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THE PROPERTIES OF THE $n-$CONJUGATE FUNCTIONS

Abstract

The conception of conjugate convex functions belongs to Fenxel (see [1,2]). Later on, this conception was advanced to infinite dimensionality by Brennstedom and Morro (see [3-5]). The conception of n-conjugate functions is determined and its some properties are studied in the work. An existence theorem of intervening B−affine functional is proved.

Let $E_i, i = \overline{1,n}$ be real vector spaces and $\tilde{B}(E_1 \times ... \times E_n, R)$ be the vector spaces of n-linear functionals from $E_1 \times ... \times E_n$ in R. For any $(x_1, ..., x_n) \in E_1 \times ... \times E_n$ the mapping $l \to l(x_1, ..., x_n)$ is a linear functional on $\tilde{B}(E_1 \times ... \times E_n, R)$ - and it is easy to see that the mapping $\chi : (x_1, ..., x_n) \to u_{(x_1,...,x_n)}$ of the product $E_1 \times ... \times E_n$ on $\tilde{B}(E_1 \times ... \times E_n, R)$ is $n-$linear, where an algebraic conjugate space to $\tilde{B}(E_1 \times ... \times E_n, R)$ is denoted by $\tilde{B}(E_1 \times ... \times E_n, R)^*$. The linear span of the sets $\chi(E_1 \times ... \times E_n)$ in $\tilde{B}(E_1 \times ... \times E_n, R)^*$ is denoted by $E_1 \otimes ... \otimes E_n$ (see [6]) and are called tenzor product $E_i,\ i = \overline{1,n}$. The mapping χ is called canonical n-linear mapping from $E_1 \times ... \times E_n$ in $E_1 \otimes ... \otimes E_n$. An element $u_{(x_1,...,x_n)}$ from $E_1 \otimes ... \otimes E_n$ is denoted by $x_1 \otimes ... \otimes x_n$.

Let $X_i,\ i = \overline{1,n}$, be real Banach spaces and $q : X_1 \times ... \times X_n \to \overline{R} = R \cup \{+\infty\}$. If the functions $x_i \to q(x_1, ..., x_i, ..., x_n)$ are positively homogenous and $q(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n) = 0$ for $i = \overline{1,n}$, then the function q is called n-positively homogenous. If q n-positively homogenous function and the functions $x_i \to q(x_i, ..., x_n)$ are convex, then the function q is called n-sublinear. The sets of the all continuous n-linear mappings from $X_1 \times ... \times X_n$ in R denote by $B(X_1 \times ... \times X_n, R)$. The tensor product of the spaces $X_i,\ i = \overline{1,n}$, is denoted by $X_1 \otimes ... \otimes X_n$. Consider that $X_1 \otimes ... \otimes X_n$ is supplied with projection topology (see [6]). If the function $x_i \to q(x_1, ..., x_i, ..., x_n)$ are convex, then the function q is called n-convex. If

$$q(x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_{j-1}, x_j, x_{j+1}, ..., x_n) =$$
$$= q(x_1, ..., x_{i-1}, -x_i, x_{i+1}, ..., x_{j-1}, -x_j, x_{j+1}, ..., x_n)$$

for $i, j \in \overline{1,n}$, where $i < j$, then the function $q : X_1 \times ... \times X_n \to \overline{R}$ is called even.

If $domq = \{(x_1, ..., x_n) \in X_1 \times ... \times X_n : q(x_1, ..., x_n) < +\infty\} \neq \oslash$ and $x \in B(X_1 \times ... \times X_n, R)$, then we put

$$q^*(x^*) = \sup_{(x_1,...,x_n) \in X_1 \times ... \times X_n} \{x^*(x_1, ..., x_n) - q(x_1, ..., x_n)\},$$

$$q^{**}(x_1, ..., x_n) = \sup_{x^* \in B(X_1 \times ... \times X_n, R)} \{x^*(x_1, ..., x_n) - q^*(x^*)\}.$$

As $q^*(x^*) \geq x^*(x_1,...,x_n) - q(x_1,...,x_n)$, then $q^{**}(x_1,...,x_n) \leq q(x_1,...,x_n)$ for $(x_1,...,x_n) \in X_1 \times ... \times X_n$. It is clear that

$$q^*(0) = \sup_{(x_1,...,x_n)\in X_1\times...\times X_n} \{-q(x_1,...,x_n)\},$$

$$q^{**}(x_1,...,x_{i-1},0,x_{i+1},...,x_n) = \sup\{-q^*(x^*) : x^* \in B(X_1 \times ... \times X_n, R)\}.$$

