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EXISTENCE OF A GLOBAL WEAK SOLUTION OF THE CAUCHY PROBLEM FOR SYSTEMS OF SEMILINEAR HYPERBOLIC EQUATIONS

Abstract

We consider the Cauchy problem for systems of second and fourth order semilinear hyperbolic equations.

In the paper we established the criterion analogous to Fujit's criterion for global solvability: the conditions on nonlinearity increase ensuring the existence of global solution are obtained. The mentioned example shows that the conditions on nonlinearity increase are exact.

In the domain $R_+ \times R^n$ we consider the Cauchy problem for the systems of semilinear hyperbolic equations

$$\left.\begin{array}{l} u_{tt} + u_t - \Delta u = f_1(u,v), \\ \\ v_{tt} + v_t + \Delta^2 v = f_2(u,v) \end{array}\right\} \tag{1}$$

with the initial conditions

$$\left.\begin{array}{l} u(0,x) = \varphi_1(x), \;\; u_t(0,x) = \psi_1(x), \\ \\ v(0,x) = \varphi_2(x), \;\; v_t(0,x) = \psi_2(x), \end{array}\right\}, \tag{2}$$

where $f_1, f_2 \in C^1\left(R^2\right)$.

It is known that at any $\varphi_i \in W_2^i(R^n), \psi_i \in L_2(R^n), i = 1,2$ the problem has a local solution $(u,v) \in C[0,T];\, W_2^1(R^n) \times W_2^2(R^n) \cap C^1([0,T]; L_2(R^n) \times L_2(R^n))$, where $T = T\left(\|\varphi_1\|_{W_2^1(R^n)}, \|\varphi_2\|_{W_2^2(R^n)}, \|\psi_1\|_{L_2(R^n)}, \|\psi_2\|_{L_2(R^n)}\right)$.

Let T' be length of maximum interval of existence of the solutions $(u,v) \in C\left([0,T');\, W_2^1(R^n) \times W_2^2(R^n)\right) \cap C^1([0,T');\, L_2(R^n) \times L_2(R^n))$, then either $T' = +\infty$, or $\lim\limits_{t \to T'-0} E(t) = +\infty$, where

$$E(t) = \|u(t,\cdot)\|_{W_2^1(R^n)} + \|v(t,\cdot)\|_{W_2^2(R^n)} + \|u_t(t,\cdot)\|_{L_2(R^n)} + \|v_t(t,\cdot)\|_{L_2(R^n)}.$$

Hence if follows that for global extension of the given solution it is sufficient to obtain a priori estimate

$$E(t) \le c, t \in \left[0, T'\right). \tag{3}$$

The existence and absence of global solutions of the Cauchy problem for semilinear dissipative wave equations have been investigated in the papers of various authors (see [1-7]), and existence of global solutions for higher order pseudo-hyperbolic equations has been investigated in the paper [8].

In the present paper the existence and uniqueness of global solutions of the problem (1), (2) are investigated.

At the end of the paper it is cited an example of the Cauchy problem for systems of hyperbolic equations of the form (1), on absence of global solutions that shows the exactness of the obtained result.

Assume that the following conditions are satisfied:

$$|f_i(u,v)| \le c\,|u|^{p_i}\,|v|^{q_i}\,,\ u,v\in R,\ i=1,2. \tag{4}$$

Here

$$P_i \ge 0,\ q_i\ge 0,\quad p_i+q_i>0$$
$$P_i+\frac{q_i}{2} > \frac{2}{n}+\varphi(p_i,q_i)\,,\quad i=1,2, \tag{5}$$

where $\varphi(p,q)=\begin{cases}\frac{1}{2} & p\ge 0,\quad q>0\\ 1 & p>0,\quad q=0\end{cases}.$

Theorem 1. *Let the condition (4), (5) be satisfied, then there exists such $\delta>0$ that at any*

$$((\varphi_1,\psi_1),(\varphi_2,\psi_2))\in U_\delta \equiv \{((\varphi_1,\psi_1),(\varphi_2,\psi_2)):$$
$$(\varphi_1,\psi_1)\in\ \left[W_2^1(R^n)\cap L_1(R^n)\right]\times\left[L_2(R^n)\cap L_1(R^n)\right]\}$$
$$(\varphi_2,\psi_2)\in\ \left[W_2^2(R^n)\cap L_1(R^n)\right]\times\left[L_2(R^n)\cap L_1(R^n)\right],$$
$$\sum_{i=1}^{2}\left[\{\left\|\nabla^i\varphi_i\right\|_{L_2(R^n)}+\|\varphi_i\|_{L_1}+\|\psi_i\|_{L_1(R^n)}+\|\psi_i\|_{L_2(R^n)}\right]<\delta\}$$

