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ON REMOVABLE SETS OF SOLUTIONS OF ELLIPTIC EQUATIONS

Abstract

In this paper we consider a nondivergent elliptic equation of second order whose leading coefficients are from some weight space. The sufficient condition of removability of a compact with respect to this equation in the weight space of Hölder functions was found.

Let D be a bounded domain situated in n-dimensional Euclidean space E_n of the points $x=(x_1,...,x_n)$, $n\geq 3$, ∂D be its boundary. Consider in D the following elliptic equation

$$\mathcal{L}u=\sum_{i,j=1}^{n} a_{ij}(x)\,u_{ij}+\sum_{i=1}^{n} b_i(x)\,u_i+c(x)\,u=0, \tag{1}$$

in supposition that $\|a_{ij}(x)\|$ is a real symmetric matrix, moreover $\omega(x)$ is a positive measurable function satisfying the doubling condition: for concentric balls B_R^x or R and $2R$ radius, there exists such a constant γ

$$\omega\left(B_R^x\right)\geq\gamma\omega\left(B_{2R}^x\right),$$

where for the measurable sets E $\omega(E)$ means $\int\limits_E \omega(y)\,dy$

$$\gamma\,|\xi|^2\,\omega(x)\leq\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\xi_j\leq\gamma^{-1}\omega(x)\,|\xi|^2;\ \xi\in E_n,\ x\in D, \tag{2}$$

$$a_{ij}(x)\in C_\omega^1\left(\overline{D}\right);\ i,j,1,...,n, \tag{3}$$

$$|b_i(x)|\leq b_0;\ -b_0\leq c(x)\leq 0;\ i=1,...,n;\ x\in D. \tag{4}$$

Here $u_i=\dfrac{\partial u}{\partial x_i}$, $u_{ij}=\dfrac{\partial^2 u}{\partial x_i\partial x_j}$; $i,j=1,...,n$; $\gamma\in(0,1]$ and $b_0\geq 0$ are constants. Besides we'll suppose that the lower coefficients of the operator $\mathcal{L}$ are measurable functions in D. Let $\lambda\in(0,1)$ be a number. Denote by $C^{0,\lambda}(D)$ a Banach space of the functions $u(x)$ defined in D with the finite norm.

$$\|u\|_{C_\omega^\lambda(D)}=\sup_{x\in D}\omega(x)\,|u(x)|+\sup_{\substack{x,y\in D\\ x\neq y}}\frac{|u(x)-u(y)|\,\omega}{|x-y|^\lambda}.$$

The compact $E\subset\overline{D}$ is called removable with respect to the equation (1) in the space $C_\omega^\lambda(D)$ if from

$$\mathcal{L}u=0,\ x\in D\backslash E;\ u|_{\partial D\backslash E}=0;\ u(x)\in C_\omega^\lambda(D) \tag{5}$$

it follows that $u(x) \equiv 0$ in D.

The aim of the given paper is finding sufficient condition of removability of a compact with respect to the equation (1) in the space $C_\omega^\lambda(D)$. This problem have been investigated by many researchers. For the Laplace equation the corresponding result was found by L. Carleson [1]. Concerning the second order elliptic equations of divergent structure, we show in this direction the papers [2], [3]. For a class of non-divergent elliptic equations of the second order with discontinuous coefficients the removability condition for a compact in the space $C^\lambda(D)$ was found in [4]. Mention also papers [5-7] in which the conditions of removability for a compact in the space of continuous functions have been obtained.The removable sets of solutions of the second order elliptic and parabolic equations in nondivergent form were considered in [10]-[12]. In [13], T. Kilpelainen and X. Zhong have studied the divergent quasilinear equation without minor members, proved the removability of a compact. Removable sets for pointwise solutions of elliptic partial differential equations were found by J. Diederich [14]. Removable singularities of solutions of linear partial differential equations were considered in R. Harvey, J. Polking paper [15]. Removable sets at the boundary for subharmonic functions have been investigated by B. Dahlberg [16].

