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THE INITIAL-BOUNDARY VALUE PROBLEM FOR ONE SEMILINEAR HYPERBOLIC EQUATION WITH MEMORY OPERATOR

Abstract

In this work we consider the initial-boundary value problem for one semilinear hyperbolic equation with memory operator. We prove the existence and uniqueness of solutions of this problem.

Let $\Omega \subset R^N\ (N \geq 1)$ be a bounded, connected set with a smooth boundary Γ. We consider the following problem:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial t}\left[u + F(u)\right] - \Delta u + |u|^p u = f \quad \text{in} \quad Q = \Omega \times (0,T), \tag{1}$$

$$u = 0 \quad \text{on} \quad \Gamma \times [0,T], \tag{2}$$

$$\left[u + F(u)\right]_{t=0} = u^{(0)} + w^{(0)}, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = u^{(1)} \text{ in } \Omega, \tag{3}$$

where $p > 0$ and F is a memory operator (at any instant t, $F(u)$ may depend not only on $u(t)$ but also on the previous evolution of u) which acts from $M\left(\Omega; C^0([0,T])\right)$ to $M\left(\Omega; C^0([0,T])\right)$. Here $M\left(\Omega; C^0([0,T])\right)$ is a space of strongly measurable functions $\Omega \to C^0([0,T])$. We assume that the operator F is applied at each point $x \in \Omega$ independently: the output $[F(u)](x,t)$ depends on $u(x,\cdot)|_{[0,t]}$, but not on $u(y,\cdot)|_{[0,t]}$ for any $y \neq x$.

We assume that

$$\begin{cases} \forall \upsilon_1, \upsilon_2 \in M\left(\Omega; C^0([0,T])\right),\ \forall t \in [0,T], \text{if } \upsilon_1 = \upsilon_2 \text{in } [0,t], \text{a.e. in } \Omega, \\ \text{then } [F(\upsilon_1)](\cdot,t) = [F(\upsilon_2)](\cdot,t) \text{ a.e. in } \Omega, \end{cases} \tag{4}$$

$$\begin{cases} \forall \left\{\upsilon_n \in M\left(\Omega; C^0([0,T])\right)\right\}_{n\in N}, \text{ if } \upsilon_n \to \upsilon \text{ uniformly in } [0,T], \\ \text{a.e. in } \Omega, \text{ then } F(\upsilon_n) \to F(\upsilon) \text{ uniformly in } [0,T], \text{ a.e. in } \Omega, \end{cases} \tag{5}$$

$$\begin{cases} \exists L \in R^+,\ \exists g \in L^2(\Omega) : \forall \upsilon \in M\left(\Omega; C^0([0,T])\right), \\ \|[F(\upsilon)](x,\cdot)\|_{C^0([0,T])} \leq L \|\upsilon(x,\cdot)\|_{C^0([0,T])} + g(x) \text{ a.e. in } \Omega, \end{cases} \tag{6}$$

$$\begin{cases} \forall \upsilon \in M\left(\Omega; C^0([0,T])\right),\ \forall [t_1,t_2] \subset [0,T], \\ \text{if } \upsilon(x,\cdot) \text{ is affine in } [t_1,t_2] \text{ a.e. in } \Omega, \text{ then} \\ \left\{[F(\upsilon)](x,t_2) - [F(\upsilon)](x,t_1)\right\} \cdot [\upsilon(x,t_2) - \upsilon(x,t_1)] \geq 0 \text{ a.e. in } \Omega, \end{cases} \tag{7}$$

$$\begin{cases} \exists \overline{L} \in R^+ : \forall \upsilon \in M\left(\Omega; C^0([0,T])\right),\ \forall [t_1,t_2] \subset [0,T], \\ \text{if } \upsilon(x,\cdot) \text{ is affine in } [t_1,t_2] \text{ a.e. in } \Omega, \text{ then} \\ \left|[F(\upsilon)](x,t_2) - [F(\upsilon)](x,t_1)\right| \leq \overline{L}\,|\upsilon(x,t_2) - \upsilon(x,t_1)| \text{ a.e. in } \Omega. \end{cases} \tag{8}$$

