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A PROBLEM FOR A COMPOSITE TYPE EQUATION OF THIRD ORDER WITH GENERAL LINEAR BOUNDARY CONDITIONS

Abstract

The considered problem is devoted to the solution of a boundary value problem for a third order model equation. The boundary conditions are not local and are obtained by means of necessary conditions. In the domain, the equation of the considered problem has both real and complex characteristics.

The form of non-local boundary conditions for which the stated boundary value problem is Fredholm is determined.

1. Introduction

Composite type equations of third order are considered.(see[2,3]). Unlike the classic papers [1]-[4], the boundary conditions are not local. Our goal is to define the kind of linear non-local boundary conditions under which the stated problem is Fredholm. The problem is investigated proceeding from the refined kind of fundamental solution [4] of the considered equation. Necessary conditions similar to ones in [9], [10] are obtained. After regularizing and combining them with the boundary conditions, sufficient condition of Fredholm property of the given problem is found. Similar problem for the Schrodinger equation and fist order parabolic type equation was considered in [7]-[10].

Proceeding from the special kind of the considered equation, the boundary conditions are chosen so that for the boundary values of the second derivative we get normal kind relations. Further, as for boundary values for minor derivatives we get second order Ferdholm integral equation for these unknowns proceeding from necessary conditions.

2. Problem Statement

Let D be a bounded, convex in the direction x_2, plane domain with sufficiently smooth boundary Γ (Lyapunov line [4]).

Considered the following problem:

$$\frac{\partial^3 u(x)}{\partial x_2^3} + \frac{\partial^3 u(x)}{\partial x_1^2 \partial x_2} = 0, \qquad x \in D \subset R^2, \tag{2.1}$$

$$\begin{cases} \left.\dfrac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_k(x_1)} = \sum\limits_{p=1}^{2}\sum\limits_{j=1}^{2} \alpha_{kjp}(x_1) \left.\dfrac{\partial u(x)}{\partial x_j}\right|_{x_2=\gamma_p(x_1)} + \\ + \sum\limits_{p=1}^{2} \alpha_{kp}(x_1) u\left(x_1 \gamma_p(x_1)\right) + \sum\limits_{p=1}^{2}\sum\limits_{j=1}^{2}\int\limits_{a_1}^{b_1} \alpha_{kjp}(x_1,\eta_1) \left.\dfrac{\partial u(\eta)}{\partial \eta_j}\right|_{\eta_2=\gamma_p(\eta_1)} d\eta_1 + \\ + \sum\limits_{p=1}^{2}\int\limits_{a_1}^{b_1} \alpha_{kp}(x_1,\eta_1) u\left(\eta_1,\gamma_p(\eta_1)\right) d\eta_1 + f_k(x_1),\ k=1,2;\ x_1\in[a_1,b_1], \quad (2.2) \\ \left.\dfrac{\partial^2 u(x)}{\partial x_1^2}\right|_{x_2=\gamma_2(x_1)} = \sum\limits_{p=1}^{2}\sum\limits_{j=1}^{2} \alpha_{3jp}(x_1) \left.\dfrac{\partial u(x)}{\partial x_j}\right|_{x_2=\gamma_p(x_1)} + \\ + \sum\limits_{p=1}^{2} \alpha_{3p}(x_1) u\left(x_1 \gamma_p(x_1)\right) + \sum\limits_{p=1}^{2}\sum\limits_{j=1}^{2}\int\limits_{a_1}^{b_1} \alpha_{3jp}(x_1,\eta_1) \left.\dfrac{\partial u(\eta)}{\partial \eta_j}\right|_{\eta_2=\gamma_p(\eta_1)} d\eta_1 + \\ + \sum\limits_{p=1}^{2}\int\limits_{a_1}^{b_1} \alpha_{3p}(x_1,\eta_1) u\left(\eta_1,\gamma_p(\eta_1)\right) d\eta_1 + f_3(x_1), \quad x_1\in[a_1,b_1], \end{cases}$$

where all the data of boundary condition (2.2) are real-valued continuous functions, $\gamma_k(x_1)$ $(k=1,2)$ are the equations of open lines Γ_k $(\Gamma_1\cup\Gamma_2=\Gamma)$ obtained from the boundary Γ of the domain D by means of orthogonal projection of this domain onto the axis x_1, and $[a_1,b_1]=np_{x_1}\Gamma_1=np_{x_1}\Gamma_2$, moreover $\gamma_1(x_1)<\gamma_2(x_1)$, $x_1\in(a_1,b_1)$.