Denote by $B_R(z)$ and $S_R(z)$ the ball $\{x : |x-z| < R\}$ and the sphere $\{x : |x-z| = R\}$ of radius R with the center at the point $z \in E_n$ respectively. We'll need the following generalization of mean value theorem belonging to E.M. Landis and M.L. Gerver [8] in weight case.

Lemma. *Let the domain G be situated between the spheres $S_R(0)$ and $S_{2R}(0)$, moreover the intersection $\partial G \cap \{x : R < |x| < 2R\}$ be a smooth surface. Further, let in $\overline{G}$ the uniformly positive definite matrix $\|a_{ij}(x)\|$; $i,j = 1,...,n$ and the function $u(x) \in C^2(G) \cap C_\omega^1\left(\overline{G}\right)$ be given. Then there exists the piece-wise smooth surface Σ dividing in G the spheres $S_R(0)$ and $S_{2R}(0)$ such that*

$$\int_\Sigma \omega \left|\frac{\partial u}{\partial \nu}\right| ds \le K \underset{G}{osc}\, u \cdot \frac{\omega(G)}{R^2}.$$

Here $K > 0$ is a constant depending only on the matrix $\|a_{ij}(x)\|$ and n, and $\dfrac{\partial u}{\partial \nu}$ is a derivative by a conormal determined by the equality

$$\frac{\partial u(x)}{\partial \nu} = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \cos(\bar{n}, x_j)^{\frac{1}{2}},$$

where $\cos(\bar{n}, x_j)$; $j = 1,...,n$ are direction cosines of a unit external normal vector to Σ.

Denote by $W_{2,\omega}^1(D)$ the Banach space of the functions $u(x)$ defined in D with the finite norm

$$\|u\|_{W_{2,\omega}^1(D)} = \left(\int_D \omega \left(u^2 + \sum_{i=1}^n u_i^2\right) dx\right)^{\frac{1}{2}},$$

and let $\overset{o}{W}{}^{1}_{2,\omega}(D)$ be a completion of $C_0^\infty(D)$ by the norm of the space $W^1_{2,\omega}(D)$.

By $m_H^s(A)$ we'll denote the Hausdorff measure of the set A of order $s>0$. Further everywhere the notation $C(...)$ means, that the positive constant C depends only on the content of brackets.

Theorem 1. *Let D be a bounded domain in E_n, $E\subset\overline{D}$ be a compact. If with respect to the coefficients of the operator $\mathcal{L}$ the conditions (2)-(4) are fulfilled, then for removability of the compact E with respect to the equation (1) in the space $C_\omega^\lambda(D)$ it sufficies that*

$$m_H^{n-2+\lambda}(E)=0. \tag{6}$$

Proof. At first we show that without loss of generality we can suppose the condition $\partial D\in C^1$ is fulfilled. Suppose, that the condition (6) provides the removability of the compact E for the domains, whose boundary is the surface of the class C^1, but $\partial D\in C^1$ and by fulfilling (6) the compact E is not removable. Then the problem (5) has non-trivial solution $u(x)$, moreover $u|_E=f(x)$ and $f(x)\neq 0$. We always can suppose the lowest coefficients of the operator $\mathcal{L}$ are infinitely differentiable in D. Moreover, without loss of generality, we'll suppose that the coefficients of the operator $\mathcal{L}$ are extended to a ball $B\supset\overline{D}$ with saving the conditions (2)-(4). Let $f^+(x)=\max\{f(x),0\}$, $f^-(x)=\min\{f(x),0\}$, and $u^\pm(x)$ be generalized by Wiener (see [8]) solutions of the boundary value problems

$$\mathcal{L}u^\pm=0,\ x\in D\backslash E;\ u^\pm\big|_{\partial D\backslash E}=0;\ u^\pm\big|_E=f^\pm.$$

Evidently, by $u(x)=u^+(x)+u^-(x)$. Further, let D' be such a domain, that $\partial D'\in C^1$, $\overline{D}\subset D'$, $\overline{D'}\subset B$, and $\vartheta^\pm(x)$ be solutions of the problems

$$\mathcal{L}\vartheta^\pm=0,\ x\in D'\backslash E;\ \vartheta^\pm\big|_{\partial D'}=0;\ \vartheta^\pm\big|_E=f^\pm;\ \vartheta^\pm(x)\in C_\omega^\lambda(D').$$