The $n-$subdiferential functions q at the point of $(\overline{x}_1,...,\overline{x}_n) \in domq$ we call the following set

$$\partial_n q(\overline{x}_1,...,\overline{x}_n) = \{x^* \in B(X_1 \times ... \times X_n, R) : q(x_1,...,x_n) - q(\overline{x}_1,...,\overline{x}_n) \geq$$

$$\geq x^*(x_1,...,x_n) - x^*(\overline{x}_1,...,\overline{x}_n),\ (x_1,...,x_n) \in X_1 \times ... \times X_n\},$$

and if $P : X_1 \times ... \times X_n \to R$ is $n-$positively homogenous function, then we put

$$\partial_n P = \{x^* \in B(X_1 \times ... \times X_n, R) : P(x_1,...,x_n) \geq$$

$$\geq x^*(x_1,...,x_n),\ (x_1,...,x_n) \in X_1 \times ... \times X_n\}.$$

Note that the function $q : X_1 \times ... \times X_n \to R$ is called proper, if $domq \neq \oslash$. Later on, we consider that all the functions are proper.

Lemma 1. *1) If $q_1 \geq q_2$, then $q_2^* \leq q_1^*$;*

2) If Y_i are Banach spaces, $A_i \in L(X_i, Y_i)$, $i = \overline{1,n}$, $q : Y_1 \times ... \times Y_n \to \overline{R}$, A_i, $i = \overline{1,n}$, are isomorphisms, $A = (A_1,...,A_n)$, $A^{-1}\left(A_1^{-1},...,A_n^{-1}\right)$, then $(q \circ A)^*(x^*) = q^*\left(x^* \cdot A^{-1}\right)$;

3) If $g(x_1,...,x_n) = q(\lambda_1 x_1,...,\lambda_n x_n)$, then $g^*(x^*) = q^*\left(\frac{x^*}{\lambda_1...\lambda_n}\right)$ for $\lambda;_1,...,\lambda_n \neq 0$;

4) If $g(x_1,...,x_n) = \lambda q(x_1,...,x_n)$ and $\lambda > 0$, then $g^*(x^*) = \lambda q^*\left(\frac{x^*}{\lambda}\right)$,

5) If $g(x_1,...,x_n) = q(x_1,...,x_n) + \overrightarrow{x}(x_1,...,x_n) + \alpha$, $\overline{x}^* \in B(X_1 \times ... \times X_n, R)$, then $g^*(x^*) = q^*(x^* - \overline{x}^*) - \alpha$.

Proof.1) If $q_1(x_1,...,x_n) \leq q_2(x_1,...,x_n)$,then $x^*(x_1,...,x_n) - q_1(x_1,...,x_n) \geq x^*(x_1,...,x_n) - q_2(x_1,...,x_n)$ for $(x_1,...,x_n) \in X_1 \times ... \times X_n$. Therefore $q_2^*(x^*) \leq q_1^*(x^*)$.

2) If $A = (A_1,...,A_n) \in L(X_1,Y_1) \times ... \times L(X_n,Y_n)$ is isomorphism, $A^{-1} = \left(A_1^{-1},...,A_n^{-1}\right)$,then

$$(q \circ A)^*(x^*) = \sup_{(x_1,...,x_n)\in X_1\times...\times X_n} \{x^*(x_1,...,x_n) - q(A(x_1,...,x_n))\} =$$

$$= \sup_{(y_1,...,y_n)\in X_1\times...\times X_n} \left\{x^*\left(A^{-1}(y_1,...,y_n)\right) - q(y_1,...,y_n)\right\} = q^*\left(x^* \circ A^{-1}\right).$$

3) It is clear, that

$$g^*(x^*) = \sup_{(x_1,...,x_n)\in X_1\times...\times X_n} \{x^*(x_1,...,x_n) - q(\lambda_1 x_1,...,\lambda_n x_n)\} =$$

$$= \sup_{(x_1,...,x_n)\in X_1\times...\times X_n} \left\{x^*\left(\frac{x_1}{\lambda_1},...,\frac{x_n}{\lambda_n}\right) - q(x_1,...,x_n)\right\} = q^*\left(\frac{x^*}{\lambda_1...\lambda_n}\right).$$

4) If $g(x_1,...,x_n) = \lambda q(x_1,...,x_n)$ and $\lambda > 0$, then

$$g^*(x^*) = \sup_{(x_1,...,x_n)\in X_1\times...\times X_n} \{x^*(x_1,...,x_n) - \lambda q(x_1,...,x_n)\} =$$

$$= \lambda \sup_{(x_1,...,x_n)\in X_1\times...\times X_n} \left\{\frac{1}{\lambda}x^*(x_1,...,x_n) - q(x_1,...,x_n)\right\} = \lambda q^*\left(\frac{x^*}{\lambda}\right).$$

5) If $g(x_1,...,x_n) = q(x_1,...,x_n) + \overline{x}^*(x_1,...,x_n) + \alpha$, then

$$g^*(x^*) = \sup_{(x_1,...,x_n)\in X_1\times...\times X_n} \{x^*(x_1,...,x_n) - \overline{x}^*(x_1,...,x_n) - q(x_1,...,x_n)\} -$$

$$-\alpha = q^*(x^* - \overline{x}^*) - \alpha.$$

The lemma is proved.