the problem (1), (2) has a unique weak global solution

$$(u,v)\in C\left([0,\infty)\,;W_2^1(R^n)\times W_2^2(R^n)\right)\cap C^1\left([0,\infty);L_2(R^n)\times L_2(R^n)\right)$$

for which the following estimates hold:

$$\|u(t,\cdot)\|_{L_2(R^n)} \le c\,(1+t)^{-\frac{n}{4}}\,, \tag{6}$$

$$\|\nabla u(t,\cdot)\|_{L_2(R^n)} \le c\,(1+t)^{-\left(\frac{1}{2}+\frac{n}{4}\right)}\,, \tag{7}$$

$$\|u_t(t,\cdot)\|_{L_2(R^n)} \le c\,(1+t)^{-\lambda_0}\,, \tag{8}$$

$$\sum_{|\alpha|=r}\|D^\alpha v(t,\cdot)\|_{L_2(R^n)} \le c\,(1+t)^{-\frac{n+2r}{8}}\,,\ r=0,1,2, \tag{9}$$

$$\|v_t(t,\cdot)\|_{L_2(R^n)} \le c\,(1+t)^{-\lambda_1}\,, \tag{10}$$

where

$$\lambda_0=\min\left\{1+\frac{n}{4},\ \frac{n}{2}\left(p_1+\frac{q_1}{2}\right)-\frac{n}{2}\left(\frac{1}{\rho}+\frac{1}{2\rho'}\right)\right\}$$

$$\lambda_1 = \min\left\{1+\frac{n}{8}, \frac{n}{2}\left(p_1+\frac{q_1}{2}\right)-\frac{n}{2}\left(\frac{1}{\rho}+\frac{1}{2\rho'}\right)\right\}, \ \rho>1, \rho'>1, \frac{1}{\rho}+\frac{1}{\rho'}=1.$$

Proof. Let $\hat{u}_0(t,\xi), \hat{u}_1(t,\xi), \hat{v}_0(t,\xi)$ and $\hat{v}_1(t,\xi)$ be solutions of the following problems

$$L_{1\xi}\hat{u}_0 = 0; \hat{u}_0(0,\xi)=1, \ \hat{u}_{0_t}(0,\xi)=0;$$

$$L_{1\xi}\hat{u}_1 = 0; \hat{u}_1(0,\xi)=0, \ \hat{u}_{1_t}(0,\xi)=1;$$

$$L_{2\xi}\hat{v}_0 = 0; \hat{v}_0(0,\xi)=1, \ \hat{v}_{0_t}(0,\xi)=0;$$

$$L_{2\xi}\hat{v}_1 = 0; \hat{v}_1(0,\xi)=0, \ \hat{v}_{1_t}(0,\xi)=1,$$

where

$$L_{1\xi}\hat{u}_i = \hat{u}_{i_{tt}}(t,\xi)+\hat{u}_{i_t}(t,\xi)+|\xi|^2\hat{u}_i(t,\xi),$$

$$L_{2\xi}\hat{v}_i = \hat{v}_{i_{tt}}(t,\xi)+\hat{v}_{i_t}(t,\xi)+|\xi|^4\hat{v}_i(t,\xi).$$

It is known that (see [3])

$$\|u_0(t,x)*\varphi_1(\cdot)\|_{L_2(R^n)} \le c(1+t)^{-\frac{n}{4}}\left(\|\varphi_1\|_{L_1(R^n)}+\|\varphi_1\|_{L_2(R^n)}\right), \tag{11}$$

$$\|\nabla(u_0(t,\cdot)*\varphi_1(\cdot))\|_{L_2(R^n)} \le$$
$$\le c(1+t)^{-\left(\frac{1}{2}+\frac{n}{4}\right)}\left(\|\varphi_1\|_{L_1(R^n)}+\|\varphi_1\|_{W_2^1(R^n)}\right), \tag{12}$$

$$\|u_1(t,\cdot)*\psi_1(\cdot)\|_{L_2(R^n)} \le c(1+t)^{-\frac{n}{4}}\left(\|\psi_1\|_{L_1(R^n)}+\|\psi_1\|_{L_2(R^n)}\right), \tag{13}$$

$$\|\nabla u_1(t,x)*\psi_1(\cdot)\|_{L_2(R^n)} \le c(1+t)^{-\left(\frac{1}{2}+\frac{n}{4}\right)}\left[\|\psi_1\|_{L_1(R^n)}+\|\psi_1\|_{L_2(R^n)}\right]. \tag{14}$$