3. Fundamental solution and its basic properties

Proceeding from the Fourier transformations [1], [4] for the equation (2.1) we get the fundamental solution in the form

$$U(x-\xi)=\frac{1}{4\pi^2}\int\limits_{R^2}\frac{e^{i(\alpha,x-\xi)}}{\alpha_2\left(\alpha_1^2+\alpha_2^2\right)}d\alpha, \tag{3.1}$$

where $(\alpha,x-\xi)=\sum\limits_{j=1}^{2}\alpha_j\left(x_j-\xi_j\right)$ is a scalar product. Considering that the integrals (3.1) are hyper-singular, we regularize it by means of the Hormander ladder method[11], make some corrections and get:

$$U(x-\xi)=\frac{x_2-\xi_2}{2\pi}\left[\ln\sqrt{|x_1-\xi_1|^2+(x_2-\xi_2)^2}-1\right]+ \\ +\frac{|x_1-\xi_1|}{2\pi}arctg\frac{x_2-\xi_2}{|x_1-\xi_1|}. \tag{3.2}$$

As is seen from (3.1) and (3.2)

$$\frac{\partial U}{\partial x_1}=\frac{e\left(x_1-\xi_1\right)}{\pi}arctg\frac{x_2-\xi_2}{|x_1-\xi_1|}, \tag{3.3}$$

$$\frac{\partial U}{\partial x_2}=\frac{1}{2\pi}\ln\sqrt{|x_1-\xi_1|^2+(x_2-\xi_2)^2}, \tag{3.4}$$

$$\frac{\partial^2 U}{\partial x_1^2}=e\left(x_2-\xi_2\right)\delta\left(x_1-\xi_1\right)-\frac{1}{2\pi}\frac{x_2-\xi_2}{|x_1-\xi_1|^2+(x_2-\xi_2)^2}, \tag{3.5}$$

$$\Delta_x U(x-\xi) = e\left(x_2-\xi_2\right)\delta\left(x_1-\xi_1\right) \tag{3.6}$$

and finally

$$\frac{\partial^3 U}{\partial x_2^3} + \frac{\partial^3 U}{\partial x_1^2 \partial x_2} = \frac{\partial}{\partial x_2}\Delta_x U(x-\xi) = \delta(x-\xi), \tag{3.7}$$

where $\delta(x-\xi) = \delta\left(x_1-\xi_1\right)\delta\left(x_2-\xi_2\right)$ is Diracs $\delta-$function, $e\left(x_1-\xi_1\right)$ is Heavisede's symmetric function [1], [4].

4. Necessary conditions

Multiplying the equation (2.1) by the fundamental solution of the equation $U(x-\xi)$ and integrating with respect to the domain D, applying the Ostrogradskiy-Gauss formula [1], [4]and properties of Dirac's $\delta-$function [4] we have:

$$\int\limits_{\Gamma} U(x-\xi)\Delta u(x)\cos(\nu,x_2)dx - \int\limits_{\Gamma}\frac{\partial U}{\partial x_2}\cdot\frac{\partial u}{\partial \nu}dx +$$

$$+\int\limits_{\Gamma} u(x)\frac{\partial}{\partial \nu}\left(\frac{\partial U}{\partial x_2}\right)dx = \begin{cases} u(\xi), & \xi \in D, \\ \frac{1}{2}u(\xi), & \xi \in \Gamma, \\ 0, & \xi \notin \overline{D} \end{cases} \tag{4.1}$$

where ν is an external normal to the boundary Γ of domain D.

In the same way, similar to [5]-[10] we have:

$$\int\limits_{\Gamma}\frac{\partial u}{\partial x_2}\cdot\frac{\partial}{\partial \nu_x}\cdot\frac{\partial U}{\partial x_2}dx - \int\limits_{\Gamma}\frac{\partial U}{\partial x_2}\cdot\frac{\partial}{\partial \nu}\cdot\frac{\partial u}{\partial x_2}dx = \begin{cases} \frac{\partial u(\xi)}{\partial \xi_2}, & \xi \in D, \\ \frac{1}{2}\frac{\partial u(\xi)}{\partial \xi_2}, & \xi \in \Gamma, \\ 0, & \xi \notin \overline{D} \end{cases} \tag{4.2}$$

$$\int\limits_{\Gamma}\frac{\partial u(x)}{\partial x_1}\cdot\frac{\partial}{\partial \nu_x}\cdot\frac{\partial U}{\partial x_2}dx - \int\limits_{\Gamma}\frac{\partial U}{\partial x_1}\Delta u(x)\cos(\nu,x_2)dx +$$

$$+\int\limits_{\Gamma}\frac{\partial U}{\partial x_2}\left[\frac{\partial^2 u}{\partial x_2^2}\cos(\nu,x_1) - \frac{\partial^2 u}{\partial x_1 \partial x_2}\cos(\nu,x_2)\right]dx =$$

$$= \begin{cases} \frac{\partial u(\xi)}{\partial \xi_1}, & \xi \in D, \\ \frac{1}{2}\frac{\partial u(\xi)}{\partial \xi_1}, & \xi \in \Gamma, \quad 0, \quad \xi \notin \overline{D} \end{cases} \tag{4.3}$$