Let $V = H_0^1(\Omega) \cap L^{p+2}(\Omega)$ and

$$u^{(0)} \in V, w^{(0)} \in L^2(\Omega), u^{(1)} \in L^2(\Omega), \tag{9}$$

$$f=f_1+f_2,\ f_1\in L^2(Q),\ f_2\in W^{1,1}\left(0,T;V^{/}\right). \tag{10}$$

Definition. *A function $u\in L^2(0,T;V)\cap H^1\left(0,T;L^2(\Omega)\right)$ is said to be a solution of problem (1)-(3) if $F(u)\in L^2(Q)$ and*

$$\int\int_Q\left\{-\frac{\partial u}{\partial t}\cdot\frac{\partial v}{\partial t}-[u+F(u)]\frac{\partial v}{\partial t}+\nabla u\cdot\nabla v+|u|^p uv\right\}dxdt=$$

$$=\int_0^T {}_{V'}\langle f,v\rangle_V\,dt+\int_\Omega\left[u^{(0)}(x)+w^{(0)}(x)+u^{(1)}(x)\right]v(x,0)\,dx \tag{11}$$

for every $v\in L^2(0,T;V)\cap H^1\left(0,T;L^2(\Omega)\right)$ $(v(\cdot,T)=0$ a.e. in $\Omega)$.

The equation (11) yields

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial}{\partial t}[u+F(u)]-\Delta u+|u|^p u=f\quad\text{in}\quad D'\left(0,T;V'\right). \tag{12}$$

Integrating by parts in time in (11), we get

$$[u+F(u)]|_{t=0}=u^{(0)}+w^{(0)}\text{ in }V',\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=u^{(1)}\text{ in }L^2(\Omega). \tag{13}$$

In turn (12) and (13) yield (11).

Well posedness of problem (1)-(3) without F was studied in the works of different authours (see, for example [2]). The corresponding problem for the parabolic equation without nonlinear term $|u|^p u$ was studied in [1]. We have proved existence and uniqueness of solutions of problem (1)-(3).

Theorem 1 (existence). *Assume that (4)-(10) hold. Then problem (1)-(3) has at least one solution such that*

$$u\in W^{1,\infty}\left(0T;L^2(\Omega)\right)\cap L^\infty(0,T;V),\ F(u)\in H^1\left(0,T;L^2(\Omega)\right). \tag{14}$$

Proof. We prove this theorem with method of time discretization.

Let 's fix any $m\in N$, set $k=\frac{T}{m}$ and

$$f_{1m}^n(x)=\frac{1}{k}\int_{(n-1)k}^{nk}f_1(x,\tau)\,d\tau\text{ a. e. in }\Omega,\ f_{2m}^n=f_2(nk),\ f_m^n=f_{1m}^n+f_{2m}^n,\ n=1,...,m,$$

$$u_m^0=u^{(0)},w_m^0=w^{(0)},u_m^1=u^{(0)}+ku^{(1)},u_m^{-1}=u^{(0)}-ku^{(1)},$$

$$u_m^n(x,\cdot)=\text{linear time iterpolate of }u_m(x,nk)=u_m^n(x),$$

$$w_m^n(x)=[F(u_m)](x,nk),n=1,...,m,\text{a.e. in }\Omega.$$

We consider the following problem

$$\frac{u_m^n-2u_m^{n-1}+u_m^{n-2}}{k^2}+\frac{u_m^n-u_m^{n-1}}{k}+\frac{w_m^n-u_m^{n-1}}{k}-$$

$$-\Delta u_m^n+|u_m^n|^p u_m^n=f_m^n\text{ in }V',\quad n=1,...,m, \tag{15}$$

$$u_m^0 = u^{(0)},\ w_m^0 = w^{(0)},\ u_m^1 = u^{(0)} + ku^{(1)},\ u_m^{-1} = u^{(0)} - ku^{(1)}. \tag{16}$$

This problem can be solved step by step in time: for any $n \in \{3, ..., m\}$, assume that $u_m^2, ..., u_m^{n-1} \in V$ are known, and consider the problem of determining u_m^n. For almost any $x \in \Omega$, $u_m(x, \cdot)$ is affine in $[(n-1)k, nk]$; hence $[F(u_m)](x, nk)$ depends only on $u_m(x, \cdot)|_{[0,(n-1)k]}$, which is known, and on $u_m^n(x)$, which must be determined. That is $w_m^n(x) = [F(u_m)](x, nk) = \Psi_m^n(u_m^n(x), x)$ a.e. in Ω.