Similarly (see [6,8])

$$\|v_0(t,\cdot)*\varphi_2(\cdot)\|_{L_2(R^n)} \le c(1+t)^{-\frac{n}{8}}\left[\|\varphi_2\|_{L_1(R^n)}+\|\varphi_2\|_{L_2(R^n)}\right], \tag{15}$$

$$\sum_{|\alpha|=2}\|D^\alpha(v_0(t,\cdot)*\varphi_2(\cdot))\|_{L_2(R^n)} \le$$
$$\le c(1+t)^{-\left(\frac{1}{2}+\frac{n}{8}\right)}\left[\|\varphi_2\|_{L_1(R^n)}+\|\varphi_2\|_{W_2^2(R^n)}\right], \tag{16}$$

$$\|v_1(t,\cdot)*\psi_2(\cdot)\|_{L_2(R^n)} \le c(1+t)^{-\frac{n}{8}}\left[\|\varphi_2\|_{L_1(R^n)}+\|\varphi_2\|_{L_2(R^n)}\right], \tag{17}$$

$$\sum_{|\alpha|=2}\|D^\alpha(v_1(t,\cdot)*\psi_2(\cdot))\|_{L_2(R^n)} \le$$
$$\le c(1+t)^{-\left(\frac{1}{2}+\frac{n}{8}\right)}\left[\|\varphi_2\|_{L_1(R^n)}+\|\varphi_2\|_{L_2(R^n)}\right]. \tag{18}$$

Here $u_i = F^{-1}[\hat{u}_i], v_i = F^{-1}[\hat{v}_i], \ i=1,2, \ F^{-1}$ is an inverse Fourier transformation, $*$ is convolution by the variable x.

We can represent a solution of the problem (1), (2) in the following form

$$u(t,x) = u_0(t,x) * \varphi_1(x) + u_1(t,x) * \psi_1(x) + + \int_0^t u_1(t-\tau,x) * f_1(u(\tau,x), v(\tau,x))\, d\tau, \quad (19)$$

$$v(t,x) = v_0(t,x) * \varphi_2(x) + v_1(t,x) * \psi_2(x) + + \int_0^t v_1(t-\tau,x) * f_2(u(\tau,x), v(\tau,x))\, d\tau. \quad (20)$$

Using (11) – (14) from (19) we obtain the following inequalities

$$\|u(t,\cdot)\|_{L_2(R^n)} \le c(1+t)^{-\frac{n}{4}} E_1(\varphi_1,\psi_1) + c\int_0^t (1+t-\tau)^{-\frac{n}{4}} \times \times \left[\|f_1(u(\tau,\cdot), v(\tau,\cdot))\|_{L_1(R^n)} + \|f_1(u(\tau,\cdot), v(\tau,\cdot))\|_{L_2(R^n)} d\tau\right], \quad (21)$$

$$\|\nabla u(t,\cdot)\|_{L_2(R^n)} \le c(1+t)^{-\left(\frac{1}{2}+\frac{n}{4}\right)} \cdot E_1(\varphi_1,\psi_1) + c\int_0^t (1+t-\tau)^{-\left(\frac{1}{2}+\frac{n}{4}\right)} \times \times \left[\|f_2(u(\tau,\cdot), v(\tau,\cdot))\|_{L_1(R^n)} + \|f_2(u(\tau,\cdot), v(\tau,\cdot))\|_{L_2(R^n)}\right] d\tau, \quad (22)$$

where

$$E_1(\varphi_1,\psi_1) = \|\varphi_1\|_{L_1(R^n)} + \|\varphi_1\|_{W_2^1(R^n)} + \|\psi_1\|_{L_1(R^n)} + \|\psi_1\|_{L_2(R^n)}$$

Analogously, using (15)-(18) from (20) we obtain the following inequalities

$$\|v(t,\cdot)\|_{L_2(R^n)} \le c(1+t)^{-\frac{n}{8}} E_2(\varphi_2,\psi_2) + c\int_0^t (1+t-\tau)^{-\frac{n}{8}} \times \times \left[\|f_2(u(\tau,\cdot), v(\tau,\cdot))\|_{L_1(R^n)} + \|f_2(u(\tau,\cdot), v(\tau,\cdot))\|_{L_2(R^n)}\right] d\tau \quad (23)$$

$$\sum_{|\alpha|=2} \|v(t,\cdot)\|_{L_2(R^n)} \le c(1+t)^{-\left(\frac{1}{2}+\frac{n}{8}\right)} E_2(\varphi_2,\psi_2) + c\int_0^t (1+t-\tau)^{-\left(\frac{1}{2}+\frac{n}{8}\right)} \times \times \left[\|f_2(u(\tau,\cdot), v(\tau,\cdot))\|_{L_1(R^n)} + \|f_2(u(\tau,\cdot), v(\tau,\cdot))\|_{L_2(R^n)}\right] d\tau, \quad (24)$$

where

$$E_2(\varphi_2,\psi_2) = \|\varphi_2\|_{L_1(R^n)} + \|\varphi_2\|_{W_2^2(R^n)} + \|\psi_1\|_{L_1(R^n)} + \|\psi_1\|_{L_2(R^n)}.$$

Further we'll assume that $p_i > 0$, $q_i > 0$, $i = 1,2$. In other cases the proof is conducted similarly.