If $f : X_1 \times ... \times X_n \times Y_1 \times ... \times Y_n \to \overline{R}$ is a proper function, $Y_i,\ i = \overline{1,n}$, where are real Banach spaces and

$$q(x_1,...,x_n) = \inf\{f(x_1,...,x_n,y_1,...,y_n) : (y_1,...,y_n) \in Y_1 \times ... \times Y_n\},$$

then

$$q^*(x^*) = \sup_{(x_1,...,x_n)\in X_1\times...\times X_n} \{x^*(x_1,...,x_n) - q(x_1,...,x_n)\} =$$

$$= \sup_{(x_1,...,x_n)\in X_1\times...\times X_n} \{x^*(x_1,...,x_n) -$$

$$- \inf_{(y_1,...,y_n\in Y_1\times...\times Y_n} f(x_1,...,x_n,y_1,...,y_n)\Big\} =$$

$$= \sup_{(x_1,...,x_n)\in X_1\times...\times X_n} \sup_{(y_1,...,y_n)\in Y_1\times...\times Y_n} \{x^*(x_1,...,x_n) -$$

$$-f(x_1,...,x_n,y_1,...,y_n)\} = f^*(x^*,0)$$

Note that the some properties of n-conjugate functions are similar to properties of conjugate functions (see [4,5]).

Lemma 2. *If $\partial_n q(\overline{x}_1,...,\overline{x}_n) \neq \oslash$, then $q^{**}(\overline{x}_1,...,\overline{x}_n) = q(\overline{x}_1,...,\overline{x}_n)$.*

Proof. From lemma 3.3 [7] it follows that $\overline{x}^* \in \partial_n q(\overline{x}_1,...,\overline{x}_n)$ if and only if, $q^*(\overline{x}^*) + q(\overline{x}_1,...,\overline{x}_n) = \overline{x}^*(\overline{x}_1,...,\overline{x}_n)$. Therefore

$$q^{**}(\overline{x}_1,...,\overline{x}_n) = \sup_{x^*\in B(X_1\times...\times X_n,R)} \{x^*(\overline{x}_1,...,\overline{x}_n) - q^*(x^*)\} \geq$$

$$\geq \overline{x}^*(\overline{x}_1,...,\overline{x}_n) - \overline{x}^*(\overline{x}_1,...,\overline{x}_n) + q(\overline{x}_1,...,\overline{x}_n) = q(\overline{x}_1,...,\overline{x}_n),$$

i.e. $q^{**}(\overline{x}_1,...,\overline{x}_n) \geq q(\overline{x}_1,...,\overline{x}_n)$. As $q^{**}(\overline{x}_1,...,\overline{x}_n) \leq q(\overline{x}_1,...,\overline{x}_n)$ then we get that $q^{**}(\overline{x}_1,...,\overline{x}_n) = q(\overline{x}_1,...,\overline{x}_n)$. The lemma is proved.

Corollary 1. *If $q = q_1 + q_2$, $\partial_n q_1(\overline{x}_1,...,\overline{x}_n) \neq \oslash$* and *$\partial_n q_2(\overline{x}_1,...,\overline{x}_n) \neq \oslash$,* then *$q^{**}(\overline{x}_1,...,\overline{x}_n) = q_1^{**}(\overline{x}_1,...,\overline{x}_n) + q_2^{**}(\overline{x}_1,...,\overline{x}_n)$.*

Proof. Since $\partial_n q_1(\overline{x}_1,...,\overline{x}_n)+\partial_n q_2(\overline{x}_1,...,\overline{x}_n)\subset \partial_n q(\overline{x}_1,...,\overline{x}_n)$, then we get that $\partial_n q(\overline{x}_1,...,\overline{x}_n)\neq\oslash$. Then from lemma 2 it follows that $q^{**}(\overline{x}_1,...,\overline{x}_n)=q(\overline{x}_1,...,\overline{x}_n)$, $q_1^{**}(\overline{x}_1,...,\overline{x}_n)=q_1(\overline{x}_1,...,\overline{x}_n)$, $q_2^{**}(\overline{x}_1,...,\overline{x}_n)=q_2(\overline{x}_1,...,\overline{x}_n)$. Hence we get that corollary 1 is true.

