Bilal T. BILALOV, Miryasin S. EMINOV

ON EQUIVALENCE OF COMPLETENESS OF TWO SYSTEMS OF FUNCTIONS

Abstract

In the paper the system of powers with generated coefficients is considered. Under defined conditions the equivalence of completeness of this system in L_p to the completeness of system of exponent with generation is proved.

Let's consider the system of powers

$$\left\{A^+(t)\,\omega^+(t)\,\varphi^n(t)\,;A^-(t)\,\omega^-(t)\,\overline{\varphi}^n(t)\right\}_{n\geq 0}, \tag{1}$$

where $A^{\pm}(t) \equiv |A^{\pm}(t)|\,e^{i\alpha^{\pm}(t)}$ and $\varphi(t)$ are complexvalued on $[a,b]$ functions, $\omega^{\pm}(t)$ are degenerate coefficients

$$\omega^{\pm}(t) \equiv \prod_{k=1}^{r^{\pm}} \left|t - t_k^{\pm}\right|^{\beta_k^{\pm}},$$

$\left\{t_k^{\pm}\right\}^{r^{\pm}} \subset [a,b]$. The basis properties (completeness, minimality, basicity) of the systems of form (1) are considered by many mathematicians (see, for example [1-4]). These systems are natural generalizations of a classical system of exponents. At $\varphi(t) \equiv e^{it}$ the basicity (also the completeness and minimality) of system (1) in $L_p(-\pi,\pi)$ is studied in the paper [4]. In case, when degenerations are absent, the necessary and sufficient condition of completeness and minimality of system (1) in L_p is find in the paper [3].

In the present paper the completeness criterion of system (1) in $L_p \equiv L_p(a,b)$, $1 < p < +\infty$ is reduced.

1. Necessary assumptions and data. Let's make the following base assumptions:

1)$[A^+(t)]^{\pm 1}; [A^-(t)]^{\pm 1}; [\varphi'(t)]^{\pm 1} \in L_\infty$;

2) $\Gamma = \varphi\{[a,b]\}$ be closed $(\varphi(a) = \varphi(b))$ is a rectifiable, simple Jourdan curve. Γ is either Radon curve (i.e. an angle $\theta_0\varphi(t))$ between the tangent at the point $\varphi = \varphi(t)$ to the curve Γ and real axis there is a function with bounded variation on $[a,b]$), or the piece-wise Lyapunovs curve. Γ has a finite number of angle points without cusp. Denote by φ_k the break points of the function $\varphi'(t)$ on $[a,b]$.

For definiteness we'll suppose that when the point $\varphi = \varphi(t)$ by increasing t passes through the curve Γ, internal domain $D \equiv int\ \Gamma$ stays on the left.

Under the function $\arg\varphi'(t)$ we understand the following: we'll determine at each initial point φ_k, $(\left(\varphi_k, \varphi_{k+1}\right)$ is an interval of continuous function $\arg\varphi'(t))$ arm $\arg\varphi'(\varphi_k + 0)$; at the end point φ_{k+1} the value $\arg\varphi'(\varphi_k - 0)$ we'll obtain from the selected arm $\arg\varphi'(\varphi_k + 0)$ by continuous change. Without losing generality, at the points φ_k the value $\arg\varphi'(\varphi_k + 0)$ we'll determine from the conditions:

$$0 \leq \arg\varphi'(a+0) < 2\pi;$$

$$\left|\arg\varphi'(\varphi_k+0)-\arg\varphi'(\varphi_k-0)\right|<\pi.$$

We need the Smirnov's weight classes and the Riemann's problem in it.

Let $E_1(D)$ be an ordinary Smirnov's class and $\nu(\tau)$, $\tau\in\Gamma$ be some weight function. Let's assume:

$$E_{p,\nu}(D)\equiv\left\{f\in E_1(D):\int_\Gamma\left|f^+(\tau)\right|^p\nu(\tau)\,|d\tau|<+\infty\right\},$$

where $f^+(\tau)$ are non-tangential boundary values of the function $f(z)$ on Γ.

Let's consider the following conjugation problem:

$$F_1^+(\tau)+G(\tau)\overline{F_2^+(\tau)}=g(\tau),\tau\in\Gamma, \tag{2}$$

where $g(\tau)\in L_{p,\nu}(\Gamma)$ is a right part, $G(\tau)$ is a problem's coefficient, and $L_{p,\nu}(\Gamma)$ is a Lebesgue weight class with the norm

$$\|f\|_{p,\nu}=\left(\int_\Gamma|f(\tau)|^p\nu(\tau)\,|d\tau|\right)^{1/p},$$

We seek the pair of analytical in D functions $(F_1(z);F_2(z)):F_i\in E_{p,\nu}(D)$, $i=1,2$; whose nontangential boundary values almost everywhere on Γ, satisfies equality (2).