Applying the Hölder inequality, from (4) we obtain that

$$\|f_1(u,v)\|_{L_1(R^n)} \le c\,\|u\|^{p_1}_{L_{p_1\rho}(R^n)} \cdot \|v\|^{q_1}_{L_{q_1\rho'}} \tag{25}$$

where $\dfrac{1}{\rho}+\dfrac{1}{\rho'}=1,\ \rho,\rho'>1.$

Further, applying multiplicative inequality (see [9]) we have

$$\|u\|^{p_1}_{L_{p_1\rho}(R^n)} \le c\,\|u\|^{p_1(1-\theta_{11})}_{L_2(R^n)}\,\|\nabla u\|^{p_1\theta_{11}}_{L_2(R^n)} \tag{26}$$

$$\|v\|^{q_1}_{L_{q_1\rho'}(R^n)} \le c\,\|v\|^{q_1(1-\theta_{12})}_{L_2(R^n)} \cdot \left(\sum_{|\alpha|=2}\|D^\alpha v\|_{L_2(R^n)}\right)^{q_1\theta_{12}}, \tag{27}$$

where

$$\theta_{11}=n\left(\frac{1}{2}-\frac{1}{p_1\rho}\right), \tag{28}$$

$$\theta_{12}=\frac{n}{2}\left(\frac{1}{2}-\frac{1}{q_1\rho'}\right). \tag{29}$$

It follows from (25)-(27) that

$$\|f_1(u,v)\|_{L_1(R^n)} \le$$

$$\le c\,\|u\|^{p_1(1-\theta_{11})}_{L_2(R^n)}\,\|\nabla u\|^{p_1\theta_{11}}_{L_2(R^n)} \cdot \|v\|^{q_1(1-\theta_{11})}_{L_2(R^n)} \cdot \left(\sum_{|\alpha|=2}\|D^\alpha v\|_{L_2(R^n)}\right)^{q_1\theta_{12}}. \tag{30}$$

Analogously we obtain the inequality

$$\|f_1(u,v)\|_{L_2(R^n)} \le$$

$$\le c\,\|u\|^{p_1(1-\theta_{21})}_{L_2(R^n)}\,\|\nabla u\|^{p_1\theta_{21}}_{L_2(R^n)} \cdot \|v\|^{q_1(1-\theta_{22})}_{L_2(R^n)} \cdot \left(\sum_{|\alpha|=2}\|D^\alpha v\|_{L_2(R^n)}\right)^{q_1\theta_{22}}, \tag{31}$$

where

$$\theta_{21}=\frac{n}{2}\left(1-\frac{1}{p_1\rho}\right),\ \theta_{22}=\frac{n}{4}\left(1-\frac{1}{q_1\rho'}\right). \tag{32}$$

Allowing for condition (4) we obtain also the following inequalities

$$\|f_2(u,v)\|_{L_1(R^n)} \le$$

$$\le c\,\|u\|^{p_2(1-\theta'_{11})}_{L_2(R^n)}\,\|\nabla u\|^{p_2\theta'_{11}}_{L_2(R^n)} \cdot \|v\|^{q_2(1-\theta'_{12})}_{L_2(R^n)} \cdot \left(\sum_{|\alpha|=2}\|D^\alpha v\|_{L_2(R^n)}\right)^{q_2\theta'_{12}}, \tag{33}$$

$$\|f_2(u,v)\|_{L_1(R^n)} \le$$

$$\leq c\left\|u\right\|_{L_2(R^n)}^{p_2(1-\theta'_{21})}\left\|\nabla u\right\|_{L_2(R^n)}^{p_2\theta'_{21}}\cdot\left\|v\right\|_{L_2(R^n)}^{q_2(1-\theta'_{22})}\cdot\left(\sum_{|\alpha|=2}\left\|D^\alpha v\right\|_{L_2(R^n)}\right)^{q_2\theta'_{22}}, \qquad (34)$$

where

$$\theta'_{11}=n\left(\frac{1}{2}-\frac{1}{p_2\rho}\right),\ \theta'_{12}=\frac{n}{2}\left(\frac{1}{2}-\frac{1}{q_2\rho'}\right), \qquad (35)$$

