Zahira V. MAMEDOVA

ON APPROXIMATE PROPERTIES OF SYSTEMS IN BANACH SPACES

Abstract

Some problems of theory of bases are considered in the paper. The known notion for bases as a space of coefficients, natural isomorphism are carried to the case of systems possessing certain properties

The problems of minimality and basicity of the given of the given systems of functions in Lebesgue and Sobel spaces are very important in approximation theory. The same problems are of great interest from the point of view of spectral theory of differential operators. There are many papers in this direction. Some general aspect was considered in the paper [1] in the Hilbert space $L_2(a,b)$. In the present paper, some problems of close systems in Banach systems are considered. The papers of the authors [2-9] are closely related to this theme.

Some general facts. We'll need some facts from theory of bases in Banach spaces. Let X and Y be some Banach spaces. By $L(X;Y)$ we denote a Banach space of bounded operators acting from X to Y. Accept $L(X) \equiv L(X;X)$. Let $\{x_n\}_{n\in N}$ be a basis in X. If $T \in L(X;Y)$ is invertible, then $\{Tx_n\}_{n\in N}$ also forms a basis in Y with the same space of coefficient with $\{x_n\}_{n\in N}$.

Now, let $F \in L(X;Y)$ be a Fredholm operator, $\{x_n\}_{n\in N} \subset X$ be a complete and minimal in X system and $y_n = Fx_n$, $\forall n \in N$. If F is invertible, then it is clear that $\{y_n\}_{n\in N}$ is also complete and minimal in Y. Recall that the system $\{y_n\}_{n\in N}$ is said to be ω - linearly independent in Y if $\sum\limits_{n=1}^{\infty} a_n y_n = 0$ is possible only for $a_n = 0$, $\forall n \in N$. It is easy to notice that if F is invertible, then $\{y_n\}_{n\in N}$ is ω - linearly independent. Vice versa, assume that $\{y_n\}_{n\in N}$ is complete in Y. Take $\varphi^* \in KerF^*$ and consider:

$$0 = (F^*\varphi^*) = \varphi^*(Fx_n) = \varphi^*(y_n), \quad \forall n \in N.$$

From the completeness of $\{y_n\}_{n\in N}$ in Y we get $\varphi^* = 0$, i.e. $KerF^* = (0)$. Consequently, F is invertible and so $\{y_n\}_{n\in N}$ is also minimal and ω-linearly independent in Y. Further, it is clear that $\operatorname{Im} F$ is closed in Y, moreover $\dim Y/\operatorname{Im} F < +\infty$, where $Y/\operatorname{Im}/F$ is a factor space and $\operatorname{Im} F$ is the set of values in F.

By $L[M]$ we denote a linear span of the set M in the appropriate space. Let $S_{\overline{x}} \equiv \{x_n\}_{n\in N}$, $x \neq 0$, $\forall$ Define:

$$K_{\overline{x}} \equiv \left\{ \{\lambda_n\}_{n\in N} : \text{the series } \sum_{n=1}^{\infty} \lambda_n x_n \text{ converges in } X \right\}.$$

It is easy to notice that with respect to ordinary operations of addition and multiplication by a complex number, $K_{\overline{x}}$ is a linear space. In the sequel, we'll assume $x_n \neq 0$, $\forall n \in N$. Introduce a norm in $K_{\overline{x}}$:

$$\|\overline{\lambda}\|_{K_{\overline{x}}} = \sup_m \left\| \sum_{n=1}^{m} \lambda_n x_n \right\|, \quad \text{where } \overline{\lambda} = \{\lambda_n\}_{n\in N} \in K_{\overline{x}}$$

Obviously $\left\|\lambda\overline{\lambda}\right\|_{K_{\overline{x}}} = |\lambda| \cdot \left\|\overline{\lambda}\right\|_{K_{\overline{x}}}$, $\forall\lambda \in \mathbb{C}$ and $\left\|\overline{\lambda}+\overline{\mu}\right\|_{K_{\overline{x}}} \le \left\|\overline{\lambda}\right\|_{K_{\overline{x}}} + \left\|\overline{\mu}\right\|_{K_{\overline{x}}}$, $\forall\overline{\lambda},\overline{\mu} \in K_{\overline{x}}$. Let $\left\|\overline{\lambda}\right\|_{K_{\overline{x}}} = 0$. Denote $n_0 = \min\{n : \lambda_k = 0, \forall k < n\}$. If $n_0 < +\infty$, it is clear that $\sup\limits_m \left\|\sum\limits_{n=1}^{m} \lambda_n x_n\right\| \ge \left\|\sum\limits_{n=1}^{n_0} \lambda_n x_n\right\| = |\lambda_{n_0}| \cdot \|x_{n_0}\| > 0$. We get contradiction, so $n_0 = +\infty$; i.e. $\overline{\lambda} = 0$.