It should be to note a Riemann problem in Smirnov classes $E_p(D)$ is studied in (see for ex. [5]). In general, in weightless case the studying of problem (2) by means of conformal mapping may be reduced to the Riemann problem. But, in principle we don't need this. We find the other way. Namely, we'll use the following lemma, which is proved in the paper [6].

Lemma 1. *Let the functions $A^\pm(t)$, $\varphi(t)$ satisfy conditions 1), 2) and $\beta_k^\pm>-\dfrac{1}{p}$, $\forall k$; $p\in(1+\infty)$. The system (1) is complete in L_p iff the homogeneous conjugation problem*

$$F_1^+(\tau)-G(\tau)\overline{F_2^+(\tau)}=0,\ \tau\in\Gamma, \tag{3}$$

has only trivial solution in the classes $E_{q,\rho^\pm}(D):F_1\in E_{q,\rho^\pm}(D)$; $F_2\in E_{q,\rho^-}(D)$, $\dfrac{1}{p}+\dfrac{1}{q}=1$, where $\rho^\pm(\varphi(t))\equiv|\omega^\pm(t)|^{-q}$, and the coefficient $G(\tau)$ is determined by the expression

$$G(\varphi(t))=\frac{A^+(t)\cdot\omega^+(t)\cdot\overline{\varphi}'(t)}{A^-(t)\cdot\omega^-(t)\cdot\varphi'(t)},\ t\in(a,b).$$

2. The basic results. We'll consider the conjugation problem (3) on trivial solvability in the classes $E_{q,\rho^\pm}(D)$.

Denote by $z=\omega(\xi),\omega'(0)>0$, $\omega(-\pi)=\varphi(a)$ the function, realizing the conformal and one-sheeted mapping of a unique circle $\{\xi:|\xi|<1\}$ onto domain D. Let's consider the following functions analytical in a unique circle:

$$\Phi_i(\xi)\equiv F_i[\omega(\xi)]\cdot\omega'(\xi),\ i=1,2.$$

It is known that F_i belongs to the class $E_1(D)$ iff Φ_i belongs to H_1, where H_1 is an ordinary Hardy class. Let $F_i \in E_{p,\nu}(D)$, i.e., $F_i \in E_1(D)$ and $F_i^+ \in L_{p,\nu}(\Gamma)$. It is evident that $\Phi_i \in H_1$. We have:

$$\int_\Gamma \left|F_i^+(\tau)\right|^p \nu(\tau)\,|d\tau| = \int_{|\xi|=1} \left|F_i^+[\omega(\xi)]\right|^p \nu[\omega(\xi)] \left|\omega'(\xi)\right| |d\xi| =$$

$$= \int_{|\xi|=1} \left|\Phi_i^+(\xi)\right|^p \frac{\nu[\omega(\xi)]}{|\omega'(\xi)|^{p-1}} |d\xi| = \int_{|\xi|=1} \left|\Phi_i^+(\xi)\right|^p \mu(\xi)\,|d\xi| < +\infty. \quad (4)$$

Hence, it follows that $\Phi_i \in H_{p;\mu}$, where $\mu(\xi) \equiv \dfrac{\nu[\omega(\xi)]}{|\omega'(\xi)|^{p-1}}$, and the class $H_{p;\mu}$ is

$$H_{p;\mu} \equiv \left\{ \Phi \in H_1 : \int_{|\xi|=1} \left|\Phi^+(\xi)\right|^p \mu(\xi)\,|d\xi| < +\infty \right\}.$$

Thus, if $F_i \in E_{p,\nu}(D)$, then $\Phi_i \in H_{p,\mu}$. From relation (4) it immediately follows that the opposite is also true, i.e., if $\Phi_i \in H_{p,\mu}$, then $F_i \in E_{p,\nu}(D)$. Consequently, $F_i \in E_{p,\nu}(D)$ iff $\Phi_i \in H_{p,\mu}$. Taking into account this conclusion from (3) we get:

$$\Phi_1^+(\xi) - D(\xi)\,\overline{\Phi_2^+(\xi)} = 0,\ |\xi| = 1, \quad (5)$$

where $D(\xi) = \left[\omega(\xi) \cdot \dfrac{\omega'(\xi)}{\omega'(\xi)}\right]$.

As a result we obtain that the following lemma is true.