Let $q: X_1\times...\times X_n\to R$ be n-positively homogenous function $\partial_n q=\partial_n q(0,...,0)$, and

$$\overline{q}(\nu)=$$

$$=\inf\left\{\sum_i q\left(x_1^i,...,x_n^i\right):\nu=\sum_i x_1^i\otimes...\otimes x_n^i,\ \left(x_1^i,...,x_n^i\right)\in X_1\times...\times X_n\right\}$$

Easy to verify that (see [7])

$$\partial\overline{q}=\left\{z^*\in(X_1\otimes...\otimes X_n)^*: z^*(\nu)=\sum_i x^*\left(x_1^i,...,x_n^i\right)\ \text{for}\right.$$

$$\left.\nu=\sum_i x_1^i\otimes...\otimes x_n^i,\ x^*\in\partial_n q,\ \ \left(x_1^i,...,x_n^i\right)\in X_1\times...\times X_n\right\}$$

and

$$q^*(x^*)=\begin{cases}0: x^*\in\partial_n q\\ +\infty: x^*\notin\partial_n q\end{cases},\ \overline{q}^*(z^*)=\begin{cases}0: z^*\in\partial\overline{q}\\ +\infty: z^*\notin\partial\overline{q}\end{cases}.$$

Then we get that

$$\overline{q}^{**}\left(x_1^i\otimes...\otimes x_n^i\right)=\sup_{z^*\in\partial\overline{q}} z^*(x_1\otimes...\otimes x_n)=$$

$$=\sup_{x^*\in\partial_n q} x^*(x_1,...,x_n)=q^{**}(x_1,...,x_n).$$

If $\partial_n q(x_1,...,x_n)\neq\oslash$ for $(x_1,...,x_n)\in X_1\times...\times X_n$, then from lemma 2 it follows that $q(x_1,...,x_n)=q^{**}(x_1,...,x_n)=\sup\{x^*(x_1,...,x_n): x^*\in\partial_n q\}$.

If $q: X_1\times...\times X_n\to R$ is n-positively homogenous continuous functions and $\overline{q}(0)=0$, then from theorem 3.1 [5] and lemma 5.4 [7] it follows that

$$\overline{q}^{**}(x_1\otimes...\otimes x_n)=\overline{q}(x_1\otimes...\otimes x_n).$$

Therefore $q^{**}(x_1,...,x_n)=\overline{q}(x_1\otimes...\otimes x_n)$.

Note that if q is an even continuous n-sublinear function, then from theorem 2.4 [7] it follows that $q(x_1,...,x_n)=q^{**}(x_1,...,x_n)$ and $q(x_1,...,x_n)=\overline{q}(x_1\otimes...\otimes x_n)$ for $(x_1,...,x_n)\in X_1\times...\times X_n$.

As for any $(x_1,...,x_n)\in X_1\times...\times X_n$ the mapping $x^*\to x^*(x_1,...,x_n)$ is a linear continuous functional on $B(X_1\times...\times X_n,R)$, then from definition it follows that $x^*\to q^*(x^*)$ is a convex function.

An element $x_1\otimes...\otimes x_n\in X_1\otimes...\otimes X_n$ is called subgradient of the function $q^*(x^*)$ at the point $\overline{x}^*$, if

$$q^*(x^*)-q^*(\overline{x}^*)\geq x^*(x_1,...,x_n)-\overline{x}^*(x_1,...,x_n)$$

for all $x^* \in B(X_1 \times ... \times X_n, R)$, and the set of subgradients is called subdifferential of function q^* at the point $\overline{x}^*$ and is denoted by $\partial q^*(\overline{x}^*)$.

Let $(x_1 \otimes ... \otimes x_n)$, $(y_1 \otimes ... \otimes y_n) \in \partial q^*(\overline{x}^*)$, $\alpha \in (0,1)$. Then

$$q^*(x^*) - q^*(\overline{x}^*) \geq \alpha(x^*(x_1,...,x_n) - \overline{x}^*(x_1,...,x_n)) +$$
$$+(1-\alpha)(x^*(y_1,...,y_n) - \overline{x}^*(y_1,...,y_n)),$$

i.e. $\alpha x_1 \otimes ... \otimes x_n + (1-\alpha) y_1 \otimes ... \otimes y_n \in \partial q^*(\overline{x}^*)$, i.e. $\partial q^*(\overline{x}^*)$ is convex set.

Further, an element $x_1 \otimes ... \otimes x_n$ we identify with an element $(\overline{x}_1,...,\overline{x}_n)$.