Consequently $K_{\overline{x}}$ is a normed space. Show that it is complete. Let $\left\{\overline{\lambda}_n\right\}_{n\in N} \subset K_{\overline{x}}$ be some fundamental sequence, where $\overline{\lambda}_n \left\{\lambda_k^n\right\}_{k\in N} \subset \mathbb{C}$. Take $\forall k \in N$ and fix it. Consider:

$$\left|\lambda_k^n - \lambda_k^{n+p}\right| = \frac{\left\|\left(\lambda_k^n - \lambda_k^{n+p}\right) x_k\right\|}{\|x_k\|} =$$

$$= \frac{\left\|\sum\limits_{i=1}^{k}\left(\lambda_i^n - \lambda_i^{n+p}\right) x_i - \sum\limits_{i=1}^{k-1}\left(\lambda_i^n - \lambda_i^{n+p}\right) x_i\right\|}{\|x_k\|} \le \frac{2\left\|\overline{\lambda}_n - \overline{\lambda}_{n+p}\right\|_{K_{\overline{x}}}}{\|x_k\|} \to 0\,,$$

$$\text{for } n,p \to \infty.$$

We get that the sequence $\{\lambda_k^n\}_{n\in N}$ is fundamental for $\forall k \in N$ and let $\lambda_k^n \to \lambda_k$, $n \to \infty$. Take $\forall\varepsilon > 0$. It is clear that $\exists n_0 : \forall n \ge n_0$, $\forall p \in N : \left\|\overline{\lambda}_n - \overline{\lambda}_{n+p}\right\|_{K_{\overline{x}}} < \varepsilon$. Thus,

$$\left\|\sum_{k=1}^{m}\left(\lambda_k^n - \lambda_k^{n+p}\right) x_k\right\| < \varepsilon,\ \forall n \ge n_0, \forall p \in N,\ \forall m \in N.$$

Here, passing to limit as $p \to \infty$, we get:

$$\left\|\sum_{k=1}^{m}\left(\lambda_k^n - \lambda_k\right) x_k\right\| \le \varepsilon,\ \forall n \ge n_0,\ \forall m \in N. \tag{1}$$

Obviously,

$$\left\|\sum_{k=1}^{m}\left(\lambda_k^n - \lambda_k^{n+p}\right) x_k\right\| < \varepsilon,\ \forall n \ge n_0, \forall p \in N,\ \forall m \in N.$$

It follows from the convergence of the series $\sum\limits_{k=1}^{\infty} \lambda_k^n x_k$ that $\exists m_0^n : \forall m \ge m_0^n$, $\forall p \in N$:

$$\left\|\sum_{k=m}^{m+p} \lambda_k^n x_k\right\| < \varepsilon.$$

We have:

$$\left\|\sum_{k=m}^{m+p} \lambda_k x_k\right\| \le \left\|\sum_{k=m}^{m+p}\left(\lambda_k^n - \lambda_k\right) x_k\right\| + \left\|\sum_{k=m}^{m+p} \lambda_k^n x_k\right\| \le 3\varepsilon,\ \forall m \ge m_0^n,\ \forall p \in N.$$

This in its turn means that the series $\sum\limits_{k=1}^{\infty} \lambda_k x_k$ converges in X, i.e. $\overline{\lambda} = \{\lambda_k\}_{k\in N} \in K_{\overline{x}}$. It directly follows from (1) that $\left\|\overline{\lambda}_n - \overline{\lambda}\right\| \to 0$, $n \to \infty$. As a result, we get that $K_{\overline{x}}$ is a Banach space. Take $\forall\overline{\lambda} \in K_{\overline{x}}$ and consider the operator $T :\ K_{\overline{x}} \to X$:

$$T\overline{\lambda} = \sum_{n=1}^{\infty} \lambda_n x_n, \forall\overline{\lambda} = \{\lambda_n\}_{n\in N} \in K_{\overline{x}}.$$