By the maximum principle for $x\in D$

$$0\le u^+(x)\le\vartheta^+(x),\ \vartheta^-(x)\le u^-(x)\le 0.$$

But according to our supposition $\vartheta^+(x)\equiv\vartheta^-(x)\equiv 0$. Hence, it follows, that $u(x)\equiv 0$. So, we'll suppose that $\partial D\in C^1$. Now, let $u(x)$ be a solution of the problem (5), and the condition (6) be fulfilled. Give an arbitrary $\varepsilon>0$. Then there exists a sufficiently small positive number δ and a system of the balls $\left\{B_{r_k}\left(x^k\right)\right\}$, $k=1,2,...$, such that $r_k<\delta$, $E\subset\bigcup\limits_{k=1}^{\infty}B_{r_k}\left(x^k\right)$ and

$$\sum_{k=1}^{\infty}r_k^{n-2+\lambda}<\varepsilon. \tag{7}$$

Consider a system of the spheres $\left\{B_{2r_k}\left(x^k\right)\right\}$, and let $D_k=D\cap B_{2r_k}\left(x^k\right)$, $k=1,2,...,$. Without loss of generality we can suppose that the cover $\left\{B_{2r_k}\left(x^k\right)\right\}$ has a finite multiplicity $a_0(n)$. By the Landis-Gerver theorem for every k there exists a piece-wise smooth surface Σ_k dividing in D_k the spheres $S_{r_k}(x^k)$ and $S_{2r_k}(x^k)$, such that

$$\int\limits_{\Sigma_k} \omega \left| \frac{\partial u}{\partial \nu} \right| ds \leq K \underset{D_k}{osc} u \frac{\omega(D_k)}{r_k^2}. \tag{8}$$

Since $u(x) \in C_\omega^\lambda(D)$, there exists a constant $H_1 > 0$ depending only on the function $u(x)$ such that

$$\underset{D_k}{osc} u \omega \leq H_1 (2r_k)^\lambda. \tag{9}$$

Besides,

$$\omega(D_k) \leq mes_n B_{2r_k}\left(x^k\right) = \Omega_n 2^n r_k^n;\ k = 1, 2, ..., \tag{10}$$

where $\Omega_n = mes_n B_1(0)$. Using (9)-(10) in (8), we get

$$\int\limits_{\Sigma_k} \omega \left| \frac{\partial u}{\partial \nu} \right| ds \leq C_1 r_k^{n-2+\lambda};\ k = 1, 2, ..., \tag{11}$$

where $C_1 = K H_1 2^{n+\lambda}$.

Let D_Σ be an open set situated in $D \backslash E$ whose boundary consists of unification of Σ and Γ, where $\Sigma = \bigcup\limits_{k=1}^{\infty} \Sigma_k$, $\Gamma = \partial D \backslash \bigcup\limits_{k=1}^{\infty} D_k^+$, D_k^+ is a part of D_k remaining after the removing of points situated between Σ and $S_{2r_k}(x^k)$; $k = 1, 2, ...$ Denote by D'_Σ the arbitrary connected component D_Σ, and by $\mathcal{M}$ we denote the elliptic operator of divergent structure

$$\mathcal{M} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right).$$

According to Green formula for any functions $z(x)$ and $W(x)$ belonging to the intersection $C^2(D'_\Sigma) \cap C^1\left(\overline{D}'_\Sigma\right)$, we have

$$\int\limits_{D'_\Sigma} (z \mathcal{M} \beta - \beta \mathcal{M} z)\, dx = \int\limits_{\partial D'_\Sigma} \left(z \frac{\partial \beta}{\partial \nu} - \beta \frac{\partial z}{\partial \nu} \right) ds. \tag{12}$$

Since $\partial D \in C^1$, then u$(x) \in C^1(D'_\Sigma) \cap C^1\left(\overline{D'_\Sigma}\right)(x) \in C^1\left(\overline{D}_{\Sigma'}\right)$ (see [9]). From (12) choosing the functions $z = 1$, $\beta = \omega u^2$ we have

$$\int\limits_{D'_\Sigma} \mathcal{M}\left(\omega u^2\right) dx = 2 \int\limits_{\partial D'_\Sigma} \omega u \frac{\partial u}{\partial \nu} ds + \int\limits_{\partial D_\Sigma} \omega_{x_i} u^2 ds.$$