Let's set

$$U_m^{n-1}(x) = \max_{[0,(n-1)k]} |u_m(x, \cdot)| = \max_{j=0,1,...,n-1} \left|u_m^j(x)\right| \text{ a.e. in } \Omega. \tag{17}$$

Thus $U_m^{n-1} \in L^2(\Omega)$, and (6) yields

$$|\Psi_m^n(\upsilon(x), x)| \le L \max\left\{\left|U_m^{n-1}(x)\right|, |\upsilon(x)|\right\} + g(x) \text{ a.e. in } \Omega, \tag{18}$$

for every $\upsilon \in M(\Omega)$.

We define the operator $\widehat{\Psi}_m^n : M(\Omega) \to M(\Omega)$, $\upsilon \to \Psi_m^n(\upsilon(\cdot), \cdot)$. By (5) and (18)

$$\widehat{\Psi}_m^n : L^2(\Omega) \to L^2(\Omega) \text{ is affinely bounded and strongly continuous.} \tag{19}$$

(7) yields

$$\left(\widehat{\Psi}_m^n(\upsilon) - w_m^{n-1}\right)\left(\upsilon - u_m^{n-1}\right) \ge 0 \text{ a.e. in } \Omega$$

for every $\upsilon \in L^2(\Omega)$; by (18) and the latter inequality there exist $c_1, c_2 \in R^+$ (depending on m, n, but not on υ) such that

$$\int_\Omega \widehat{\Psi}_m^n(\upsilon)\, \upsilon dx \ge -c_1 \|\upsilon\|_{L^2(\Omega)} - c_2 \tag{20}$$

for every $\upsilon \in L^2(\Omega)$.

Omitting the fixed indexes m and n, (15) can be written in the form

$$(1+k)\, u + k\widehat{\Psi}(u) - k^2 \Delta u + k^2 |u|^p u = \varphi \text{ in } V', \tag{21}$$

where $\varphi = k^2 f_m^n + (2+k)\, u_m^{n-1} + kw_m^{n-1} - u_m^{n-2}$. We use a standard procedure to show that this equation has at least one solution. Let $\{V_j\}_{j\in N}$ be a sequence of finite dimensional subspaces invading V; for any $j \in N$ we consider the following finite-dimensional problem:

$$\begin{cases} \text{to find } u_j \in V_j \text{ such that} \\ Z(u_j) = (1+k)\, u_j + k\widehat{\Psi}(u_j) - k^2 \Delta u_j + k^2 |u_j|^p u_j = \varphi \text{ in } V_j'. \end{cases} \tag{22}$$

By (19), Z is strongly continuous as an operator from V to V'; by (20) it is also coercive:

$$\frac{1}{\|\upsilon\|_V}\, {}_{V'}\langle Z(\upsilon), \upsilon\rangle_V \to +\infty \text{ as } \|\upsilon\|_V \to +\infty. \tag{23}$$

Hence problem (22) has at least one solution; this can be easily checked by an argument based on the Brower fixed point theorem (see [2], chap. 1, sect. 4.3). By multiplying (22) by u_j and using (23), we get that the sequence $\{u_j\}$ is uniformly bounded in V. Hence there exists u such that, possibly extracting a subsequence, $u_j \to u$ weakly in V. By the compactness of the inclusion $V \subset L^2(\Omega)$ and by (19), we have

$$\widehat{\Psi}(u_j) \to \widehat{\Psi}(u) \text{ strongly in } L^2(\Omega).$$

Therefore taking $j \to \infty$ in (22), we get (21).

In order to obtain a priori estimates we multiply (15) by $u_m^n - u_m^{n-1}$ and sum for $n = 1, ..., l$, for any $l \in \{1, ..., m\}$:

$$\sum_{n=1}^{l} \int_{\Omega} \left(\frac{u_m^n - u_m^{n-1}}{k} - \frac{u_m^{n-1} - u_m^{n-2}}{k} \right) \frac{u_m^n - u_m^{n-1}}{k} dx +$$

$$+k \sum_{n=1}^{l} \int_{\Omega} \left(\frac{u_m^n - u_m^{n-1}}{k} \right)^2 dx + \frac{1}{k} \sum_{n=1}^{l} \int_{\Omega} \left(w_m^n - w_m^{n-1} \right) \left(u_m^n - u_m^{n-1} \right) dx +$$

$$+ \sum_{n=1}^{l} \int_{\Omega} \nabla u_m^n \left(\nabla u_m^n - \nabla u_m^{n-1} \right) dx +$$

$$+ \sum_{n=1}^{l} \int_{\Omega} |u_m^n|^p u_m^n \left(u_m^n - u_m^{n-1} \right) dx = \sum_{n=1}^{l} {}_{V'} \left\langle f_m^n, u_m^n - u_m^{n-1} \right\rangle_V . \tag{24}$$