$$\theta'_{21}=\frac{n}{2}\left(1-\frac{1}{p_2\rho}\right),\ \theta'_{22}=\frac{n}{4}\left(1-\frac{1}{q_2\rho'}\right), \qquad (36)$$

$$\frac{1}{\rho}+\frac{1}{\rho'}=1,\ \rho,\rho'>1.$$

Let's introduce the notation

$$\varepsilon_1(t)=(1+t)^{\frac{n}{4}}\left\|u(t,\cdot)\right\|_{L_2(R^n)}+(1+t)^{\left(\frac{1}{2}+\frac{n}{4}\right)}\left\|\nabla u(t,\cdot)\right\|_{L_2(R^n)},$$

$$\varepsilon_2(t)=(1+t)^{\frac{n}{8}}\left\|v(t,\cdot)\right\|_{L_2(R^n)}+(1+t)^{\left(\frac{1}{2}+\frac{n}{4}\right)}\sum_{|\alpha|=2}\left\|D^\alpha v(t,\cdot)\right\|_{L_2(R^n)}.$$

Allowing for (30)-(36) from (21)-(24) we obtain that

$$\varepsilon_1(t)=c_1E_1(\varphi_1,\psi_1)+c(1+t)^{\frac{n}{4}}\int_0^t(1+t-\tau)^{-\frac{n}{4}}(1+\tau)^{-\gamma_1}\varepsilon_1^{p_1}(\tau)\varepsilon_2^{q_1}(\tau)\,d\tau+$$

$$+c(1+t)^{\frac{1}{2}+\frac{n}{4}}\int_0^t(1+t-\tau)^{-\left(\frac{1}{2}+\frac{n}{4}\right)}(1+\tau)^{-\gamma_2}\varepsilon_1^{p_1}(\tau)\varepsilon_2^{q_1}(\tau)\,d\tau. \qquad (37)$$

$$\varepsilon_2(t)=cE_2(\varphi_2,\psi_2)+c(1+t)^{\frac{n}{8}}\int_0^t(1+t-\tau)^{-\frac{n}{8}}(1+\tau)^{-\gamma'_1}\varepsilon_1^{p_2}(\tau)\varepsilon_2^{q_2}(\tau)\,d\tau+$$

$$+c(1+t)^{\frac{1}{2}+\frac{n}{8}}\int_0^t(1+t-\tau)^{-\left(\frac{1}{2}+\frac{n}{8}\right)}(1+\tau)^{-\gamma'_2}\varepsilon_1^{p_2}(\tau)\varepsilon_2^{q_2}(\tau)\,d\tau, \qquad (38)$$

where

$$\gamma_1=p_1(1-\theta_{11})\frac{n}{4}+p_1\theta_{11}\left(\frac{1}{2}+\frac{n}{4}\right)+q_1(1-\theta_{12})\frac{n}{8}+q_1\theta_{12}\left(\frac{1}{2}+\frac{n}{8}\right);$$

$$\gamma_2=p_1(1-\theta_{21})\frac{n}{4}+p_1\theta_{21}\left(\frac{1}{2}+\frac{n}{4}\right)+q_1(1-\theta_{22})\frac{n}{8}+q_1\theta_{22}\left(\frac{1}{2}+\frac{n}{8}\right);$$

$$\gamma'_1=p_2(1-\theta'_{11})\frac{n}{4}+p_2\theta'_{11}\left(\frac{1}{2}+\frac{n}{4}\right)+q_2(1-\theta'_{12})\frac{n}{8}+q_2\theta'_{12}\left(\frac{1}{2}+\frac{n}{8}\right);$$

$$\gamma'_2=p_2(1-\theta'_{22})\frac{n}{4}+p_2\theta'_{22}\left(\frac{1}{2}+\frac{n}{4}\right)+q_2(1-\theta'_{22})\frac{n}{8}+q_2\theta'_{22}\left(\frac{1}{2}+\frac{n}{8}\right).$$

It follows from (28),(29),(32) and (36) that

$$\gamma_1 = \frac{n}{2}\left(p_1 + \frac{q_1}{2}\right) - \frac{n}{2}\left(\frac{1}{\rho} + \frac{1}{2\rho'}\right) > 1;$$

$$\gamma_2 = \frac{n}{2}\left(p_1 + \frac{q_1}{2}\right) - \frac{n}{4}\left(\frac{1}{\rho} + \frac{1}{2\rho'}\right) > 1;$$

$$\gamma_1' = \frac{n}{2}\left(p_2 + \frac{q_2}{2}\right) - \frac{n}{2}\left(\frac{1}{\rho} + \frac{1}{2\rho'}\right) > 1;$$

$$\gamma_2' = \frac{n}{2}\left(p_2 + \frac{q_2}{2}\right) - \frac{n}{4}\left(\frac{1}{\rho} + \frac{1}{2\rho'}\right) > 1;$$