But $|u(x)| \leq M < \infty$ for $x \in \overline{D}$. Let's put the condition:

$$\omega_{x_i} < c\omega. \tag{$*$}$$

By virtue of condition (*) and $\int\limits_{\partial D_\Sigma} \omega u^2 ds < C_3 M\varepsilon$, subject to (11) and (7) we conclude

$$\int\limits_{D'_\Sigma} \mathcal{M}\left(\omega u^2\right) dx \le 2Ma_0 \sum_{k=1}^{\infty} \int\limits_{\Sigma_k} \omega \left|\frac{\partial u}{\partial \nu}\right| ds + \int\limits_{D'_\Sigma} \omega u^2 dx \le$$

$$\le 2Ma_0 C_1 \sum_{k=1}^{\infty} r_k^{n-2+\alpha} + \varepsilon M c_2 < C_3 \varepsilon, \tag{13}$$

where $C_3 = 2Ma_0C_1$.

On the other hand

$$\mathcal{M}\left(\omega u^2\right) = 6u\omega\mathcal{M}(u) + 2\sum_{i,j=1}^{n} \omega a_{ij} u_i u_j + (2u+1)\sum_{i,j=1}^{n} a_{ij} u_{x_j} \omega_{x_i} +$$

$$+ \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_i} u \omega_{x_j} + \sum_{i,j=1}^{n} a_{ij} u \omega_{x_i x_j}$$

and besides,

$$\mathcal{M}u = \mathcal{L}u + \sum_{i=1}^{n} d_i(x) u_i - c(x) u,$$

where

$$d_i(x) = \sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j} - b_i(x), \;\; i = 1, ..., n.$$

It is evident that by virtue of conditions (3)-(4) $|d_i(x)| \le d_0 < \infty; \;\; i = 1, ...n$. Thus, from (13) we obtain

$$6\int\limits_{D'_\Sigma} u\omega \sum_{i=1}^{n} d_i(x) u_i dx - 6\int\limits_{D'_\Sigma} u^2 c(x) dx + 2\int\limits_{D'_\Sigma} \sum_{i,j=1}^{n} \omega(x) a_{ij} u_i u_j dx +$$

$$+ (2u+1)\int\limits_{D'_\Sigma} \sum_{i,j=1}^{n} a_{ij} u_j \omega_{x_i} dx + \int\limits_{D'_\Sigma} \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_j} u \omega_{x_i} dx + |\nabla u|^2 dx +$$

$$+ \int\limits_{D'_\Sigma} \sum_{i,j=1}^{n} a_{ij} u \omega_{x_i x_j} dx < C_3 \varepsilon.$$

Hence, for any $\alpha > 0$ it follows that

$$2\gamma \int\limits_{D'_\Sigma} \omega |\nabla u|^2 dx < 6d_0 \int\limits_{D'_\Sigma} \omega |u| |u_i| dx + 6\int\limits_{D'_\Sigma} u^2 \omega(x) + (2u+1)\int\limits_{D'_\Sigma} a_{ij} u_j \omega_{x_i} dx +$$

$$+ d_0 \int\limits_{D'_\Sigma} u \omega_{x_i}^2 dx + \int\limits_{D'_\Sigma} a_{ij} u \omega_{x_i x_j} + C_3 \varepsilon \le 6\frac{d_0}{\varepsilon} \int\limits_{D'_\Sigma} |u|^2 dx + 6\frac{d_0 \varepsilon}{2} \int\limits_{D'_\Sigma} \omega^2 |\nabla u|^2 dx +$$

$$+\left(2n+1\right)\int\limits_{D'_\Sigma}u_j\omega dx+d_0\int\limits_{D'_\Sigma}u\omega dx+\gamma C_4\varepsilon\leq 6\frac{d_0}{\varepsilon}Mmes_nD+$$

$$+\frac{\left(2M+1\right)\gamma}{\varepsilon}mes_nD+d_0M\omega\left(D\right)+\gamma C_4M\omega\left(D\right)+C_3\varepsilon. \tag{14}$$