Lemma 3. $(\overline{x}_1,...,\overline{x}_n) \in \partial q^*(\overline{x}^*)$ *if and only if*

$$q^*(\overline{x}^*) + q^{**}(\overline{x}_1,...,\overline{x}_n) = \overline{x}^*(\overline{x}_1,...,\overline{x}_n).$$

Proof. If $(\overline{x}_1,...,\overline{x}_n) \in \partial q^*(\overline{x}^*)$, then from definition it follows that

$$\overline{x}^*(\overline{x}_1,...,\overline{x}_n) - q^*(\overline{x}^*) \geq x^*(\overline{x}_1,...,\overline{x}_n) - q^*(x^*)$$

for $x^* \in B(X_1 \times .... \times X_n, R)$. Hence it follows that $\overline{x}^*(\overline{x}_1,...,\overline{x}_n) - q^*(\overline{x}^*) = q^{**}(\overline{x}_1,...,\overline{x}_n)$.

On the contrary, if $q^*(\overline{x}^*) + q^{**}(\overline{x}_1,...,\overline{x}_n) = \overline{x}^*(\overline{x}_1,...,\overline{x}_n)$, then we get that

$$\overline{x}^*(\overline{x}_1,...,\overline{x}_n) - q^*(\overline{x}^*) = \sup_{x^* \in B(X_1 \times ... \times X_n, R)} \{x^*(\overline{x}_1,...,\overline{x}_n) - q^*(x^*)\}.$$

Therefore $\overline{x}^*(\overline{x}_1,...,\overline{x}_n) - q^*(\overline{x}^*) \geq x^*(\overline{x}_1,...,\overline{x}_n) - q^*(x^*)$
for $x^* \in B(X_1 \times ... \times X_n, R)$, i.e. $(\overline{x}_1,...,\overline{x}_n) \in \partial q^*(\overline{x}^*)$. The lemma is proved.

Lemma 4. *For any functions* $q : X_1 \times ... \times X_n \to \overline{R}$ *it is fulfilled an equality* $q^* = q^{***}$.

Proof. As $q^{**}(x_1,...,x_n) \leq q(x_1,...,x_n)$ for $(x_1,...,x_n) \in X_1 \times ... \times X_n$, then from 1) of lemma 1 it follows that $q^{**} \geq q^*$. On the contrary, from definition q^{**} it follows that $q^{**}(x_1,...,x_n) \geq x^*(x_1,...,x_n) - q^*(x^*)$ for $(x_1,...,x_n) \in X_1 \times ... \times X_n$. Therefore

$$q^{***}(x^*) = \sup_{(x_1,...x_n) \in X_1 \times ... \times X_n} \{x^*(x_1,...,x_n) - q^{**}(x_1,...,x_n)\} \leq q^*(x^*).$$

As $q^{***} \geq q^*$ and $q^{***} \leq q^*$, then we get $q^* = q^{***}$. The lemma is proved.

By analogy of lemma 3 we verify that, $x^* \in \partial_n q(\overline{x}_1,...,\overline{x}_n)$ if and only if $q^*(x^*) + q(\overline{x}_1,...,\overline{x}_n) = x^*(\overline{x}_1,...,\overline{x}_n)$; $x^* \in \partial_n q^{**}(\overline{x}_1,...,\overline{x}_n)$ if and only if $q^{***}(x^*) + q^{**}(\overline{x}_1,...,\overline{x}_n) = x^*(\overline{x}_1,...,\overline{x}_n)$.

Hence it follows the following corollary.

Corollary 2. *If* $q(\overline{x}_1,...,\overline{x}_n) = q^{**}(\overline{x}_1,...,\overline{x}_n)$, *then*

$$\partial_n q(\overline{x}_1,...,\overline{x}_n) = \partial_n q^{**}(\overline{x}_1,...,\overline{x}_n).$$

Corollary 3. *For any functions* $q : X_1 \times ... \times X_n \to \overline{R}$ *is* $x^* \in \partial_n q(\overline{x}_1,...,\overline{x}_n)$ *it follows that* $(\overline{x}_1,...,\overline{x}_n) \in \partial q^*(x^*)$.

Proof. As $x^* \in \partial_n q(\overline{x}_1, ..., \overline{x}_n)$, then we get $q^*(x^*)+q(\overline{x}_1, ..., \overline{x}_n) = x^*(\overline{x}_1, ..., \overline{x}_n)$. Also by lemma 2 we get that $q^{**}(\overline{x}_1, ..., \overline{x}_n) = q(\overline{x}_1, ..., \overline{x}_n)$. Then

$$q^*(x^*) + q^{**}(\overline{x}_1, ..., \overline{x}_n) = x^*(\overline{x}_1, ..., \overline{x}_n).$$

By lemma 3, hence it follows that $(\overline{x}_1, ..., \overline{x}_n) \in \partial q^*(x^*)$. The corollary is proved.