It is easy to notice that T is a linear operator. Let $x = T\overline{\lambda}$. We have:

$$\left\|T\overline{\lambda}\right\| = \|x\| = \left\|\sum_{n=1}^{\infty}\lambda_n x_n\right\| \leq \sup_m \left\|\sum_{n=1}^{m}\lambda_n x_n\right\| = \left\|\overline{\lambda}\right\|_{K_{\overline{x}}}.$$

Thus, $T \in L\left(K_{\overline{k}}; X\right)$ and $\|T\| \leq 1$. It is obvious that if the system $\{x_n\}_{n\in N}$ is ω - linearly independent in X, then $KerT = \{0\}$, where $KerT = \left\{\overline{\lambda} : T\overline{\lambda} = 0\right\}$. In this case $\exists T^{-1} : \operatorname{Im} T \to TK_{\overline{x}}$, where $\operatorname{Im} T$ is an image of the operator T, i.e. a space of values of T. If in this case $\operatorname{Im} T$ is close, then by Banach theorem on an inverse operator we get $T^{-1} \in L\left(\operatorname{Im} T; K_{\overline{x}}\right)$. It is clear that the reasonings cited above are valid in the case when $\{x_n\}$n$\in N$ is minimal in X. The system $\{x_n\}_{n\in N}$ is called non-degenerate if $x_n \neq 0$, $\forall n \in N$. So, we arrive at the following conclusion.

To the non-degenerate system $S_{\overline{x}}$ there corresponds a Banach space of coefficients $S_{K_{\overline{x}}}$ and the operator $T \in L\left(K_{\overline{x}}; X\right)$; $\|T\| \leq 1$. Furthermore, if $S_{\overline{x}}$ is ω - linearly independent in X, then $\exists T^{-1}$. But if $\operatorname{Im} T$ is closed, then $T^{-1} \in L\left(T; K_x\right)$.

Denote by $\{e_n\}_{n\in N} \subset K_{\overline{x}}$, where $e_n = \{\delta_{nk}\}_{n\in N}$ (δ_{nk} -is Kronecker's symbol) a canonical system in $K_{\overline{x}}$. Obviously, $Te_n = x_n$, $\forall n \in N$. Prove that $\{e_n\}_{n\in N}$ forms a basis in $K_{\overline{x}}$. Take $\forall\overline{\lambda} = \{\lambda_n\}_{n\in N} \in K_{\overline{x}}$. Show that the series $\sum\limits_{n=1}^{\infty}\lambda_n e_n$ converges to $K_{\overline{x}}$. Really, it follows from the convergence of the series $\sum\limits_{n=1}^{\infty}\lambda_n x_n$ in X that $\forall\varepsilon > 0$, $\exists m_0 \in N : \forall m \geq m_0$, $\forall p \in N$:

$$\left\|\sum_{n=m}^{m+p}\lambda_n x_n\right\| < \varepsilon.$$

We have:

$$\left\|\sum_{n=m}^{m+p}\lambda_n e_n\right\|_{K_{\overline{x}}} = \left\|\{\lambda_n\}_{n=m}^{m+p}\right\| = \sup_{\nu}\left\|\sum_{n=m}^{\nu}\lambda_n x_n\right\| \leq \varepsilon,\ \forall m \geq m_0, \forall p \in N.$$

Hence it follows that the series $\sum\limits_{n=1}^{\infty}\lambda_n e_n$ converges in $K_{\overline{x}}$. Furthermore,

$$\left\|\overline{\lambda} - \sum_{n=1}^{m}\lambda_n e_n\right\|_{K_{\overline{x}}} = \left\|\{\lambda_n\}_{n\in N} - \{\lambda_n\}_{n=1}^{m}\right\|_{K_{\overline{x}}} = \left\|\{\lambda_n\}_{n=m+1}^{\infty}\right\|_{K_{\overline{x}}} =$$

$$= \sup_{\nu}\left\|\sum_{n=m+1}^{\nu}\lambda_n x_n\right\| \to 0 \text{ as } m \to \infty.$$

Consequently, $\overline{\lambda} = \sum\limits_{n=1}^{m}\lambda_n e_n$.