Lemma 2. *The homogeneous problem (3) is trivially solvable in the class $E_{q,\rho^+}(D) \times E_{q,\rho^-}(D)$ (i.e. $F_1 \in E_{q,\rho^-}(D)$), iff the problem (5) is trivially solvable in the class $H_{q,\mu^+} \times H_{q,\mu^-}$ (i.e. $\Phi_1 \in H_{q,\mu^+;\ \Phi_2 \in H_{q,\mu^-}}$), where* $\mu^\pm(\xi) \equiv \dfrac{\rho^\pm[\omega(\xi)]}{|\omega'(\xi)|^{q-1}}$.

Denote by $\xi = \omega_{-1}(z)$ the inverse to $z = \omega(\xi)$ function, realizing the conformal and one-sheeted mapping of the domain D on a unique circle. Let $\tau_k = \omega_{-1}[\varphi_k]$, $k = \overline{1,r}$, where φ_k is a corner point of the curve $\Gamma \backslash \{\varphi(a)\}$. It is known that $\omega'(\xi)$ has a discontinuities at the points τ_k, moreover near these points it holds (see, for ex. [7]):

$$\left|\omega'(\xi)\right| \sim |\xi - \tau_k|^{\nu_k - 1}, \xi \to \tau_k;$$

$$\left(|\varphi(\xi)| \sim |\psi(\xi)| \Leftrightarrow 0 < \delta \le \frac{|\varphi(\xi)|}{|\psi(\xi)|} \le \delta^{-1} < +\infty, \delta > 0\right),$$

where $\nu_k\pi$ are internal angles at the points φ_k of the curve Γ. Consequently,

$$\left|\omega'(\xi)\right| \sim \prod_{k=1}^r |\xi - \tau_k|^{\nu_k - 1},\ |\xi| = 1.$$

Let's assume

$$A_1^+(t) \equiv A^+(t)\,\overline{\varphi}'(t);\ A_1^-(t) \equiv A^-(t)\,\varphi'(t);$$

$$\tilde{A}(\xi) \equiv \xi^{-1} A_1^{+}\left[\varphi_{-1}(\omega(\xi))\right] \frac{\omega^{+}\left[\varphi_{-1}(\omega(\xi))\right]}{|\omega'(\xi)|^{-\frac{1}{p}}}$$

$$\widetilde{B}(\xi) \equiv \xi^{-1} A_1^{-}\left[\varphi_{-1}(\omega(\xi))\right] \frac{\omega^{-}\left[\varphi_{-1}(\omega(\xi))\right]}{|\omega'(\xi)|^{-\frac{1}{p}}} \cdot \frac{\overline{\omega'(\xi)}}{\omega'(\xi)}$$

where $\varphi_{-1} : \Gamma \backslash \{\varphi(a)\} \rightarrow (a,b)$ is an inverse to $\varphi = \varphi(t)$ function. Let's consider the system

$$\left\{\tilde{A}\left(e^{ix}\right) e^{inx}; \widetilde{B}\left(e^{ix}\right) e^{-inx}\right\}_{n\geq 0}. \tag{6}$$

Absolutely analogously to lemma 1 the validity of the following lemma is proved.

Lemma 3. *The system (6) is complete in $L_p(-\pi,\pi)$ iff the homogeneous conjugation problem (5) has only a trivial solution in the classes $H_{q,\mu^+} \times H_{q,\mu^-}$.*

In fact, allowing the existence of the function $f \in L_q(-\pi,\pi)$, annihilating the system (6), we have:

$$\int_{-\pi}^{\pi} \tilde{A}\left(e^{ix}\right) e^{inx} \overline{f(x)} dx = 0,$$

$$\int_{-\pi}^{\pi} \tilde{B}\left(e^{ix}\right) e^{-inx} \overline{f(x)} dx = 0, \ \forall n \geq 0.$$

From the first equality we obtain:

$$\int_{-\pi}^{\pi} \tilde{A}\left(e^{ix}\right) e^{-ix} \overline{f(x)\, e^{inx}} de^{ix} = \int_{|\xi|=1} \tilde{A}(\xi)\, \bar{\xi} \cdot \overline{f(\arg \xi)} \xi^{n} d\xi =$$

$$= \int_{|\xi|=1} f_1(\xi)\, \xi^{n} d\xi = 0, \tag{7}$$

where $f_1(\xi) = \tilde{A}(\xi)\, \bar{\xi} \overline{f(\arg \xi)}$. From the accepted assumtions it follows that $f_1 \in L_1(\gamma)$, where $\gamma \equiv \{\xi : |\xi| = 1\}$. It is known that (see, for ex. [8]) equalities (7) are equivalent to the existence $\Phi_1 \in H_1 : \Phi_1^{+}(\xi) = f_1(\xi)$, a.e. on γ. Thus,

$$\Phi_1^{+}(\xi) = \tilde{A}(\xi)\, \bar{\xi} \overline{f(\arg \xi)} =$$

$$= A_1^{+}\left[\varphi_{-1}(\omega(\xi))\right] \frac{\omega^{-}\left[\varphi_{-1}(\omega(\xi))\right]}{|\omega'(\xi)|^{-\frac{1}{p}}} \cdot \overline{f(\arg \xi)}.$$