Finally, for the second derivatives we get:

$$\int_{\Gamma} \frac{\partial^2 U}{\partial x_1^2} \Delta u(x) \cos(\nu, x_2) dx - \int_{\Gamma} \left[\frac{\partial^2 U}{\partial x_1 \partial x_2} \cdot \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_2^2} \cdot \frac{\partial^2 u}{\partial x_1 \partial x_2} \right] \cos(\nu, x_1) dx +$$

$$+ \int_{\Gamma} \left[\frac{\partial^2 U}{\partial x_1 \partial x_2} \cdot \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 U}{\partial x_2^2} \cdot \frac{\partial^2 u}{\partial x_1^2} \right] \cos(\nu, x_2) dx =$$

$$= \begin{cases} \dfrac{\partial^2 u(\xi)}{\partial \xi_1^2}, & \xi \in D, \\ \dfrac{1}{2} \dfrac{\partial^2 u(\xi)}{\partial \xi_1^2}, & \xi \in \Gamma, \quad 0, \qquad \xi \notin \overline{D} \end{cases} \tag{4.4}$$

$$\int_{\Gamma} \frac{\partial^2 U}{\partial x_2^2} \cdot \frac{\partial}{\partial \nu} \left(\frac{\partial u}{\partial x_2} \right) dx + \int_{\Gamma} \left[\frac{\partial^2 u}{\partial x_2^2} \cos(\nu, x_2) - \frac{\partial^2 u}{\partial x_1 \partial x_2} \cos(\nu, x_2) \right] \frac{\partial^2 U}{\partial x_1 \partial x_2} dx =$$

$$= \begin{cases} \dfrac{\partial^2 u(\xi)}{\partial \xi_2^2}, & \xi \in D, \\ \dfrac{1}{2} \dfrac{\partial^2 u(\xi)}{\partial \xi_2^2}, & \xi \in \Gamma, \\ 0, & \xi \notin \overline{D} \end{cases} \tag{4.5}$$

$$\int_{\Gamma} \frac{\partial^2 U}{\partial x_1 \partial x_2} \cdot \frac{\partial}{\partial \nu_x} \cdot \frac{\partial u}{\partial x_2} dx + \int_{\Gamma} \left[\frac{\partial^2 u}{\partial x_1 \partial x_2} \cos(\nu, x_2) - \frac{\partial^2 u}{\partial x_2^2} \cos(\nu, x_2) \right] \frac{\partial^2 U}{\partial x_2^2} dx =$$

$$= \begin{cases} \dfrac{\partial^2 u}{\partial \xi_1 \partial \xi_2}, & \xi \in D, \\ \dfrac{1}{2} \dfrac{\partial^2 u}{\partial \xi_1 \partial \xi_2}, & \xi \in \Gamma, \\ 0, & \xi \notin \overline{D} \end{cases} . \tag{4.6}$$

This establishes the following statement:

Theorem 1. *Let a plane bounded domain D be convex in x_2, the boundary be Lypunov's Γ-line, then each solution of equation (2.1) determined in the D satisfies the basic relations (4.1)-(4.6).*

Now, considering that for $x \in \Gamma_1$

$$\begin{cases} dx_1 = dx \cos(x_1, \tau_1), \\ \cos(\nu_1, x_1) = \sin(x_1, \tau_1), \\ \cos(\nu_1, x_2) = -\cos(x_1, \tau_1) \end{cases}$$

and for $x \in \Gamma_2$

$$\begin{cases} dx_1 = dx\cos(x_1, \tau_2), \\ \cos(\nu_2, x_1) = -\sin(x_1, \tau_2), \\ \cos(\nu_2, x_2) = \cos(x_1, \tau_2). \end{cases}$$

from (4.1)-(4.6) we get:

$$u\left(\xi_1, \gamma_k\left(\xi_1\right)\right) = ..., \quad k = 1,2;\ \xi_1 \in (a_1, b_1) \tag{4.7}$$

$$\left.\frac{\partial u(\xi)}{\partial \xi_2}\right|_{\xi_2=\gamma_k(\xi_1)} = ..., \quad k = 1,2; \quad \xi_1 \in (a_1, b_1) \tag{4.8}$$

$$\left.\frac{\partial u(\xi)}{\partial \xi_1}\right|_{\xi_2=\gamma_k(\xi_1)} = ..., \quad k = 1,2; \quad \xi_1 \in (a_1, b_1) \tag{4.9}$$

where the dots denote the sums of non-singular addends. Taking into account that necessary conditions obtained for a second order derivative contain singular integrals, here we give their expressions in detail.