Consider the functionals $e_n^*\left(\overline{\lambda}\right) = \lambda_n$, $\forall n \in N$, $\forall\overline{\lambda} = \{\lambda_k\}_{k\in N} \in K_{\overline{x}}$. It follows from the relation

$$\left|e_n^*\left(\overline{\lambda}\right)\right| = |\lambda_n| = \frac{\|\lambda_n x_n\|}{\|x_n\|} \leq \frac{\left\|\sum\limits_{k=1}^{n}\lambda_x x_k\right\|}{\|x_n\|} + \frac{\left\|\sum\limits_{k=1}^{n-1}\lambda_x x_k\right\|}{\|x_n\|} \leq \frac{2}{\|x_n\|}\left\|\overline{\lambda}\right\|_{K_{\overline{x}}},$$

that $\{e_n^*\}_{n\in N} \subset K_{\overline{x}}^*$, where $K_{\overline{x}}^*$ is a space conjugated to $K_{\overline{x}}$. On the other hand, it is easy to see that $e_n^*(e_k) = \delta_{nk}$, $\forall n, k \in N$, i.e. $\{\overline{e}_n^*\}_{n\in N}$ is a system conjugated to $\{e_n\}_{n\in N}$. As a result, we get that $\{e_n\}_{n\in N}$ is minimal in $K_{\overline{x}}$ and at the same time it forms a basis in it. Thus, it is valid

Statement 1. Let $\{x_n\}_{n\in N} \subset X$ be a non-degenerate system. Then the appropriate space of coefficients $K_{\overline{x}}$ is a Banach space with a canonical basis $\{e_n\}_{n\in N}$, in other words, each non-degenerate system $S_{\overline{x}}$ generates a Banach space of coefficients $K_{\overline{x}}$ with a canonical basis.

In addition to what has been said, assume that $\{x_n\}_{n\in N}$ is ω - linearly independent (minimal) and $\operatorname{Im} T$ is closed. Then it is easy to notice that $\{x_n\}_{n\in N}$ forms a basis in $\operatorname{Im} T$ and in the case of its completeness in X it is a basis in it. Consequently, $K_{\overline{x}}$ and X are isomorphic and T is an isomorphism between them. The inverse is also true, i.e. if the operator T defined above is an isomorphism of $K_{\overline{x}}$ to X, then $\{x_n\}_{n\in N}$ forms a basis in X. T is said to be a coefficient operator.

Statement 2. Let $S_{\overline{x}}$ be a non-degenerate system, $K_{\overline{x}}$-be an appropriate space of coefficients, $T : K_{\overline{x}} \to S_{\overline{x}}$ be a coefficient operator. $S_{\overline{x}}$ form a basis X if T is an isomorphism between $K_{\overline{x}}$ and X.

Let X be some Banach space and $T : X \to X$ be a completely continuous operator. Consider $\Phi_\lambda = I + \lambda T, \lambda \in C$ is a complex parameter. It is known that Φ_λ is a Fredholm operator. If λ is a regular value of T, then Φ_λ is invertible and consequently it takes any basis $\{x_n\}_{n\in N} \subset X$ to basis $\{\Phi_\lambda x_n\}_{n\in N}$. But if λ is an eigen value of T, then the system $\{\Phi_\lambda x_n\}_{n\in N}$ simultaneously is not complete and is not minimal in X and it has a finite defect. The set of such values $\{\lambda_k\}_{k\in N}$ is discrete and $\lim\limits_{k\to\infty} |\lambda_k| = \infty$.

Assume that $\{x_n\}_{n\in N} \subset X$ is a basis in Banach space X and $S_x \equiv \{x_n^*\}_{n\in N} \subset X^*$ is its conjugated system, where X^* is a space conjugated to X. Consider the operator $\Phi : X \to X$:

$$\Phi x = \sum_{n=1}^{\infty} x_n^*(x)\, y_n, \tag{2}$$

where $S_{\overline{y}} \equiv \{y_n\}_{n\in N} \subset X$ is some system. Obviously, the domain of definition D_Φ of the operator Φ consists of those $x \in X$, for which series (2) converges in X.

It is clear that $\Phi = I + T$, where

$$Tx = \sum_{n=1}^{\infty} x_n^*(x)\,(y_n - x_n)\,, \forall x \in D_\Phi. \tag{3}$$

Accept the following

Definition 1. *We call the system $S_{\overline{y}}$ $S_{\overline{x}}^*-$close to the system $S_{\overline{x}}$ if for $\forall x \in X$ series (3) converges, i.e. $D_T = X$. Thereby, if the operator T determined by expression (3) is completely continuous, we call this closeness a $\sigma S_{\overline{x}}^*$-closeness. It is easy to notice that if for $\forall x \in X$:*

$\{x_n^*(x)\}_{n\in N} \in l_p$ and $\{\|y_n - x_n\|\}_{n\in N} \in l_n$, where $\frac{1}{p} + \frac{1}{q} = 1, 1 \le p \le +\infty$ system $S_{\overline{y}}$ and $S_{\overline{x}}$, $\sigma S_{\overline{x}}^*$ then the systems $S_{\overline{y}}$ and $S_{\overline{x}}$ are $\sigma S_{\overline{x}}^*$ - close.