By (7) we have

$$\left(w_m^n - w_m^{n-1} \right) \left(u_m^n - u_m^{n-1} \right) \geq 0 \text{ a.e. in } \Omega, n = 1, ..., l; \tag{25}$$

moreover

$$\sum_{n=1}^{l} \int_{\Omega} \left(\frac{u_m^n - u_m^{n-1}}{k} - \frac{u_m^{n-1} - u_m^{n-2}}{k} \right) \frac{u_m^n - u_m^{n-1}}{k} dx = \frac{1}{2} \sum_{n=1}^{l} \int_{\Omega} \left(\frac{u_m^n - u_m^{n-1}}{k} \right)^2 dx +$$

$$+ \frac{1}{2} \sum_{n=1}^{l} \int_{\Omega} \left[\left(\frac{u_m^n - u_m^{n-1}}{k} \right)^2 - 2 \frac{u_m^{n-1} - u_m^{n-2}}{k} \cdot \frac{u_m^n - u_m^{n-1}}{k} \right] dx \geq$$

$$\geq \frac{1}{2} \sum_{n=1}^{l} \int_{\Omega} \left(\frac{u_m^n - u_m^{n-1}}{k} \right)^2 dx - \frac{1}{2} \sum_{n=1}^{l} \int_{\Omega} \left(\frac{u_m^{n-1} - u_m^{n-2}}{k} \right)^2 dx =$$

$$= \frac{1}{2} \int_{\Omega} \left[\left(\frac{u_m^l - u_m^{l-1}}{k} \right)^2 - \left(\frac{u^{(0)} - u_m^{-1}}{k} \right)^2 \right] dx =$$

$$= \frac{1}{2} \int_{\Omega} \left[\left(\frac{u_m^l - u_m^{l-1}}{k} \right)^2 - \left| u^{(1)} \right|^2 \right] dx, \tag{26}$$

$$\sum_{n=1}^{l} \int_{\Omega} \nabla u_m^n \left(\nabla u_m^n - \nabla u_m^{n-1} \right) dx \geq \sum_{n=1}^{l} \int_{\Omega} \left(|\nabla u_m^n|^2 - \frac{1}{2} |\nabla u_m^n|^2 - \frac{1}{2} \left| \nabla u_m^{n-1} \right|^2 \right) dx =$$

$$=\frac{1}{2}\sum_{n=1}^{l}\int_{\Omega}\left(|\nabla u_m^n|^2-\left|\nabla u_m^{n-1}\right|^2\right)dx=\frac{1}{2}\int_{\Omega}\left(\left|\nabla u_m^l\right|^2-\left|\nabla u^{(0)}\right|^2\right)dx, \tag{27}$$

$$\sum_{n=1}^{l}\int_{\Omega}|u_m^n|^{p+2}\,dx-\sum_{n=1}^{l}\int_{\Omega}|u_m^n|^p\,u_m^n u_m^{n-1}dx\geq\sum_{n=1}^{l}\int_{\Omega}|u_m^n|^{p+2}\,dx-$$

$$-\sum_{n=1}^{l}\left(\int_{\Omega}|u_m^n|^{(p+1)\frac{p+2}{p+1}}\,dx\right)^{\frac{p+1}{p+2}}\cdot\left(\int_{\Omega}\left|u_m^{n-1}\right|^{p+2}dx\right)^{\frac{1}{p+2}}=$$

$$=\sum_{n=1}^{l}\int_{\Omega}|u_m^n|^{p+2}-\sum_{n=1}^{l}\left(\int_{\Omega}|u_m^n|^{p+2}\,dx\right)^{\frac{p+1}{p+2}}\cdot\left(\int_{\Omega}\left|u_m^{n-1}\right|^{p+2}dx\right)^{\frac{1}{p+2}}\geq$$

$$\geq\sum_{n=1}^{l}\int_{\Omega}|u_m^n|^{p+2}\,dx-\frac{p+1}{p+2}\sum_{n=1}^{l}\int_{\Omega}|u_m^n|^{p+2}\,dx-\frac{1}{p+2}\sum_{n=1}^{l}\int_{\Omega}\left|u_m^{n-1}\right|^{p+2}dx=$$

$$=\frac{1}{p+2}\int_{\Omega}\sum_{n=1}^{l}\left(|u_m^n|^{p+2}-\left|u_m^{n-1}\right|^{p+2}\right)dx=\frac{1}{p+2}\int_{\Omega}\left(\left|u_m^l\right|^{p+2}-\left|u^{(0)}\right|^{p+2}\right)dx. \tag{28}$$