$$\frac{1}{2}\left.\frac{\partial^2 u(\xi)}{\partial \xi_1^2}\right|_{\xi_2=\gamma_k(\xi_1)} = -\int_{a_1}^{b_1} \left.\frac{\partial^2 U}{\partial x_1^2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \left.\Delta u(x)\right|_{x_2=\gamma_1(x_1)} dx_1 +$$

$$+\int_{a_1}^{b_1} \left.\frac{\partial^2 U}{\partial x_1^2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \left.\Delta u(x)\right|_{x_2=\gamma_2(x_1)} dx_1 -$$

$$-\int_{a_1}^{b_1} \left[\left.\frac{\partial^2 U}{\partial x_1 \partial x_2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \left.\frac{\partial^2 u}{\partial x_2^2}\right|_{x_2=\gamma_1(x_1)} + \right.$$

$$\left. + \left.\frac{\partial^2 U}{\partial x_2^2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \left.\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_1(x_1)} \right] \gamma_1'(x_1) dx_1 +$$

$$+\int_{a_1}^{b_1} \left[\left.\frac{\partial^2 U}{\partial x_1 \partial x_2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \left.\frac{\partial^2 u}{\partial x_2^2}\right|_{x_2=\gamma_2(x_1)} + \right.$$

$$\left. + \left.\frac{\partial^2 U}{\partial x_2^2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \left.\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_2(x_1)} \right] \gamma_2'(x_1) dx_1 -$$

$$-\int_{a_1}^{b_1} \left[\left.\frac{\partial^2 U}{\partial x_1 \partial x_2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \left.\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|_{x_2=\gamma_1(x_1)} + \right.$$

$$\left. + \left.\frac{\partial^2 U}{\partial x_2^2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}} \left.\frac{\partial^2 u}{\partial x_1^2}\right|_{x_2=\gamma_1(x_1)} \right] dx_1 +$$

$$+\int_{a_1}^{b_1}\left[\left.\frac{\partial^2 U}{\partial x_1\partial x_2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}}\left.\frac{\partial^2 u}{\partial x_1\partial x_2}\right|_{x_2=\gamma_2(x_1)}+\right.$$

$$\left.+\left.\frac{\partial^2 U}{\partial x_2^2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}}\left.\frac{\partial^2 u}{\partial x_1^2}\right|_{x_2=\gamma_2(x_1)}\right]dx_1,\ k=1,2;\ \xi_1\in(a_1,b_1) \tag{4.10}$$

Similar to the one mentioned above we also have:

$$\frac{1}{2}\left.\frac{\partial^2 u}{\partial \xi_2^2}\right|_{\xi_2=\gamma_k(\xi_1)}=\int_{a_1}^{b_1}\left.\frac{\partial^2 U}{\partial x_2^2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}}\frac{\partial}{\partial\nu_1}\cdot\left.\frac{\partial u}{\partial x_2}\right|_{x_2=\gamma_1(x_1)}\frac{dx_1}{\cos(x_1,\tau_1)}+$$

$$+\int_{a_1}^{b_1}\left.\frac{\partial^2 U}{\partial x_2^2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}}\frac{\partial}{\partial\nu_2}\cdot\left.\frac{\partial u}{\partial x_2}\right|_{x_2=\gamma_2(x_1)}\frac{dx_1}{\cos(x_1,\tau_2)}+$$

$$+\int_{a_1}^{b_1}\left[\left.\frac{\partial^2 U}{\partial x_2^2}\right|_{x_2=\gamma_1(x_1)}\gamma_1'(x_1)+\left.\frac{\partial^2 u}{\partial x_1\partial x_2}\right|_{x_2=\gamma_1(x_1)}\right]\left.\frac{\partial^2 U}{\partial x_1\partial x_2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}}dx_1-$$

$$-\int_{a_1}^{b_1}\left[\left.\frac{\partial^2 u}{\partial x_2^2}\right|_{x_2=\gamma_2(x_1)}\gamma_2'(x_1)+\left.\frac{\partial^2 u}{\partial x_1\partial x_2}\right|_{x_2=\gamma_2(x_1)}\right]\left.\frac{\partial^2 U}{\partial x_1\partial x_2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}}dx_1,$$