Thus, if the system $S_{\overline{y}}$ is $\sigma S_{\overline{x}}^*$- close to the minimal system S_x, then the operator Φ is Fredholm. In this case it holds

Theorem 1. *Let system $S_{\overline{x}}$ form a basis in X and the system $S_{\overline{y}}$ be $S_{\overline{x}}^*$ - close to it. Then the following statements are equivalent:*

1) $S_{\overline{y}}$ is complete in X;

2) $S_{\overline{y}}$ is minimal in X;

3) $S_{\overline{y}}$ is ω-linearly independent in X;

4) $S_{\overline{y}}$ forms a basis in X isomorphic to the basis $S_{\overline{x}}$;

5) the operator $\Phi = I + T$ is invertible in $L(X)$, where $L(X)$ is an algebra of bounded operators acting from X to X.

Validity of this statement directly follows from the reasonings cited above and relations $\Phi x_n = y_n$, $\forall n \in N$.

Now, let $\lambda \in \rho(T)$ be a regular value of the operator T. Thus, in this case the Fredholm operator $\Phi_\lambda = I + \lambda T$ is invertible.

We have: $\Phi_\lambda x_n = x_n + \lambda(y_n - x_n) = (1-\lambda)x_n + \lambda y_n$, $\forall n \in N$. But if $0 \neq \lambda \in \sigma(T)$ is an eigen value of the operator T, the system $S_{\overline{y}}^{\lambda} \equiv \{x_n + \lambda(y_n - x_n)\}_{n\in N}$ simultaneously is not complete and is not minimal (it is not ω- linearly independent), in X . It is clear that if $S_{\overline{y}}^{\lambda}$ is complete (minimal or is ω- linearly independent), then Φ_λ is invertible.Thus, it is valid

Theorem 2. *Let $S_{\overline{x}}$ form a basis in X and the system $S_{\overline{y}}$ be $\sigma S_{\overline{x}}^{*}$ - close to it. Then the following statements are equivalent:*

1) $S_{\overline{y}}^{\lambda}$ is complete in X;

2) $S_{\overline{y}}^{\lambda}$ is minimal in X;

3) $S_{\overline{y}}^{\lambda}$ is ω - linearly independent in X;

4) $S_{\overline{y}}^{\lambda}$ forms a basis in X isomorphic to the basis $S_{\overline{x}}$;

5) λ belongs to the resolvent set T.

References

[1]. Kazmin Yu. A. *On bases and complete systems of functions in Hilbert space.* // Mat. Sbornik. 1953, vol. 42 (84), No 4, pp. 513-522 (Russian).

[2]. Zinger I. *Bases in Banach spaces* I, SVBH, New York, 1970, 672 p.

[3]. Bilalov B.T. *Bases from some system of exponents in L_p.* // Dokl. RAN. 2003, vol. 392, No 5 (Russian).

[4]. Bilalov B.T. *Bases from exponents, cosines and sines being eigen functions of differential operator* // Differen. Uravn. 2003, vol. 39, No 5, pp. 1-5 (Russian).

[5]. Bilalov B.T., Muradov T.R. *On equivalent bases in Banach space* // Ukr. Mat zh. 2007, vol. 59, No 4, pp. 551-554 (Russian).

[6]. Casazza P.G., Christensen O. *Frames containing a Riesz basis and preservation of this property under perturbation.* SIAMJ. Math. Anal. 1998, 29, No 1, pp. 266-278.

[7]. Christensen O., Jensen T.K. *An introduction to the theory of bases, frames, and waveletes.* DTU, 2000, 165 p.

[8]. He X., Volkmer H. *Riesz bases of solutions of Sturm-Lioville equations.* J. Fourier Anal. Appl., 2001, v. 7, No 3, pp. 297-307.

[9]. Christensen O. Frames *Riesz bases and discrete Gabor-wavelet expansions.* Bull Amer. Math. Soc., 38, 2001, pp. 279-291.

Zahira V. Mamedova
Institute of Mathematics and Mechanics of NAS of Azerbaijan.
9, F.Agayev str., AZ1141, Baku, Azerbaijan.
Tel.: (99412) 439 47 20 (off.).

Received September 09, 2009; Revised November 24, 2009.