Then by virtue of Sigal lemma (see [10])

$$\left.\begin{array}{l} (1+t)^{\frac{n}{4}} \int\limits_0^t (1+t-\tau)^{-\frac{n}{4}} (1+\tau)^{-\gamma_1} d\tau \leq c, \\ (1+t)^{\frac{1}{2}+\frac{n}{4}} \int\limits_0^t (1+t-\tau)^{-\left(\frac{1}{2}+\frac{n}{4}\right)} (1+\tau)^{-\gamma_2} d\tau \leq c, \\ (1+t)^{\frac{n}{8}} \int\limits_0^t (1+t-\tau)^{-\frac{n}{8}} (1+\tau)^{-\gamma_1'} d\tau \leq c, \\ (1+t)^{\frac{1}{2}+\frac{n}{8}} \int\limits_0^t (1+t-\tau)^{-\left(\frac{1}{2}+\frac{n}{8}\right)} (1+\tau)^{-\gamma_2'} d\tau \leq c \end{array}\right\} \quad (39)$$

Allowing for (39), from (37) and (38) we obtain that

$$\widetilde{\varepsilon}_1(t) = cE_1(\varphi_1, \psi_1) + c_1 \widetilde{\varepsilon}_1^{p_1}(t)\, \widetilde{\varepsilon}_2^{q_1}(t), \quad (40)$$

$$\widetilde{\varepsilon}_2(t) = cE_2(\varphi_2, \psi_2) + c_1 \widetilde{\varepsilon}_1^{p_2}(t)\, \widetilde{\varepsilon}_2^{q_2}(t), \quad (41)$$

where $\widetilde{\varepsilon}_1(t) = \max\limits_{0\leq\tau\leq t} \varepsilon_1(\tau)$, $\widetilde{\varepsilon}_2(t) = \max\limits_{0\leq\tau\leq t} \varepsilon_2(\tau)$.

Denoting by $X(t) = \widetilde{\varepsilon}_1(t) + \widetilde{\varepsilon}_2(t)$, $X_0 = cE_1(\varphi_1, \psi_1) + cE_2(\varphi_2, \psi_2)$ from (40) and (41) we obtain that

$$X(t) \leq X_0 + c_1\left(X^{p_1+q_1}(t) + X^{p_2+q_2}(t)\right). \quad (42)$$

Let $P = \max(p_1 + q_1, p_2 + q_2)$, then applying the Young inequality from (42) we obtain that

$$X(t) \leq (X_0 + \varepsilon) + \left(c_1 + \frac{c_1}{\varepsilon}\right) X^P(t).$$

Hence it follows that

$$X(t) \leq M, t \in [0, T'). \quad (43)$$

It follows from (43) that for $u(t,x)$ and $v(t,x)$ the estimations (6),(7) and (9) are valid, respectively.

Further, from (19) and (20) we obtain that

$$\|u_t(t,\cdot)\|_{L_2(R^n)} \leq c(1+t)^{-\left(1+\frac{n}{4}\right)} E_1(\varphi_1, \psi_1) + c\int\limits_0^t (1+t-\tau)^{-\left(1+\frac{n}{4}\right)} \times$$

$$\times\left[\|f_1(u(\tau,\cdot),v(\tau,\cdot))\|_{L_1(R^n)}+\|f_1(u(\tau,\cdot),v(\tau,\cdot))\|_{L_2(R^n)}\right]d\tau, \quad (44)$$

$$\|v_t(t,\cdot)\|_{L_2(R^n)}\le c(1+t)^{-\left(1+\frac{n}{8}\right)}E_2(\varphi_2,\psi_2)+c\int_0^t(1+t-\tau)^{-\left(1+\frac{n}{8}\right)}\times$$

$$\times\left[\|f_1(u(\tau,\cdot),v(\tau,\cdot))\|_{L_1(R^n)}+\|f_2(u(\tau,\cdot),v(\tau,\cdot))\|_{L_2(R^n)}\right]d\tau. \quad (45)$$

On the other hand, from (4),(30),(31) and (43) it follows

$$\|f_1(u(\tau,\cdot),v(\tau,\cdot))\|_{L_1(R^n)}\le$$

$$\le C_M(1+t)^{-\gamma_1},\|f_1(u(\tau,\cdot),v(\tau,\cdot))\|_{L_2(R^n)}\le C_M(1+t)^{-\gamma_2};$$

$$\|f_2(u(\tau,\cdot),v(\tau,\cdot))\|_{L_1(R^n)}\le$$

$$\le C_M(1+t)^{-\gamma_1'},\|f_2(u(\tau,\cdot),v(\tau,\cdot))\|_{L_2(R^n)}\le C_M(1+t)^{-\gamma_2'}.$$