The sets $C_1, C_2 \subset (X_1 \times R) \times ... \times (X_n \times R)$ are called strict n-separable if there exist $x^* \in B(X_1 \times ... \times X_n, R)$ and $b, c \in R$, where $(x^*, b) \neq 0$, such that

$$x^*(x_1, ..., x_n) + b\alpha_1...\alpha_n > c \geq x^*(y_1, ..., y_n) + b\beta_1...\beta_n$$

for all $((x_1, \alpha_1), ..., (x_n, \beta_n)) \in C_1$, $((y_1, \beta_1), ..., (y_n, \beta_n)) \in C_2$.

Let $f_1 : X_1 \times ... \times X_n \to \overline{R}$, $f_2 : X_1 \times ... \times X_n \to R \cup \{-\infty\}$. Put

$$D_1 = \{((x_1, \alpha_1), ..., (x_n, \alpha_n)) \in$$

$$\in (X_1 \times R) \times ... \times (X_n \times R) : f_1(x_1, ..., x_n) < \alpha_1...\alpha_n\},$$

$$D_2 = \{((y_1, \beta_1), ..., (y_n, \beta_n)) \in$$

$$\in (X_1 \times R) \times ... \times (X_n \times R) : f_2(y_1, ..., y_n) > \beta_1...\beta_n\}.$$

Theorem 1. *Let $dom f_1 \cap dom|f_2| \neq \oslash$, D_1 and D_2 be strictly n-separable. Then there exists B-affine functional $b : X_1 \times ... \times X_n \to R$ such that*

$$f_1(x_1, ..., x_n) \geq b(x_1, ..., x_n) \text{ for all } (x_1, ..., x_n) \in X_1 \times ... \times X_n,$$

$$b(x_1, ..., x_n) \geq f_2(x_1, ..., x_n) \text{ for all } (x_1, ..., x_n) \in X_1 \times ... \times X_n.$$

Proof. As D_1 and D_2 are strictly n-separable, then there exists $x^* \in B(X_1 \times ... \times X_n, R)$ and $b \in R$, where $(x^*, b) \neq 0$, such that

$$x^*(x_1, ..., x_n) + b\alpha_1...\alpha_n > c \geq x^*(y_1, ..., y_n) + b\beta_1...\beta_n$$

for any $((x_1, \alpha_1), ..., (x_n, \beta_n)) \in D_1$, $((y_1, \beta_1), ..., (y_n, \beta_n)) \in D_2$. By condition $dom f_1 \cap dom|f_2| \neq \oslash$ we get that $b \neq 0$. Let $(\overline{x}_1, ..., \overline{x}_1) \in dom f_1 \cap dom|f_2|$. Then $b\overline{\alpha}_1...\overline{\alpha}_n > c \geq b\overline{\beta}_1...\overline{\beta}_n$ for $f_1(\overline{x}_1, ..., \overline{x}_n) < \overline{\alpha}_1...\overline{\alpha}_n$, $f_2(\overline{y}_1, ..., \overline{y}_n) > \overline{\beta}_1...\overline{\beta}_n$. Hence for $\overline{\alpha}_1...\overline{\alpha}_n \to +\infty$ $\left(\text{or } \overline{\beta}_1...\overline{\beta}_n \to -\infty\right)$ we get that $b > 0$. Putting $\overline{x}^*(x_1, ..., x_n) = -\frac{1}{b}x^*(x_1, ..., x_n)$, $\alpha_1...\alpha_n = f_1(x_1, ..., x_n)$ and $\beta_1...\beta_n = f_2(y_1, ..., y_n)$ we get that $f_1(x_1, ..., x_n) - \overline{x}^*(x_1, ..., x_n) \geq \frac{c}{b} \geq f_2(y_1, ..., y_n) - \overline{x}^*(y_1, ..., y_n)$ for all $(x_1, ..., x_n) \in X_1 \times ... \times X_n$ and $(y_1, ..., y_n) \in X_1 \times ... \times X_n$. If we put $b(x_1, ..., x_n) = \overline{x}^*(x_1, ..., x_n) + \frac{c}{b}$, then we obtain that the theorem is true.

The teorem is proved.