From this relation we immediately obtain that:

$$\frac{\Phi_1^{+}(\xi)}{\omega^{+}\left[\varphi_{-1}(\omega(x))\right] \cdot |\omega'(\xi)|^{-\frac{1}{p}}} \in L_q(\gamma), \text{ i.e, } \Phi_1 \in H_{q,\mu^+}.$$

Absolutely analogously it is constructed that
$\exists \Phi_2 \in H_{q,\mu^-} : \Phi_2^{+}(\xi) = \overline{\tilde{B}(\xi)} \cdot \bar{\xi} f(\arg \xi)$, a.e. on γ.

Consequently,

$$\overline{f(\arg\xi)}=\frac{\overline{\Phi_2^+(\xi)}}{\tilde{B}(\xi)\xi}=\frac{\overline{\Phi_2^+(\xi)}}{A_1^-\left[\varphi_{-1}(\omega(\xi))\right]\dfrac{\omega^-\left[\varphi_{-1}(\omega(\xi))\right]}{|\omega'(\xi)|^{-\frac{1}{p}}}\cdot\dfrac{\overline{\omega'(\xi)}}{\omega'(\xi)}}\equiv g(\xi).$$

From the obtained two relations we obtain:

$$\frac{\Phi_1^+(\xi)}{A^+\left[\varphi_{-1}(\omega(\xi))\right]\dfrac{\omega^+\left[\varphi_{-1}(\omega(\xi))\right]}{|\omega'(\xi)|^{-\frac{1}{p}}}}=g(\xi),$$

i.e.,

$$\Phi_1^+(\xi)-\frac{A_1^+\left[\varphi_{-1}(\omega(\xi))\right]\cdot\omega^+\left[\varphi_{-1}(\omega(\xi))\right]}{A_1^-\left[\varphi_{-1}(\omega(\xi))\right]\cdot\omega^-\left[\varphi_{-1}(\omega(\xi))\right]}\cdot\frac{\omega'(\xi)}{\overline{\omega'(\xi)}}\cdot\overline{\Phi_2^+(\xi)}=0;$$

$$\Phi_1^+(\xi)-D(\xi)\cdot\overline{\Phi_2^+(\xi)}=0,\ \text{a.e. on }\gamma.$$

Thus, we obtained relation (5). The opposite is proved analogously to lemma 1.

The results of these lemmas is the following.

Theorem 1. *Let the conditions 1), 2) and* $\beta_k^\pm>-\dfrac{1}{p}$, $\forall k; p\in(1,+\infty)$ *be fulfilled. The system (1) is complete in* $L_p(a,b)$ *iff the system (6) is complete in* $L_p(-\pi,\pi)$.

References

[1]. Walsh J.L. *Interpolation and approximation by rational functions in complex domain.* M.: "IL", 1961 (Russian).

[2]. Kazmin Yu.A. *Closing of linear shell of one system of function.* Sib. Mat. Jurn., 1977, v.18, No4, pp.799-805 (Russian).

[3]. Bilalov B.T. *Necessary and sufficient condition of completeness and minimality of the system of the form* $\{A\varphi^n; B\overline{\varphi}^n\}$. Dokl. RAN, 1992, v.322, No6, pp.1019-1021 (Russian).

[4]. Veliev S.G. *Bases from sub-sets of eigen-functions of two discontinuous differential operators.* Matem. Fizika, analiz, geometriya, 2005, v.12, No 2 (Russian).

[5]. Danilyuk I.I. *Nonregular boundary-value problems on a plane.* M.: "Nauka", 1975, v.12, No 2 (Russian).

[6]. Bilalov B.T., Eminov M.S. *On completeness of a a system of powers.* Trans. of NAS of Azerb., 2006, v. XXVI, No1, pp.45-50.

[7]. Vekua I.N. *Generalized analytical functions.* M.: "Nauka", 1988, 510p. (Russian).

[8]. Privalov I.I. *Boundary properties of analytical functions.* M-L., 1950 (Russian).

Bilal T.Bilalov, Miryasin S. Eminov
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, F.Agayev str., AZ1141, Baku, Azerbaijan
Tel.: (99412) 439 47 20 (off.)

Received April 02, 2009; Revised June 17, 2009.