Allowing for these estimations in the inequalities (43) and (44), we obtain that

$$\|u_t(t,\cdot)\|_{L_2(R^n)}\le c(1+t)^{-\left(1+\frac{n}{4}\right)}E_1(\varphi_1,\psi_1)+$$

$$+c\cdot c_M\int_0^t(1+t-\tau)^{-\left(1+\frac{n}{4}\right)}\left[(1+\tau)^{-\gamma_1}+(1+\tau)^{-\gamma_2}\right]d\tau\le c_1(1+t)^{-\min\left\{1+\frac{n}{4},\gamma_1\right\}}.$$

Analogously it follows from (43),(45) that

$$\|v_t(t,\cdot)\|_{L_2(R^n)}\le c_1(1+t)^{-\min\left\{1+\frac{1}{8},\gamma_1'\right\}},t\ge 0.$$

The following example shows that conditions (5) are exact.

In the domain $R_+\times R^3$ we consider the Cauchy problem

$$\left.\begin{array}{l} u_{tt}+u_t-\Delta u=|v|^{7/3} \\ v_{tt}+v_t+\Delta^2 u=|u|^{5/3} \end{array}\right\} \quad (46)$$

$$\left.\begin{array}{l} u(0,x)=\varphi_1(x),u_t(0,x)=\psi_1(x) \\ v(0,x)=\varphi_2(x),v_t(0,x)=\psi_2(x) \end{array}\right\} \quad (47)$$

and assume that

$$\varphi_j(x),\psi_j(x)\in L_1\left(R^3\right),j=1,2 \quad (48)$$

$$\sum_{i=1}^2\int_{R^3}(\varphi_i(x)+\psi_i(x))\,dx\ge 0. \quad (49)$$

The function $(u,v) \in L_{5/3,lok}\left(R^3\right) \cap L_{7/3,lok}\left(R^3\right)$ satisfying the system

$$-\int\limits_{R^3} \psi_1(x)\,\xi_1(0,x)\,dx + \int\limits_{R^3} \varphi_1(x)\,\frac{\partial \xi_1}{\partial t}(0,x)\,dx + \int\limits_0^\infty \int\limits_{R^3} u\frac{\partial^2 \xi_1}{\partial t^2}dxdt -$$

$$-\int\limits_0^\infty \int\limits_{R^3} u\frac{\partial \xi_1}{\partial t}dxdt + \int\limits_0^\infty \int\limits_{R^3} u\Delta\xi_1 dxdt = \int\limits_0^\infty \int\limits_{R^3} |v|^{7/3}\,\xi_1(0,x)\,dxdt,$$

$$-\int\limits_{R^3} \psi_2(x)\,\xi_2(0,x)\,dx + \int\limits_{R^3} \varphi_2(x)\,\frac{\partial \xi_2}{\partial t}(0,x)\,dx + \int\limits_0^\infty \int\limits_{R^3} v\frac{\partial^2 \xi_2}{\partial t^2}dxdt - \int\limits_0^\infty \int\limits_{R^3} v\frac{\partial \xi_2}{\partial t}dxdt +$$

$$+\int\limits_0^\infty \int\limits_{R^3} v\Delta^2\xi_2 dxdt = \int\limits_0^\infty \int\limits_{R^3} |u|^{5/3}\,\xi_2(0,x)\,dxdt,$$

for any non-negative functions $\xi_1(t,x), \xi_2(t,x) \in C^\infty(R_+ \times R^n)$ with compact support is called a weak solution of systems of equations with the initial data (47)-(49).

Following [1], let's introduce the trial function

$$\xi_1(t,x) = \xi_2(t,x) = h\left(\frac{t^\mu + |x|^{2\mu}}{d^2}\right),\ h(\cdot) \in C_0^\infty(R),$$

$$0 \le h(r) \le 1,\ h(r) = \begin{cases} 0, r \ge 2 \\ 1\ \ r \le 1, \end{cases}$$

where $d > 0$ and $\mu > 1$ are some parameters. From (47),(48) we obtain that

$$\sum_{i=1}^2 \int\limits_{|x|<2d^2} \left[\varphi_i(x) + \psi_i(x)\right] h\left(\frac{|x|^{2\mu}}{d^2}\right) dx +$$

$$+\int\limits_0^\infty \int\limits_{R^3} |v|^{7/3}\,\xi(t,x)\,dxdt + \int\limits_0^\infty \int\limits_{R^3} |u|^{5/3}\,\xi(t,x)\,dxdt =$$

$$= \int\limits_0^\infty \int\limits_{R^3} u\left(\xi_{tt} - \xi_t - \Delta\xi\right) dxdt + \int\limits_0^\infty \int\limits_{R^3} v\left(\xi_{tt} - \xi_t + \Delta^2\xi\right) dxdt. \tag{50}$$