Corollary 4. If $f : X_1 \times ... \times X_n \to \overline{R}$, $(\overline{x}_1, ..., \overline{x}_n) \in dom\ f$, the sets $\{((x_1, \alpha_1), ..., (x_n, \alpha_n)) \in (X_1 \times R) \times ... \times (X_n \times R) : f(x_1, ..., x_n) < \alpha_1...\alpha_n\}$ and $((\overline{x}_1, f(\overline{x}_1, ..., \overline{x}_n)), (\overline{x}_2, 1), ..., (\overline{x}_n, 1))$ are strictly n-separable, then

$$\partial_n f(\overline{x}_1, ..., \overline{x}_n) \neq \oslash.$$

Proof. Denote $f_1(x_1,...,x_n) = f(x_1,...,x_n)$ and

$$f_2(x_1,...,x_n) = \begin{cases} f(\overline{x}_1,...,\overline{x}_n) : (x_1,...,x_n) = (\overline{x}_1,...,\overline{x}_n) \\ \\ -\infty \qquad\qquad : (x_1,...,x_n) \neq (\overline{x}_1,...,\overline{x}_n) \end{cases}$$

Then by theorem 1 there exists B - affine functional $b : X_1 \times ... \times X_n \to R$, where $b(x_1,...,x_n) = x^*(x_1,...,x_n) + b$, such that $f(x_1,...,x_n) \geq x^*(x_1,...,x_n) + b$ for all $(x_1,...,x_n) \in X_1 \times ... \times X_n$ and $x^*(\overline{x}_1,...,\overline{x}_n) + b \geq f(\overline{x}_1,...,\overline{x}_n)$. Summing these inequalities we obtain that $f(x_1,...,x_n) - f(\overline{x}_1,...,\overline{x}_n) \geq x^*(x_1,...,x_n) - x^*(\overline{x}_1,...,\overline{x}_n)$ for $(x_1,...,x_n) \in X_1 \times ... \times X_n$, i.e $x^* \in \partial_n f(\overline{x}_1,...,\overline{x}_n)$.The corollary is proved.

Lemma 5. *If* $q = q_1 + q_2$ *and* $x^* = x_1^* + x_2^*$, *then*

$$q^*(x^*) \leq \inf_{x^*=z_1^*+z_2^*} \{q_1^*(z_1^*) + q_2^*(z_2^*)\} \leq q_1^*(x_1^*) + q_2^*(x_2^*),$$

$$q^{**}(x,y) \geq q_1^{**}(x,y) + q_2^{**}(x,y).$$

Proof. If $z_1^*, z_2^* \in B(X_1 \times ... \times X_n, R)$ and $x^* = z_1^* + z_2^*$, then

$$q^*(x^*) = \sup_{(x_1,...,x_n)\in X_1\times....\times X_n} \{z_1^*(x_1,...,x_n) +$$

$$+z_2^*(x_1,...,x_n) - q_1(x_1,...,x_n) - q_2(x_1,...,x_n)\} \leq$$

$$\leq \sup_{(x_1,...,x_n)\in X_1\times....\times X_n} \{z_1^*(x_1,...,x_n) - q_1(x_1,...,x_n)\} +$$

$$+ \sup_{(x_1,...,x_n)\in X_1\times....\times X_n} \{z_2^*(x_1,...,x_n) - q_2(x_1,...,x_n)\} = q_1^*(z_1^*) + q_2^*(z_2^*).$$

Hence it follows that

$$q^*(x^*) \leq \inf_{z_1^*+z_2^*=x^*} \{q_1^*(z_1^*) + q_2^*(z_2^*)\} \leq q_1^*(x_1^*) + q_2^*(x_2^*).$$

Therefore

$$q^{**}(x_1,...,x_n) = \sup_{x^*\in B(X_1\times....\times X_n,R)} \{x^*(x_1,...,x_n) - q^*(x^*)\} \geq$$

$$\geq \sup_{x_1^*,x_2^*\in B(X_1\times....\times X_n,R)} \{x_1^*(x_1,...,x_n) + x_2^*(x_1,...,x_n) - q_1^*(x_1^*) - q_2^*(x_2^*)\} =$$

$$= \sup_{x_1^*\in B(X_1\times....\times X_n,R)} \{x_1^*(x_1,...,x_n) - q_1^*(x_1^*)\} +$$

$$+ \sup_{x_2^*\in B(X_1\times....\times X_n,R)} \{x_2^*(x_1,...,x_n) - q_2^*(x_2^*)\} =$$

$$= q_1^{**}(x_1,...,x_n) + q_2^{**}(x_1,...,x_n).$$

The lemma is proved.

Let $f_1 : X_1 \times ... \times X_n \to \overline{R}$ and $f_2 : X_1 \times ... \times X_n \to \overline{R}$. In lemma 5 it is proved that $(f_1 + f_2)^*(x_1^* + x_2^*) \leq f_1^*(x_1^*) + f_2^*(x_2^*)$. Let's show that in some case the inverse inequality is true is well.