$$k=1,2;\ \xi_1\in(a_1,b_1) \tag{4.11}$$

Finally, for the mixed derivative we have:

$$\frac{1}{2}\left.\frac{\partial^2 u}{\partial \xi_1\partial\xi_2}\right|_{\xi_2=\gamma_k(\xi_1)}=\int_{a_1}^{b_1}\left.\frac{\partial^2 U}{\partial x_1\partial x_2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}}\frac{\partial}{\partial\nu_1}\cdot\left.\frac{\partial u}{\partial x_2}\right|_{x_2=\gamma_1(x_1)}\frac{dx_1}{\cos(x_1,\tau_1)}+$$

$$+\int_{a_1}^{b_1}\left.\frac{\partial^2 U}{\partial x_1\partial x_2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}}\frac{\partial}{\partial\nu_2}\cdot\left.\frac{\partial u}{\partial x_2}\right|_{x_2=\gamma_2(x_1)}\frac{dx_1}{\cos(x_1,\tau_2)}-$$

$$-\int_{a_1}^{b_1}\left[\left.\frac{\partial^2 U}{\partial x_1\partial x_2}\right|_{x_2=\gamma_1(x_1)}+\left.\frac{\partial^2 u}{\partial x_2^2}\right|_{x_2=\gamma_1(x_1)}\gamma_1'(x_1)\right]\left.\frac{\partial^2 U}{\partial x_2^2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}}dx_1+$$

$$+\int_{a_1}^{b_1}\left[\left.\frac{\partial^2 U}{\partial x_1\partial x_2}\right|_{x_2=\gamma_2(x_1)}+\left.\frac{\partial^2 u}{\partial x_2^2}\right|_{x_2=\gamma_2(x_1)}\gamma_2'(x_1)\right]\left.\frac{\partial^2 U}{\partial x_2^2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}}dx_1,$$

$$k=1,2;\ \xi_1\in(a_1,b_1)\,. \tag{4.12}$$

Thus, we establish the following statement:

Theorem 2. *Let D be a bounded palne domain convex in the direction x_2 with Lyapunov's $\Gamma-$line. The boundary values of each solution of equation (2.1) determined in domain D satisfy the necessary conditions (4.7)-(4.12), where $\gamma_k(x_1)$ $k = 1,2$; $x_1 \in (a_1, b_1)$ are the equations of the part Γ_k $(k = 1,2)$ of the boundary Γ of domain D that are obtained under orthogonal projection of this domain onto the axis x_1,*

$$\gamma_1(x_1) < \gamma_2(x_1), \quad x_1 \in (a_1, b_1).$$

Remark 1. Integrating equation (2.1) with respected to the variable x_2 from $\gamma_k(x_1)$ $(k = 1,2)$ to x_2, we have the following necessary conditions are easily obtained also from (4.10), (4.11):

$$\Delta u(x) = \left.\Delta u(x)\right|_{x_2=\gamma_1(x_1)} = \left.\Delta u(x)\right|_{x_2=\gamma_2(x_1)} \quad x_1 \in [a_1, b_1]. \tag{4.13}$$

Remark 2. The obtained conditions (4.7)-(4.9) don't contain singular integrals, i.e. these conditions are regular.

Remark 3. Necessary conditions (4.10)-(4.12) are not regular, i.e. they contain singular integrals.

5. Regularization of singularities in necessary conditions

On the base of (3.2)-(3.7) we get the formulas (5.1)-(5.4) and considering them we will not need to differentiate these expressions. Substitute in them $(x_1 - \xi_1)^2$ for $|x_1 - \xi_1|^2$:

$$\left.\frac{\partial^2 U}{\partial x_2^2}\right|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_p(\xi_1)}} = \frac{1}{2\pi} \cdot \frac{x_2 - \xi_2}{(x_1 - \xi_1)^2 - (x_2 - \xi_2)^2} =$$

$$= \frac{1}{2\pi} \cdot \frac{\gamma_p'(\tau_p(x_1, \xi_1))}{(x_1 - \xi_1)\left[1 + \gamma_p'^2(\tau_p)\right]}, \tag{5.1}$$

$$\left.\frac{\partial^2 U}{\partial x_1 \partial x_2}\right|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_p(\xi_1)}} = \frac{e(x_1 - \xi_1)}{\pi} \cdot \left.\frac{|x_1 - \xi_1|}{|x_1 - \xi_1|^2 + (x_2 - \xi_2)^2}\right|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_p(\xi_1)}} =$$