Using (25)-(28) in (24) and denoting by C_1, C_2 suitable constants independent of m, we get

$$\frac{1}{2}\int_{\Omega}\left[\left(\frac{u_m^l-u_m^{l-1}}{k}\right)^2-\left|u^{(1)}\right|^2\right]dx+k\sum_{n=1}^{l}\int_{\Omega}\left(\frac{u_m^n-u_m^{n-1}}{k}\right)^2dx+$$

$$+\frac{1}{2}\int_{\Omega}\left(\left|\nabla u_m^l\right|^2-\left|\nabla u^{(0)}\right|^2\right)dx+\frac{1}{p+2}\int_{\Omega}\left(\left|\nabla u_m^l\right|^{p+2}-\left|\nabla u^{(0)}\right|^{p+2}\right)dx\leq$$

$$\leq\sum_{n=1}^{l}{}_{V'}\left\langle f_m^n,u_m^n-u_m^{n-1}\right\rangle_V=\sum_{n=1}^{l}\int_{\Omega}f_{1m}^n\left(u_m^n-u_m^{n-1}\right)dx+{}_{V'}\left\langle f_{2m}^l,u_m^l\right\rangle_V-$$

$$-{}_{V'}\left\langle f_2(0),u^{(0)}\right\rangle_V-\sum_{n=2}^{l}{}_{V'}\left\langle f_{2m}^n-f_{2m}^{n-1},u_m^{n-1}\right\rangle_V\leq$$

$$\leq\left(k\sum_{n=1}^{l}\int_{\Omega}\left(f_{1m}^n\right)^2dx\right)^{\frac{1}{2}}\left[k\sum_{n=1}^{l}\int_{\Omega}\left(\frac{u_m^n-u_m^{n-1}}{k}\right)^2dx\right]^{\frac{1}{2}}+$$

$$+\left(\max_{n=1,\dots,l}\|f_{2m}^n\|_{V'}+k\sum_{n=2}^{l}\left\|\frac{f_{2m}^n-f_{2m}^{n-1}}{k}\right\|_{V'}\right)\max_{n=1,\dots,l}\|u_m^n\|_V+$$

$$+\|f_2(0)\|_{V'}\left\|u^{(0)}\right\|_V\leq\frac{1}{2}\|f_1\|_{L^2(Q)}^2+\frac{k}{2}\sum_{n=1}^{l}\int_{\Omega}\left(\frac{u_m^n-u_m^{n-1}}{k}\right)^2dx+$$

$$+C_1 \|f_2\|^2_{W^{1,1}(0,T;V')} + \frac{1}{4}\max_{n=0,...,l} \|u_m^n\|^2_V ; \qquad (29)$$

a simple calculation then yields

$$\left\| \frac{u_m^n - u_m^{n-1}}{k} \right\|_{L^2(\Omega)}, \quad \max_{n=1,...,m} \|u_m^n\|_V \le C_2. \qquad (30)$$

We introduce some further notations. A.e. in Ω, we set
$w_m(x;\cdot)$ =linear time interpolate of $w_m(x,nk) = w_m^n(x)$, for $n = 1,...,m$,
$\widetilde{u}_m(x,t) = u_m^n(x)$ if $(n-1)k < t \le nk$, for $n = 1,...,m$,
and define $\widetilde{w}_m$, $\widetilde{f}_m$ similarly. Thus (15) and (30) yield

$$\frac{\partial^2 u_m}{\partial t^2} + \frac{\partial}{\partial t}(u_m + w_m) - \Delta\widetilde{u}_m + |\widetilde{u}_m|^p \widetilde{u}_m = \widetilde{f}_m \text{ in } V\prime\text{,a.e.in } (0,T), \qquad (31)$$

$$\|u_m\|_{W^{1,\infty}(0,T;L^2(\Omega))\cap L^\infty(0,T;V)}, \quad \|\widetilde{u}_m\|_{L^\infty(0,T;V)} \le const. \qquad (32)$$