If we'll take into account that

$$\left|\omega_{x_ix_j}\right|<C_4\omega\left(x\right),$$

then from here we have that

$$\int\limits_{D'_\Sigma}\omega^2\left|\nabla u\right|^2dx\leq C_5,$$

where $C_5=\left(6d_0+\left(2M+1\right)\right)Mmes_nD+\left(d_0M+\gamma C_4M\right)\omega\left(D\right)+\dfrac{C_3}{\gamma}$. Without loss of generality we assume that $\varepsilon\leq 1$. Hence we have

$$\int\limits_{D}\omega^2\left|\nabla u\right|^2dx\leq C_6.$$

Thus $u\left(x\right)\in W^1_{2,\omega}\left(D\right)$. From the boundary condition and $mes_{n-1}\left(\partial D\cap E\right)=0$ we get $u\left(x\right)\in \overset{\circ}{W}{}^1_{2,\omega}\left(D\right)$. Now, let $\sigma\geq 2$ be a number which will be chosen later, $D^+_\Sigma=\{x:x\in D'_\Sigma,u\left(x\right)>0\}$. Without loss of generality, we suppose that the set D^+_Σ isn't empty. Supposing in (12) $z=1,\ \beta=\omega u^\sigma$, we get

$$\int\limits_{D^+_\Sigma}\mathcal{M}\left(\omega u^\sigma\right)dx=\sigma\int\limits_{\partial D^+_\Sigma}\left(\omega_\nu u^\sigma+\sigma u^{\sigma-1}\frac{\partial u}{\partial\nu}\right)ds\leq$$

$$\leq M^\sigma\int\limits_{\partial D^+_\Sigma}\omega ds+\sigma M^{\sigma-1}\int\limits_{\partial D^+_\Sigma}\left|\frac{\partial u}{\partial\nu}\right|ds\leq C_5\left(a_0,M,\sigma,C_1\right)\varepsilon.$$

But, on the other hand

$$\mathcal{M}\left(u^\sigma\right)=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial\omega u^\sigma}{\partial x_j}\right)=$$

$$=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}\omega\left(\sigma u^{\sigma-1}\frac{\partial u}{\partial x_j}\right)+\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}\omega_{x_i}\frac{\partial u^\sigma}{\partial x_j}\right)\right)$$

$$=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}\omega\sigma u^{\sigma-1}\frac{\partial u}{\partial x_j}\right)+\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}\sigma u^{\sigma-1}\omega x\frac{\partial u}{\partial x_j}\right)=$$

$$=\sigma\omega u^{\sigma-1}\mathcal{M}\left(u\right)+\sigma\omega\frac{\partial}{\partial x_i}\left(a_{ij}u^{\sigma-1}\frac{\partial u}{\partial x_j}\right)+\sigma u^{\sigma-1}\frac{\partial}{\partial x_i}\left(a_{ij}\omega\frac{\partial u}{\partial x_j}\right)+\beta=$$

$$= \sigma\omega u^{\sigma-1}\mathcal{M}(u) + \sigma\omega u^{\sigma-1}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sigma\omega a_{ij}u_{x_j}(\sigma-1)u^{\sigma-2}u_{x_i} +$$

$$+\sigma u^{\sigma-1}\omega_{x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sigma u^{\sigma-1}\omega\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \beta =$$

$$= 3\sigma\omega u^{\sigma-1}M(u) + \sigma(\sigma-1)a_{ij}ux_iu_{x_j}u^{\sigma-2}\omega + \sigma u^{\sigma-1}\omega_{x_i}a_{ij}u_{x_j} + \beta =$$

$$= \sigma\int\limits_{D_\Sigma^+} d_i(x)\,u_{x_i}u\omega dx - \sigma(\sigma-1)\int\limits_{D_\Sigma^+} u^\sigma\omega(x)\,c(x)\,dx +$$

$$+\sigma(\sigma-1)\int\limits_{D_\Sigma^+}\sum_{i,j=1}^n u^{\sigma-2}\omega(x)\,a_{ij}u_iu_jdx + (2u+1)\int\limits_{D_\Sigma^+}\sum_{i,j=1}^n a_{ij}u_j\omega_{x_j}u^{\sigma-1}.$$