$$= \frac{1}{2\pi} \cdot \frac{1}{(x_1 - \xi_1)\left[1 + \gamma_p'^2(\tau_p)\right]}, \tag{5.2}$$

$$\left.\frac{\partial^2 U}{\partial x_1^2}\right|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_p(\xi_1)}} = \delta(x_1 - \xi_1)\, e\left(\gamma_p(x_1) - \gamma_p(\xi_1)\right) - \left.\frac{\partial^2 U}{\partial x_2^2}\right|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_p(\xi_1)}} =$$

$$= -\frac{1}{2\pi} \cdot \frac{\gamma_p'(\tau_p)}{(x_1 - \xi_1)\left[1 + \gamma_p'^2(\tau_p)\right]}, \tag{5.3}$$

$$\left.\frac{\partial^2 U}{\partial x_1^2}\right|_{\substack{x_2=\gamma_p(x_1)\\ \xi_2=\gamma_q(\xi_1)}} = \delta(x_1 - \xi_1)\, e\left(\gamma_p(x_1) - \gamma_q(\xi_1)\right) -$$

$$-\frac{1}{2\pi}\cdot\frac{\gamma_p(x_1)-\gamma_q(\xi_1)}{(x_1-\xi_1)^2+\left[\gamma_p(x_1)-\gamma_q(\xi_1)\right]^2}. \tag{5.4}$$

Proceeding from these expressions, necessary condition (4.10) for $k=1$ takes the form:

$$\left.\frac{\partial^2 u(\xi)}{\partial\xi_1^2}\right|_{\xi_2=\gamma_1(\xi_1)}-\left.\frac{\partial^2 u(\xi)}{\partial\xi_1^2}\right|_{\xi_2=\gamma_2(\xi_1)}-\left.\frac{\partial^2 u(\xi)}{\partial\xi_2^2}\right|_{\xi_2=\gamma_2(\xi_1)}=$$

$$=-\frac{1}{\pi}\int_{a_1}^{b_1}\left.\frac{\partial^2 u(x)}{\partial x_1\partial x_2}\right|_{x_2=\gamma_1(x_1)}\frac{dx_1}{x_1-\xi_1}+... \tag{5.5}$$

Exactly in the same way, from (4.10) for $k=2$ we have:

$$\left.\frac{\partial^2 u(\xi)}{\partial\xi_1^2}\right|_{\xi_2=\gamma_2(\xi_1)}-\left.\frac{\partial^2 u(\xi)}{\partial\xi_1^2}\right|_{\xi_2=\gamma_1(\xi_1)}-\left.\frac{\partial^2 u(\xi)}{\partial\xi_2^2}\right|_{\xi_2=\gamma_1(\xi_1)}=$$

$$=\frac{1}{\pi}\int_{a_1}^{b_1}\left.\frac{\partial^2 u(x)}{\partial x_1\partial x_2}\right|_{x_2=\gamma_2(x_1)}\frac{dx_1}{x_1-\xi_1}+... \tag{5.6}$$

From (4.11) for $k=1$ we have:

$$\left.\frac{\partial^2 u(\xi)}{\partial\xi_2^2}\right|_{\xi_2=\gamma_1(\xi_1)}=\frac{1}{\pi}\int_{a_1}^{b_1}\left.\frac{\partial^2 u(x)}{\partial x_1\partial x_2}\right|_{x_2=\gamma_1(x_1)}\frac{dx_1}{x_1-\xi_1}+... \tag{5.7}$$

also for $k=2$ from (4.11) we have:

$$\left.\frac{\partial^2 u(\xi)}{\partial\xi_2^2}\right|_{\xi_2=\gamma_2(\xi_1)}=-\frac{1}{\pi}\int_{a_1}^{b_1}\left.\frac{\partial^2 u(x)}{\partial x_1\partial x_2}\right|_{x_2=\gamma_2(x_1)}\frac{dx_1}{x_1-\xi_1}+... \tag{5.8}$$

In the same way, from (4.12) we have:

$$\left.\frac{\partial^2 u(\xi)}{\partial\xi_1\partial\xi_2}\right|_{\xi_2=\gamma_1(\xi_1)}=-\frac{1}{\pi}\int_{a_1}^{b_1}\left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_1(x_1)}\frac{dx_1}{x_1-\xi_1}+... \tag{5.9}$$

$$\left.\frac{\partial^2 u(\xi)}{\partial\xi_1\partial\xi_2}\right|_{\xi_2=\gamma_2(\xi_1)}=\frac{1}{\pi}\int_{a_1}^{b_1}\left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_2(x_1)}\frac{dx_1}{x_1-\xi_1}+... \tag{5.10}$$

