APPLIED PROBLEMS OF MATHEMATICS AND MECHANICS

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MINIMAL ENERGY CONTROLLABILITY PROBLEM FOR STRING'S VIBRATION EQUATION

Abstract

In the paper the controllability problem is considered for the string vibration equation with control at the right hand side with minimal energy. The solution of the considered problem i.e. optimal control is found in the form of series in two cases using the explicit form of the solution of boundary problem and moments method.

Let the state of the controllable system be described by the function $u(x,t)$ that inside the domain $Q = \{0 < x < 1, 0 < t < T\}$ satisfies the equation

$$u_{tt} = u_{xx} + \upsilon(x,t), \tag{1}$$

initial and boundary conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad 0 < x < 1, \tag{2}$$

$$u_x(0,t) = 0, \qquad u_x(1,t) = 0, \quad 0 < t < T, \tag{3}$$

where $u_0(x) \in W_2^1(0,1)$, $u_1(x) \in L_2(0,1)$ are the given functions, $\upsilon(x,t)$ is a control function.

In the paper, the optimal control problems are considered for the following different cases:

a) $\upsilon(x,t) = q(x)\upsilon(t)$, where $q(x)$ is a given function from $L_2(0,1)$ and $\upsilon(t) \in L_2(0,T)$ is an admissible control that are arbitrary function.

b) $\upsilon(x,t) = \upsilon(t)\delta(x - x_0)$, where x_0 is an arbitrary point from the interval $(0,1)$, $\upsilon(t) \in L_2(0,T)$ is an admissible controls, and $\delta(x)$ is Dirac's function.

Under the solution of problem (1)-(3), for each admisible control $\upsilon(x,t)$ we understand the function $u(x,t)$ from $W_2^1(Q)$, that for any function $\psi(x,t) \in W_2^1(Q), \psi(x,T) = 0$ satisfies the integral identity

$$\iint\limits_Q \left(-\frac{\partial u}{\partial t}\frac{\partial \psi}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial \psi}{\partial x} - \upsilon\psi \right) dxdt -$$

$$-\int_0^1 u_1(x)\,\psi(x,0)\,dx = 0,$$

and fulfillment of the condition $u(x,0) = u_0(x)$ is understood in the ordinary sense.

These cases will be spectrately studied below.

Case a). Each admissible control in this case defines a unique generalized solution and by means of the Fourier method the solution of problem (1)-(3) may be represented in the following form [1], [2]

$$u(x,t) = \int_0^1 u_0(\xi) G_t(x,\xi,t)\, d\xi + \int_0^1 u_1(\xi) G(x,\xi,t)\, d\xi +$$

$$+\int_0^t \int_0^1 q(\xi) \upsilon(s) G(x,\xi,t-s)\, d\xi ds, \tag{4}$$

where

$$G(x,\xi,t) = \sum_{n=1}^{\infty} \frac{1}{\pi n} \sin \pi n t \cos \pi n x \cos \pi n \xi \tag{5}$$

and $\{\cos \pi n x\}$ is an eigenfunction of the following spectral problem

$$X'' + \lambda^2 X = 0, \tag{6}$$

$$X'(0) = X'(1) = 0.$$

The problem is: to find such a control $\upsilon(t) \in L_2(0,T)$ that the corresponding solution $u(x,t)$ of problem (1)-(3) represented in the form (4) satisfies the conditions

$$u(x,T) = \varphi_0(x),\, u_t(x,T) = \varphi_1(x) \tag{7}$$

and the functional

$$I(\upsilon) = \|\upsilon\|^2_{L_2(0,T)} \tag{8}$$

takes the least possible value, where $\varphi_0 \in W_2^1(0,1)$, $\varphi_1 \in L_2(0,1)$ [3].

We can write conditions (7) in the form

$$\left\{ \begin{array}{l} \displaystyle\int_0^T \int_0^1 G(x,\xi;T-s) q(\xi) \upsilon(s)\, d\xi ds = \psi_0(x), \\ \displaystyle\int_0^T \int_0^1 G_t(x,\xi;T-s) q(\xi) \upsilon(s)\, d\xi ds = \psi_1(x), \end{array} \right. \tag{9}$$

where

$$\psi_0(x) = \varphi_0(x) - \int_0^1 G_t(x,\xi;T) u_0(\xi)\, d\xi - \int_0^1 G(x,\xi;T) u_1(\xi)\, d\xi,$$

$$\psi_1(x) = \varphi_1(x) - \int_0^1 G_{tt}(x,\xi;T) u_0(\xi)\, d\xi - \int_0^1 G_t(x,\xi;T) u_1(\xi)\, d\xi.$$

From the problem data it follows that $\psi_0 \in W_2^1(0,1)$, $\psi_1 \in L_2(0,1)$. Then $\psi_0(x)$ and $\psi_1(x)$ may be expanded by the eigenfunctions of the spectral problem (6) as follows

$$\psi_0(x) = \sum_{n=1}^{\infty} \psi_n^0 \cos \pi n x,$$

$$\psi_1(x) = \sum_{n=1}^{\infty} \psi_n^1 \cos \pi n x, \tag{10}$$

where $\psi_n^0 = \int_0^1 \psi_0(x) \cos \pi n x dx$, $\psi_n^1 = \int_0^1 \psi_1(x) \cos \pi n x dx$.

