)}\,,\ m = 1, 2, ... \tag{27}$$

is fulfilled for any function $u(x) \in W^p_{2,\widetilde{\omega}^m}(D)$.

Here, the constant C is independent of m and $u(x)$.

Consider the family of Dirichlet problems

$$L^m u^m = f^m, \; x \in D; \, u^m|_{\partial D} = 0. \tag{28}$$

Then problem (28) has a unique solution $u^m \in W^p_{2,\widetilde{\omega}^m}(D)$ by [7, 9] (since the operator L^m is not degenerated). By increase of the function $\dfrac{\omega_i^{-1}(t)}{t}$, the space $W^p_{2,\widetilde{\omega}^m}(D) \subset W^p_{2,\widetilde{\omega}}(D)$. Therefore,

$$\|u^m\|_{W^p_{2,\widetilde{\omega}}(D)} \le \|u^m\|_{W^p_{2,\widetilde{\omega}^m}(D)} \le C\,\|Lu^m\|_{L^q(D)} = C\,\|f^m\|_{L^q(D)} \tag{29}$$

Here, $\{f^m\}$ is some sequence approximating the function f in the norm of the space $L^q(d)$, i.e. $\|f^m - f\|_{L^q(D)} \to 0$ as $m \to \infty$. Then $\|u^m\|_{W^p_{2,\widetilde{\omega}}(D)} \le k$, where $k > 0$ is a number independent of m. The sequence $\{u^m\}$ is bounded by the norm of the space $W^p_{2,\widetilde{\omega}}(D)$. Then from this sequence we can isolate a converging subsequence $\{u^{m_k}\}$ that weakly converges to some function $u \in W^p_{2,\widetilde{\omega}}(D)$. Then, for any function $\varphi \in C_0^\infty(D)$ it holds the equality

$$\lim_{k\to\infty} \int_D \varphi(x)\, Lu^{m_k} dx = \int_D \varphi Lu dx, \tag{30}$$

$$\lim_{k\to\infty} \int_D f^{m_k} \varphi dx = \int_D \varphi f dx. \tag{31}$$

On the other hand

$$\int_D \varphi(x)\, Lu^{m_k} dx = \int_D (L - L^{m_k})\, u^{m_k} \varphi dx + \int_D L^{m_k} u^{m_k} \varphi dx + \int_D f^{m_k} \varphi dx, \tag{32}$$

and also

$$\left| \int_D (L - L^{m_k})\, u^{m_k} \varphi(x)\, dx \right| \le C \int_{\pi_{1/m}} \sum_{i,j=1}^n \sqrt{\lambda_i^m(x)\, \lambda_j^m(x)} \left| \frac{\partial^2 u^m}{\partial x_i \partial x_j} \right| dx \le$$

$$\le C\,\|u^m\|_{W^p_{2,\widetilde{\omega}^m}\left(\pi_{1/m}\right)} \cdot \|\varphi\|_{L^p\left(\pi_{1/m}\right)} \le$$

$$\le C\,\|f^m\|_{L^q(D)} \cdot \|\varphi\|_{L^p(1/m)} \to 0, \quad \text{as } m \to \infty \tag{33}$$

It follows from (30)-(33) that

$$\int_D Lu\varphi dx = \int_D f\varphi dx$$

Hence, from $Lu - f \in L^q(D)$ and density of the class of functions $C_0^\infty(D)$ in $L^q(D)$ we get

$$Lu = f \;\text{ a.e. } x \in D.$$

i.e. $u(x)$ is a strong solution of the Dirichlet problem. Estimate (26) follows from inequality (29) and weak convergence $u^m \to u$ in $W^p_{2,\widetilde{\omega}}(D)$ (we used the inequality $\lim\limits_{m\to\infty} \|x_m\| \geq \|x\|$ for the sequence $\{x_m\}$ weakly converging to x in Banach space).

Above we implicitly used the following reasonings. From the belonging of $u \in W^p_{2,\widetilde{\omega}}(D)$ it follows $u_{x_i}u_{x_j} \in L^1(D)$ by the fact that from (5)

$$\int\limits_0^{\omega_i^{-1}(d)} \left(\frac{\omega_i(t)}{t}\right)^{p'} dt < \infty$$

for $1 \leq p' \leq q$. Then, by the Holder inequality

$$\int\limits_D \left|u_{x_ix_j}\right| dx \leq \left(\int\limits_D \left(\sqrt{\lambda_i(x)\lambda_j(x)}\left|u_{x_ix_j}\right|\right)^p dx\right)^{1/p} \cdot \left(\int\limits_D \frac{dx}{\left(\sqrt{\lambda_i(x)\lambda_j(x)}\right)^{p'}}\right)^{1/p'}. \tag{34}$$

By lack of increase of the functions $\dfrac{\omega_i(t)}{t}$ and that $\lambda_i(x) = \left[\dfrac{\omega_i^{-1}(\rho(x))}{\rho(x)}\right]$ we have $\lambda_i(x) \geq \dfrac{|x_i|}{\omega_i|x_i|}$, therefore

$$\int\limits_D \frac{dx}{\left(\sqrt{\lambda_i(x)\lambda_j(x)}\right)^{p'}} \leq \int\limits_D \left(\frac{\omega_i(|x_i|)}{|x_i|}\right)^{p'} \cdot \left(\frac{\omega_j(|x_j|)}{|x_j|}\right)^{p'} dx =$$

$$= \left(\int\limits_0^{\omega_i^{-1}(d)} \left(\frac{\omega_i(t)}{t}\right)^{p'} dt\right)\left(\int\limits_0^{\omega_i^{-1}(d)} \left(\frac{\omega_j(t)}{t}\right)^{p'} dt\right) \leq C_1,$$

where C is independent of $u(x)$. Then from (34) we get

$$\int\limits_D \left|u_{x_ix_j}\right| dx \leq C\left(\int\limits_D \left(\sqrt{\lambda_i(x)\lambda_j(x)}\left|u_{x_ix_j}\right|\right)^p dx\right)^{1/p} \leq C\|u\|_{W^p_{2,\widetilde{\omega}}(D)} \tag{35}$$

Taking into account (35) in all $i,j = 1,...,n$ for all $u \in W^p_{2,\widetilde{\omega}}(D)$, $f \in L^q(D)$ well have $Lu - f \in L^1(D)$. Then, from the identity

$$\int\limits_D (Lu - f)\varphi dx = 0$$

for any function $\varphi \in C_0^\infty(D)$ we extract the equality

$$Lu = f \quad \text{a.e. } x \in D.$$

We can justify it by the following way.

Let φ_j be a sequence of averagings by means of any smooth positive kernel with a compact support of the function $F(x) = sgn\,(Lu - f)$. Then $\varphi_i \to F$ a.e. $x \in D$, where $|\varphi_j| \leq 2$ a.e. $x \in D$ (see for example [10]). Therefore,

$$\int\limits_D \varphi_j\,(Lu - f)\,dx = 0, \quad j = 1, 2, \ldots\,. \tag{36}$$

By means of the Lebesgue theorem we pass to limit as $j \to \infty$ in equality (36). Then we get $\int\limits_D |Lu - f|\,dx = 0$, whence it follows $F \equiv 0$ a.e. $x \in D$.

Theorem 2 is proved.

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