Hence, we conclude

$$\sigma(\sigma-1)\int\limits_{D_\Sigma^+}\omega^2u^{\sigma-2}|\nabla u|^2dx \le d_0\int\limits_{D_\Sigma^+}u^{\sigma-1}\omega u_idx \le$$

$$\le d_0\int\limits_{D_\Sigma^+}u^{\sigma-1}\omega u_idx \le \frac{d_0\varepsilon}{2}\int\limits_{D_\Sigma^+}u^\sigma dx. \tag{15}$$

Let $D^+ = \{x : x \in D, u(x) > 0\}$, D_1^+ be an arbitrary connected component of D^+. Subject to the arbitrariness of ε from (15) we get

$$(\sigma-1)\gamma\int\limits_{D_1^+}\omega u^{\sigma-2}|\nabla u|^2dx \le d_0\int\limits_{D_1^+}\omega u^{\sigma-1}\sum_{i=1}^n|u_i|\,dx.$$

Thus, for any $\mu > 0$

$$(\sigma-1)\gamma\int\limits_{D_1^+}\omega u^{\sigma-2}|\nabla u|^2dx \le \frac{d_0\mu}{2}\int\limits_{D_1^+}\omega u^{\sigma-2}\left(\sum_{i=1}^n|u_i|\right)^2dx + \frac{d_0}{2\mu}\int\limits_{D_1^+}\omega u^\sigma dx \le$$

$$\le \frac{d_0\mu n}{2}\int\limits_{D_1^+}\omega u^{\sigma-2}|\nabla u|^2dx + \frac{d_0}{2\mu}\int\limits_{D_1^+}\omega u^\sigma dx. \tag{16}$$

But, on the other hand

$$I = -\sigma\sum_{i=1}^n\int\limits_{D_1^+}x_i\omega u^{\sigma-1}u_idx = -\sum_{i=1}^n\int\limits_{D_1^+}x_i\omega\,(u^\sigma)_i\,dx = n\int\limits_{D_1^+}\omega u^\sigma dx,$$

and besides, for any $\beta > 0$

$$I = \frac{\sigma\beta}{2}\int\limits_{D_1^+}r^2\omega u^\sigma dx + \frac{\sigma}{2\beta}\int\limits_{D_1^+}u^{\sigma-2}\omega^2|\nabla u|^2dx.$$

Then

$$I \le \frac{\sigma\beta}{2} \int\limits_{D_1^+} r^2 \omega u^\sigma dx + \frac{\sigma}{2\beta} \int\limits_{D_1^+} \omega^2 \left|\nabla u\right|^2 u^{\sigma-2} dx,$$

where $r = |x|$. Denote by $k(D)$ the quantity $\sup\limits_{x\in D} |x|$. Without loss of generality we'll suppose, that $k(D) = 1$. Then

$$I \le \frac{\sigma}{2\beta} \int\limits_{D_1^+} \omega u^\sigma dx + \frac{\sigma}{2\beta} \int\limits_{D_1^+} \omega^2 u^{\sigma-2} \left|\nabla u\right|^2 dx.$$

Thus,

$$\left(n - \frac{\sigma\beta}{2}\right) \int\limits_{D_1^+} \omega u^\sigma dx + \frac{\sigma}{2\beta} \int\limits_{D_1^+} \omega^2 u^{\sigma-2} \left|\nabla u\right|^2 dx.$$

Now, choosing $\beta = \dfrac{n}{\sigma}$, we finally obtain

$$\int\limits_{D_1^+} \omega u^\sigma dx \le \frac{\sigma^2}{n^2} \int\limits_{D_1^+} \omega^2 u^{\sigma-2} \left|\nabla u\right|^2 dx. \tag{17}$$