Substituting expansion (10) into relation (9), taking into account (5) and assuming

$$q(x) = \sum_{n=1}^{\infty} q_n \cos \pi n x, \tag{11}$$

where $q_n = \int_0^1 q(x) \cos \pi n x dx$, from equation (9) we obtain the following system of equations

$$\begin{cases} \dfrac{q_n}{\pi n} \displaystyle\int_0^T \sin \pi n (T-s) \upsilon(s) ds = \psi_n^0, \\ q_n \displaystyle\int_0^T \cos \pi n (T-s) \upsilon(s) ds = \psi_n^1, n = 1, 2, ... \end{cases} \tag{12}$$

Thus, a minimal energy control problem is formulated in the following way.

To find such $\upsilon(t) \in L_2(0,T)$ that satisfies equations (12) and gives minimum to the functional

$$I = \int_0^T \upsilon^2(t) dt. \tag{13}$$

From (12), assuming

$$q_n \neq 0, \forall n = 1, 2, ... \tag{14}$$

we have

$$\begin{cases} \displaystyle\int_0^T \sin \pi n (T-s) \upsilon(s) ds = \frac{\pi n \psi_n^0}{q_n} = a_n, \\ \displaystyle\int_0^T \cos \pi n (T-s) \upsilon(s) ds = \frac{\psi_n^1}{q_n} = b_n, \ n = 1, 2, ... \end{cases} \tag{15}$$

System (15) is an infinite dimensional moments problem [4].

Assuming that the condition

$$\sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) = \sum_{n=1}^{\infty} \left(\left(\frac{\pi n \psi_n^0}{q_n} \right)^2 + \left(\frac{\psi_n^1}{q_n} \right)^2 \right) < \infty, \tag{16}$$

is fulfilled in addition to (14), we show that the moments problem (15) and consequently the control problem (1)-(3), (7), (8) is solvable for $T = 2$.

In the integrals in (15) we replace $T - s = \tau$. Taking into accont $T = 2$, we obtain

$$\begin{cases} \int\limits_0^2 \upsilon(2-\tau)\sin \pi n\tau d\tau = a_n, \\ \int\limits_0^2 \upsilon(2-\tau)\cos \pi n\tau d\tau = b_n, \ n = 1, 2, ... \end{cases} \tag{17}$$

Since $\{\sin \pi n\tau, \cos \pi n\tau\}$ is an orthonormed system in $L_2(0,2)$, then by Riesz-Fisher theorem [5], under condition (16) the function

$$\upsilon(2-\tau) = \sum_{n=1}^{\infty} (a_n \sin \pi n\tau + b_n \cos \pi n\tau), \ 0 \le \tau \le 2 \tag{18}$$

will be the solution of the moments problem (17). The series (18) converges in the norm of $L_2(0,2)$ and its sum $\upsilon(2-\tau) \in L_2(0,2)$. Passing in (18) to the variable $t = 2 - \tau$ and taking into account that $\cos(2-\tau)\pi n = \cos(2\pi n - \pi n\tau) = \cos \pi n\tau$, $\sin(2-\tau)\pi n = \sin(2\pi n - \pi n\tau) = -\sin \pi n\tau$, we get

$$\upsilon(t) = \sum_{n=1}^{\infty} (a_n \sin \pi n\tau - b_n \cos \pi n\tau), \ 0 \le t \le 2, \tag{19}$$

where the coefficients a_n, b_n are taken from (15). Notice that

$$\int\limits_0^2 \upsilon(t)\, dt = 0, \tag{20}$$

since the functions $\{\sin \pi nt, \cos \pi nt, n = 1, 2, ...\}$ are orthogonal in $L_2(0,2)$ to the element 1. By this, it is shown that for $T = 2$, the control problem (1)-(3), (7), (8) has the solution (19) possessing the property of (20).

Case b). Since for any function $\alpha(\xi)$ continuous with respect to ξ at the point x_0 it holds the equality $\int\limits_0^1 \delta(\xi - x_0)\alpha(\xi)\, dx = \alpha(x_0)$, in this case the solution of problem (1)-(3) may be represented in the form:

$$u(x,t) = \int\limits_0^1 u_0(\xi) G_t(x,\xi,t)\, d\xi + \int\limits_0^1 u_1(\xi) G(x,\xi,t)\, d\xi +$$

$$+ \int\limits_0^t \upsilon(s) G(x, x_0, t-s)\, ds. \tag{21}$$

In this case the problem is: to find such a control $\upsilon(t) \in L_2(0,T)$ that corresponding solution $u(x,t)$ of problem (1)-(3) represented in the from of (21) satisfy conditions (7) and the functional (8) take minimal value.

We can write condition (7) in the form

$$\int_0^T G(x, x_0; T-s)\, \upsilon(s)\, ds = \psi_0(x),$$

$$\int_0^T G_t(x, x_0; T-s)\, \upsilon(s)\, ds = \psi_1(x). \tag{22}$$

As in the previous case, we get the system

$$\int_0^T \upsilon(s) \sin \pi n (T-s) = \alpha_n,$$

$$\int_0^T \upsilon(s) \cos \pi n (T-s)\, ds = \beta_n, n = 1, 2, ..., \tag{23}$$

where

$$\alpha_n = \frac{\pi n \psi_n^0}{\cos \pi n x_0},$$

$$\beta_n = \frac{\psi_n^1}{\cos \pi n x_0},$$

moreover, $\cos \pi n x_0 \neq 0$, $0 < x_0 < 1$.

System (23) also represents an infinite dimensional moments problem.

Assuming that the condition

$$\sum_{n=1}^{\infty} \left(\alpha_n^2 + \beta_n^2\right) < \infty,$$

is fulfilled, similar to the case a) it may be shown that the moments problem (23) and consequently the control problem (1)-(3), (7), (8) is solvable for $T = 2$ and has the solution

$$\upsilon(t) = \sum_{n=1}^{\infty} \left(\alpha_n \sin \pi n t - \beta_n \cos \pi n t\right).$$

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