Applying the Young and Hölder inequalities, hence we obtain the following estimation

$$\sum_{i=1}^2 \int\limits_{|x|^{2\mu}<2d^2} \left(\varphi_i(x) + \psi_i(x)\right) h\left(\frac{|x|^{2\mu}}{d^2}\right) dx +$$

$$+(1-3\varepsilon)\int\limits_0^\infty \int\limits_{R^3} |v|^{7/3}\,\xi(t,x)\,dxdt + (1-3\varepsilon)\int\limits_0^\infty \int\limits_{R^3} |u|^{5/3}\,\xi(t,x)\,dxdt \le$$

$$\leq \frac{1}{\varepsilon}\int_0^\infty\int_{R^n} u\left[|\xi_{tt}|^{p'}\xi^{1-p'}+|\xi_t|^{p'}\xi^{1-p'}+|\Delta\xi|^{p'}\xi^{1-p'}\right]dxdt+$$

$$+\frac{1}{\varepsilon}\int_0^\infty\int_{R^n}\left[|\xi_{tt}|^{q'}+|\xi_t|^{q'}+\left|\Delta^2\xi\right|^{q'}\right]\xi^{1-q'}dxdt.$$

Making substitution $t=d^{\frac{2}{\mu}}\tau, x_i=d^{\frac{1}{\mu}}y_i,\ i=1,2,3$, we obtain that

$$\sum_{i=1}^{2}\int_{|x|^{2\mu}<2d^2}(\varphi_i(x)+\psi_i(x))\,h\left(\frac{|x|^{2\mu}}{d^2}\right)dx+$$

$$+(1-3\varepsilon)\int_0^\infty\int_{R^3}|v|^{7/3}\xi dxdt+(1-3\varepsilon)\int_0^\infty\int_{R^3}|u|^{5/3}\xi dxdt\leq c_0+c_1d^{-\frac{p'}{\mu}}+c_2d^{-\frac{q'}{\mu}}.$$

Hence, taking into account (50)

$$\int_0^\infty\int_{R^3}|u|^{5/3}dxdt<+\infty,\quad \int_0^\infty\int_{R^3}|v|^{7/3}dxdt<+\infty. \tag{51}$$

From (51) we obtain that

$$\sum_{i=1}^{2}\int_{|x|^{2\mu}<2d^2}(\varphi_i(x)+\psi_i(x))\,h\left(\frac{|x|^{\mu}}{d^2}\right)dx+\int_0^\infty\int_{R^3}|u|^{5/3}\xi dxdt+\int_0^\infty\int_{R^3}|v|^{7/3}\xi dxdt\leq$$

$$\leq\left(\iint_{d^2\leq t^\mu+|x|^{2\mu}\leq 2d^2}|u|^{5/3}dxdt\right)^{3/5}\times\left(\iint_{d^2\leq t^\mu+|x|^{2\mu}\leq 2d^2}|\Phi_1(t,x)|^{5/2}dx\right)^{2/5}+$$

$$+\left(\iint_{d^2\leq t^\mu+|x|^{2\mu}\leq 2d^2}|v|^{7/3}dxdt\right)^{3/7}\cdot\left(\iint_{d^2\leq t^\mu+|x|^{2\mu}\leq 2d^2}|\Phi_2(t,x)|^{7/4}dxdt\right)^{4/7}, \tag{52}$$

where $\Phi_1(t,x)=\xi_{tt}-\xi_t-\Delta\xi,\ \Phi_2(t,x)=\xi_{tt}-\xi_t-\Delta^2\xi$. Taking into account (51) we get

$$\lim_{d\to\infty}\iint_{d^2\leq t^\mu+|x|^{2\mu}\leq 2d^2}\left(|u|^{5/3}+|v|^{7/3}\right)dxdt=0. \tag{53}$$

It follows from (52) and (53) that

$$\sum_{i=1}^{2}\int_{R^3}(\varphi_i(x)+\psi_i(x))\,dx+\int_0^\infty\int_{R^n}\left(|u|^{5/3}+|v|^{7/3}\right)dxdt=0$$

i.e.

$$u(t,x)\equiv 0, v(t,x)\equiv 0, (t,x)\in R_+\times R^3.$$

Thus, if condition (49) is satisfied, then the problem (47),(48) has not a nontrivial weak global solution.

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