As $H^1(0,T;L^2(\Omega)) = L^2(\Omega;H^1(0,T)) \subset L^2(\Omega;C^0([0,T]))$ with continuous injection, by (6) and (32) we have

$$\|w_m\|_{L^2(\Omega;C^0([0,T]))} \le L\|u_m\|_{L^2(\Omega;C^0([0,T]))} + \|g\|_{L^2(\Omega)} \le const. \qquad (33)$$

Moreover (see [2], chap. 1, sect. 1.3)

$$|\widetilde{u}_m|^p \widetilde{u}_m \in L^\infty\left(0,T;L^{p'}(\Omega)\right). \qquad (34)$$

(31)-(34) yield

$$\|u_{m_{tt}}\|_{L^2(Q)} \le const. \qquad (35)$$

By these estimates and applying Proposition XII. 2.1 with $D = L^1(0,T)$ from [1], we conclude that there exist u, w such that, possible taking $m \to \infty$ along a subsequence,

$$u_m \to u \qquad \text{weakly star in } H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V), \qquad (36)$$

$$\widetilde{u}_m \to u \qquad \text{weakly star in } L^\infty(0,T;V), \qquad (36)$$

$$w_m \to w \qquad \text{weakly star in } L^2_{w^*}(\Omega;L^\infty(0,T)) = \left(L^2(\Omega;L^1(0,T))\right)', \qquad (38)$$

$$u_m + w_m \to u + w \qquad \text{weakly star in } L^2_{w^*}(\Omega;L^\infty(0,T)) \cap H^1\left(0,T;V'\right), \qquad (39)$$

$$u_{m_{tt}} \to u_{tt} \qquad \text{weakly star in } L^2(Q). \qquad (40)$$

So, by taking $m \to \infty$ in (31), we get (12) in the sense of $L^2(0,T;V')$; (13) is also easily obtained. As we saw, this yields (11).

By interpolation (see [3], chap. 4, p. 378) we have

$$H^1\left(0,T;L^2(\Omega)\right) \cap L^\infty(0,T;V) \subset H^1(Q) \subset H^\varepsilon\left(\Omega;H^{1-\varepsilon}(0,T)\right) \subset L^2\left(\Omega;C^0([0,T])\right)$$

for $\forall\varepsilon \in \left(0,\frac{1}{2}\right)$ with continuous injections, and the latter one is also compact. Hence, possibly extracting a further subsequence, we have

$$u_m \to u \text{ iniformly in } (0,T), \text{ a. e. in } \Omega.$$

Then by (5) $F(u) \in M\left(\Omega; C^0([0,T])\right)$ and

$F(u_m) \to F(u)$ uniformly in $[0,T]$, a.e. in Ω.

As $w_m(x,\cdot)$ is the linear time interpolate of $w_m(x,nk) = [F(u_m)](x,nk)$ $(n = 0,...,m)$ for a.e. $x \in \Omega$, we have

$$w_m \to F(u) \quad \text{uniformly in } [0,T], \text{ a.e. in } \Omega.$$

Therefore by (38) we get $w = F(u)$ a.e. in Q. By (6) w_m converges strongly in $L^2\left(\Omega; C^0([0,T])\right)$.

As the family of continuous, piecewise linear functions is dense in $W^{1,1}(0,T)$, (8) entails that for every $v \in M\left(\Omega; W^{1,1}(0,T)\right)$

$$F(v) \in M\left(\Omega; W^{1,1}(0,T)\right) \text{ and } \left|\frac{d}{dt}F(v)\right| \le \overline{L}\left|\frac{dv}{dt}\right| \text{a.e. in } Q.$$

Hence as $u \in H^1\left(0,T; L^2(\Omega)\right) = L^2\left(\Omega; H^1(0,T)\right)$, then $F(u) \in H^1\left(0,T; L^2(\Omega)\right)$.

The theorem is proved.