Taking into account the first two boundary conditions from (2.2) in the expression (5.9) and (5.10) we have:

$$\left.\frac{\partial^2 u(\xi)}{\partial\xi_1\partial\xi_2}\right|_{\xi_2=\gamma_k(\xi_1)}=\frac{(-1)^k}{\pi}\int_{a_1}^{b_1}\left\{\sum_{p=1}^{2}\sum_{j=1}^{2}\alpha_{kjp}(x_1)\left.\frac{\partial u(x)}{\partial x_j}\right|_{x_2=\gamma_p(x_1)}+\right.$$

$$+\sum_{p=1}^{2}\alpha_{kp}(x_1)u\left(x_1,\gamma_p(x_1)\right)+\sum_{p=1}^{2}\sum_{j=1}^{2}\int_{a_1}^{b_1}a_{kjp}\left(x_1,\eta_1\right)\left.\frac{\partial u(\eta)}{\partial\eta_j}\right|_{\eta_2=\gamma_p(\eta_1)}d\eta_1+$$

$$\left.+\sum_{p=1}^{2}\int_{a_1}^{b_1}a_{kp}\left(x_1,\eta_1\right)u\left(\eta_1,\gamma_p(\eta_1)\right)d\eta_1+f_k(x_1)\right\}\frac{dx_1}{x_1-\xi_1}+...,$$

$$k=1,2;\quad \xi_1\in[a_1,b_1] \tag{5.11}$$

a pair of regular relations. Really regularity of the last summand in the right hand side (5.11) in brackets is given under the restriction [12]

$$f_k(x_1)\in C^{(1)}\left(a_1,b_1\right),\quad f_k(a_1)=f_k(b_1)=0,\quad k=1,2, \tag{5.12}$$

As for regularity of the second and third summands from the end of (5.11) in brackets, then under regularity the kernels $a_{kp}\left(x_1,\eta_1\right)$ and $a_{kjp}\left(x_1,\eta_1\right)$ are obtained after substitution of integrals.

Finally, as for the first two summands in the edge part of (5.11), for regularization of these expressions it suffices to substitute

$$u(x_1,\gamma_p(x_1))\qquad \left.\frac{\partial u(x)}{\partial x_j}\right|_{x_2=\gamma_p(x_1)},\qquad p,j=1,2$$

from (4.7)-(4.9) and then change the order of integrals obtained after substitution.

For that, it suffices the kernels $a_{kp}\left(x_1\right),$ and $a_{kjp}\left(x_1\right),$ $k,j,p=1,2$ be nonsingular.

After regularization of above mentioned necessary conditions(5.5)-(5.8) we have:

$$\left.\Delta u\ (x)\right|_{x_2=\gamma_1(x_1)}=\left.\Delta u\ (x)\right|_{x_2=\gamma_2(x_1)}.$$

If we consider the boundary condition (2.2)in the obtained expression, we have:

$$\left.\frac{\partial^2u(x)}{\partial x_1^2}\right|_{x_2=\gamma_1(x_1)}=\left.\frac{\partial^2u(x)}{\partial x_1^2}\right|_{x_2=\gamma_2(x_1)}+\left.\frac{\partial^2u(x)}{\partial x_2^2}\right|_{x_2=\gamma_2(x_1)}-\left.\frac{\partial^2u(x)}{\partial x_2^2}\right|_{x_2=\gamma_1(x_1)}=$$

$$=\sum_{p=1}^{2}\sum_{j=1}^{2}a_{3jp}\left(x_1\right)\left.\frac{\partial u(x)}{\partial x_j}\right|_{x_2=\gamma_p(x_1)}+\sum_{p=1}^{2}a_{3p}\left(x_1\right)u(x_1,\gamma_p(x_1))+$$

$$+\sum_{p=1}^{2}\sum_{j=1}^{2}\int_{a_1}^{b_1}a_{3jp}\left(x_1,\eta_1\right)\left.\frac{\partial u(\eta)}{\partial\eta_j}\right|_{\eta_2=\gamma_p(\eta_1)}d\eta_1+$$

$$+\sum_{p=1}^{2}\int_{a_1}^{b_1}a_{3p}\left(x_1,\eta_1\right)u(\eta_1,\gamma_p(\eta_1))d\eta_1+$$

$$+\sum_{k=1}^{2}(-1)^k\left\{\sum_{p=1}^{2}\sum_{j=1}^{2}\alpha_{kjp}\left(x_1\right)\left.\frac{\partial u(x)}{\partial x_j}\right|_{x_2=\gamma_p(x_1)}+\sum_{p=1}^{2}a_{kp}\left(x_1\right)u(x_1,\gamma_p(x_1))+\right.$$