Subject to (17) in (16) , we conclude

$$(\sigma - 1)\,\gamma \int\limits_{D_1^+} \omega^2 u^{\sigma-2} \left|\nabla u\right|^2 dx \le \left(\frac{d_0 \varepsilon n}{2} + \frac{d_0 \sigma^2}{2\varepsilon n^2}\right) \int\limits_{D_1^+} \omega^2 u^{\sigma-2} \left|\nabla u\right|^2 dx. \tag{18}$$

Now choose μ such that

$$(\sigma - 1)\,\gamma > \frac{d_0 \mu n}{2} + \frac{d_0 \sigma^2}{2\mu n^2}. \tag{19}$$

Then from (17)-(19) it will follow that $u\,(x) \equiv 0$ in D_1^+, and thus $u\,(x) \equiv 0$ in D. Suppose that $\mu = \dfrac{(\sigma - 1)\,\gamma}{d_0 n}$. Then (19) is equivalent to the condition

$$n > \left(\frac{\sigma}{\sigma - 1}\right)^2 \left(\frac{d_0}{\gamma}\right)^2. \tag{20}$$

At first, suppose that

$$n > \left(\frac{d_0}{\gamma}\right)^2. \tag{21}$$

Let's choose and fix such a big $\sigma \ge 2$ that by fulfilling (21) the inequality (20) was true. Thus, the theorem is proved, if with respect to n the condition (21) is fulfilled. Show that it is true for any $n \ge 3$. For that, at first, note that if $k\,(D) \ne 1$, then condition (21) will take the form

$$n > \left(\frac{d_0 k(D)}{\gamma}\right)^2.$$

Now, let the condition (21) be not fulfilled. Denote by k the least natural number for which

$$n+k>\left(\frac{d_0}{\gamma}\right)^2. \qquad (22)$$

Consider $(n+k)$-dimensional semi-cylinder $D'=D\times(-\delta_0,\delta_0)\times...\times,\times(-\delta_0,\delta)$, where the number $\delta_0>0$ will be chosen later. Since $\omega(D)=1$, then $\omega(D')\leq 1+\delta_0\sqrt{k}$. Let's choose and fix δ_0 so small that along with the condition (22) the condition

$$n+k>\left(\frac{d_0\omega(D')}{\gamma}\right)^2 \qquad (23)$$

was fulfilled too.

Let

$$y=(x_1,...,x_n,x_{n+1},...,x_{n+k}),E'=\underbrace{E\times[-\delta_0,\delta_0]\times...\times[-\delta_0,\delta_0]}_{k\text{ times}}.$$

Consider on the domain D' the equation

$$\mathcal{L}'_{\vartheta}=\sum_{i,j=1}^{n}a_{ij}(x)\,\vartheta_{ij}+\sum_{i=1}^{k}\frac{\partial^2\vartheta}{\partial x_{n+i}^2}+\sum_{i=1}^{n}b_i(x)\,\vartheta_i+c(x)\,\vartheta=0. \qquad (24)$$

It is easy to see that the function $\vartheta(y)=u(x)$ is a solution of the equation (24) in $D'\backslash E'$. Besides, $m_H^{n+k-2+\lambda}(E')=(2\delta_0)^k\,m_H^{n-2+\lambda}(E)=0$, the function $\vartheta(y)$ vanishes on $\left(\partial D\times\underbrace{[-\delta_0,\delta_0]\times..\times[-\delta_0,\delta_0]}_{k\text{ times}}\right)\backslash E'$ and $\dfrac{\partial\vartheta}{\partial\nu'}=0$ at $x_{n+i}=\pm\delta_0$, $i=1,...,k$, where $\dfrac{\partial}{\partial\nu'}$ is a derivative by the conormal generated by the operator $\mathcal{L}'$. Noting that $\gamma(\mathcal{L}')=\gamma(\mathcal{L}),d_0(\mathcal{L}')=d_0(\mathcal{L})$ and subject to the condition (23), from the proved above we conclude that $\vartheta(y)\equiv0$, i.e. D'. The theorem is proved.

Remark. As is seen from the proof, the assertion of the theorem remains valid if instead of the condition (3) it is required that the coefficients $a_{ij}(x)\,(i,j=1,...,n)$ have to satisfy in domain D the uniform Lipschitz condition with weight.

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