Theorem 2 (uniqueness). *Assume that the hypotheses of theorem 1 hold ,*

$$p \le \frac{2}{N-2} \; (p \text{ is arbitrary and finite when } N = 2) \tag{42}$$

and for every $u,\ v \in M\left(\Omega; W^{1,1}(0,T)\right)$

$$\frac{\partial}{\partial t}[F(u) - F(v)] \le L_1 \frac{\partial}{\partial t}(u - v). \tag{43}$$

Then problem (1)-(3) has one and only one solution.

Proof. We will prove this theorem for case $N \ge 3$ (in the case when $N = 2$ the proof is obtained in the same way).

Let u_1 and u_2 be two solutions of problem (1)-(3). Then for $\theta = u_1 - u_2$ we get

$$\theta_{tt} + \theta_t + [F(u_1) - F(u_2)]_t - \Delta\theta + |u_1|^p u_1 - |u_2|^p u_2 = 0, \tag{44}$$

$$\theta|_\Gamma = 0, \tag{45}$$

$$\theta|_{t=0} = 0, \quad \theta_t|_{t=0} = 0 \tag{46}$$

$$\theta \in L^\infty(0,T;V), \tag{47}$$

$$\theta_t \in L^\infty\left(0,T; L^2(\Omega)\right). \tag{48}$$

In order to prove that $\theta = 0$, we will use a standard procedure applied in the theory of linear hyperbolic equations (see [2], p.28).

Let $s \in (0,T)$,

$$\psi(t) = \begin{cases} -\int_t^s \theta(\sigma)\,d\sigma, & t \le s, \\ 0, & t > s, \end{cases}$$

$$\theta_1(t) = \int_0^t \theta(\sigma)\,d\sigma.$$

Hence we get that $\psi(t)=\theta_1(t)-\theta_1(s)$ for $t\le s$. By multiplying (44) by $\psi(t)$:

$$-\int_0^s(\theta_t,\psi_t)\,dt+\int_0^s(\theta_t,\psi)\,dt+\int_0^s(\theta_x,\psi_x)\,dt+\int_0^s([F(u_1)-F(u_2)]_t,\psi)\,dt=$$

$$=\int_0^s(|u_1|^p u_1-|u_2|^p u_2,\psi)\,dt$$

and taking into account $\psi_t=\theta,\psi(0)=-\theta_1(s)$, we have

$$-\frac{1}{2}\|\theta(s)\|^2_{L^2(\Omega)}-\int_0^s\|\theta\|^2_{L^2(\Omega)}dt-\frac{1}{2}\|\theta_{1x}(s)\|^2_{L^2(\Omega)}+$$

$$+\int_0^s([F(u_1)-F(u_2)]_t,\psi)\,dt=\int_0^s(|u_1|^p u_1-|u_2|^p u_2,\psi)\,dt. \tag{49}$$

By (42) we can proof that

$$\left|\int_0^s(|u_1|^p u_1-|u_2|^p u_2,\psi)\,dt\right|\le\frac{1}{4}\|\theta_{1x}(s)\|^2_{L^2(\Omega)}+c\int_0^s\left(\|\theta\|^2_{L^2(\Omega)}+\|\theta_{1x}\|^2_{L^2(\Omega)}\right)dt.$$

Using this inequality and (43) in (49), we have

$$\frac{1}{2}\|\theta(s)\|^2_{L^2(\Omega)}+\frac{1}{2}\|\theta_{1x}(s)\|^2_{L^2(\Omega)}+(1+L_1)\int_0^s\|\theta\|^2_{L^2(\Omega)}dt\le$$

$$\le\frac{1}{4}\|\theta_{1x}(s)\|^2_{L^2(\Omega)}+c\int_0^s\left(\|\theta\|^2_{L^2(\Omega)}+\|\theta_{1x}\|^2_{L^2(\Omega)}\right)dt.$$

or

$$\|\theta(s)\|^2_{L^2(\Omega)}+\|\theta_{1x}(s)\|^2_{L^2(\Omega)}\le\bar{c}\int_0^s\left(\|\theta\|^2_{L^2(\Omega)}+\|\theta_{1x}\|^2_{L^2(\Omega)}\right)dt;$$

hence we get that $\theta=0$.

The theorem is proved.

References

[1]. Visintin A. *Differential Models of Hysteresis.* Springer, 1993.

[2]. Lions I. L. *Some solutions methods to nonlinear boundary value problems.* M.: Mir, 1972 (Russian).

[3]. Lions I. L. Magenes E. *Inhomogeneous boundary value problems and their applications.* M.: Mir, 1971 (Russian).

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