$$+\sum_{p=1}^{2}\sum_{j=1}^{2}\int_{a_1}^{b_1} a_{kjp}(x_1,\eta_1)\left.\frac{\partial u(\eta)}{\partial \eta_j}\right|_{\eta_2=\gamma_p(\eta_1)} d\eta_1+$$

$$\left.+\sum_{p=1}^{2}\int_{a_1}^{b_1} a_{kp}(x_1,\eta_1)\,u(\eta_1,\gamma_p(\eta_1))d\eta_1+f_k(x_1)\right\}, \tag{5.13}$$

by this for the second derivatives we get six regular relations. Further, from (4.7) we have:

$$u(\xi_1,\gamma_k(\xi_1))=-2\int_{a_1}^{b_1} U(x_1-\xi_1,\gamma_1(x_1)-\gamma_k(\xi_1))\left.\frac{\partial^2 u(x)}{\partial x_1^2}\right|_{x_2=\gamma_1(x_1)} dx_1-$$

$$-2\int_{a_1}^{b_1} U(x_1-\xi_1,\gamma_1(x_1)-\gamma_k(\xi_1))\left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_1(x_1)} dx_1+$$

$$+2\int_{a_1}^{b_1} U(x_1-\xi_1,\gamma_2(x_1)-\gamma_k(\xi_1))\left.\frac{\partial^2 u(x)}{\partial x_1^2}\right|_{x_2=\gamma_2(x_1)} dx_1+$$

$$+2\int_{a_1}^{b_1} U(x_1-\xi_1,\gamma_2(x_1)-\gamma_k(\xi_1))\left.\frac{\partial^2 u(x)}{\partial x_2^2}\right|_{x_2=\gamma_2(x_1)} dx_1-$$

$$-2\int_{a_1}^{b_1}\left.\frac{\partial U}{\partial x_2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}}\left.\frac{\partial u(x)}{\partial x_1}\right|_{x_2=\gamma_1(x_1)}\frac{\cos(\nu_1,x_1)}{\cos(x_1,\tau_1)}dx_1-$$

$$-2\int_{a_1}^{b_1}\left.\frac{\partial U}{\partial x_2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}}\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_1(x_1)}\frac{\cos(\nu_1,x_2)}{\cos(x_1,\tau_1)}dx_1-$$

$$-2\int_{a_1}^{b_1}\left.\frac{\partial U}{\partial x_2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}}\left.\frac{\partial u(x)}{\partial x_1}\right|_{x_2=\gamma_2(x_1)}\frac{\cos(\nu_2,x_1)}{\cos(x_1,\tau_2)}dx_1-$$

$$-2\int_{a_1}^{b_1}\left.\frac{\partial U}{\partial x_2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}}\left.\frac{\partial u(x)}{\partial x_2}\right|_{x_2=\gamma_2(x_1)}\frac{\cos(\nu_2,x_2)}{\cos(x_1,\tau_2)}dx_1+$$

$$+2\int_{a_1}^{b_1} u(x_1,\gamma_1(x_1))\frac{\partial}{\partial\nu_1}\left.\frac{\partial U}{\partial x_2}\right|_{\substack{x_2=\gamma_1(x_1)\\ \xi_2=\gamma_k(\xi_1)}}\frac{dx_1}{\cos(x_1,\tau_1)}+$$

$$+2\int_{a_1}^{b_1} u(x_1,\gamma_2(x_1))\frac{\partial}{\partial\nu_2}\left.\frac{\partial U}{\partial x_2}\right|_{\substack{x_2=\gamma_2(x_1)\\ \xi_2=\gamma_k(\xi_1)}}\frac{dx_1}{\cos(x_1,\tau_2)}$$

$$(k=1,2)\,;\xi_1\in[a_1,b_1]. \tag{5.14}$$

As for remaining regular relations, they are obtained from (4.8) (4.9) if instead of the second derivatives we put the obtained regular relations.

This establishes

Theorem 3. *Under condition of the Theorem, if all the data (both the coefficients and the right hand sides) of the boundary condition (2.2) are continuous functions and conditions (5.12) are valid, then the boundary value problem (2.1)-(2.2) is Fredholm.*

Really, obvious expression for boundary values of the second derivative are obtained from boundary values of minor derivatives. For the boundary values of minor derivatives we get a system of second kind Fredholm integral equations of normal form, where the kernels of these integrals do not contain singularity. They are obtained similar to (5.14).

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