Put that $(f_1+f_2)^*(x^*)=d<+\infty$. Consider the sets

$$A_1=\{((x_1,\alpha_1),...,(x_n,\alpha_n))\in$$

$$\in(X_1\times R)\times...\times(X_n\times R):f_1(x_1,...,x_n)<\alpha_1...\alpha_n\}$$

$$A_2=\{((y_1,\beta_1),...,(y_n,\beta_n))\in(X_1\times R)\times...\times(X_n\times R):\beta_1...\beta_n\leq$$

$$\leq x^*(x_1,...,x_n)-f_2(x_1,...,x_n)-d\}.$$

Show that $A_1\cap A_2=\oslash$. It is clear that, if $((x_1,\alpha_1),...,(x_n,\alpha_n))\in A_1\cap A_2$, then $f_1(x_1,...,x_n)<\alpha_1..\alpha_n\leq x^*(x_1,...,x_n)-f_2(x_1,...,x_n)-d$, i.e.

$$d<x^*(x_1,...,x_n)-f_1(x_1,...,x_n)-f_2(x_1,...,x_n)\leq(f_1+f_2)^*(x^*)=d$$

We obtained contradiction, i.e. $A_1\cap A_2=\oslash$. If the sets A_1 and A_2 are strictly n-separable, then there exists $x^*\in B(X_1\times...\times X_n,R)$ and $b\in R$, where $(x^*,b)\neq 0$, such that

$$x^*(x_1,...,x_n)+b\alpha_1\cdots\alpha_n>c\geq x^*(y_1,...,y_n)+b\beta_1\cdots\beta_n$$

for all $((x_1,\alpha_1),...,(x_n,\alpha_n))\in A_1$, $((y_1,\beta_1),...,(y_n,\beta_n))\in A_2$. Put that $dom f_1\cap dom f_2\neq\oslash$. Then by analogy of theorem 1 we obtain that $b>0$. Therefore

$$-\frac{1}{b}x^*(x_1,...,x_n)-\alpha_1\cdots\alpha_n<-\frac{c}{b}\leq-\frac{1}{b}x^*(y_1,...,y_n)-\beta_1\cdots\beta_n$$

for all $((x_1,\alpha_1),...,(x_n,\alpha_n))\in A_1$ and $((y_1,\beta_1),...,(y_n,\beta_n))\in A_2$. Having put $x_1^*=-\frac{1}{b}x^*$, we obtain

$$f_1^*(x_1^*)=\sup\{x_1^*(x_1,...,x_n)-f_1(x_1,...,x_n):(x_1,...,x_n)\in X_1\times...\times X_n\}=$$

$$=\sup\{x_1^*(x_1,...,x_n)-\alpha_1\cdots\alpha_n:((x_1,\alpha_1),...,(x_n,\alpha_n))\in A_1\}\leq$$

$$\leq\inf\{x_1^*(x_1,...,x_n)-\alpha_1\cdots\alpha_n:((x_1,\alpha_1),...,(x_n,\alpha_n))\in A_2\}=$$

$$=\inf\{x_1^*(x_1,...,x_n)-x^*(x_1,...,x_n)+f_2(x_1,...,x_n)+$$

$$+d:(x_1,...,x_n)\in dom f_2\}=-f_2^*(x^*-x_1^*)+d.$$

From these relations it follows the following inequality

$$(f_1+f_2)^*(x^*)\geq f_1^*(x_1^*)+f_2^*(x^*-x_1^*).$$

By lemma 5, we get

$$(f_1+f_2)^*(x^*)\leq f_1^*(x_1^*)+f_2^*(x^*-x_1^*).$$

Therefore we obtain that

$$(f_1+f_2)^*(x^*)=f_1^*(x_1^*)+f_2^*(x^*-x_1^*).$$

In particular, hence it follows that if $x^* \in \partial_n (f_1 + f_2)(\overline{x}_1, ..., \overline{x}_n)$, then $x_1^* \in \partial_n f_1 (\overline{x}_1, ..., \overline{x}_n)$ and $x^* - x_1^* \in \partial_n f_2 (\overline{x}_1, ..., \overline{x}_n)$.

Denote

$$\partial_2^k q(\overline{x}_1, ..., \overline{x}_n) = \{(x_1^*, .., x_n) \in X_1^* \times ... \times X_n^* : q(x_1, ..., x_n) - q(\overline{x}_1, ..., \overline{x}_n) \geq$$

$$\geq x_1^*(x_1) \cdots x_n^*(x_n) - x_1^*(\overline{x}_1) \cdots x_n^*(\overline{x}_n) \text{ for } (x_1, ..., x_n) \in X_1 \times ... \times X_n\}.$$

n-sublinear function $f(x_1, ..., x_n) = \sup\limits_{(x_1^*, ..., x_n^*) \in D} x_1^*(x_1) \cdots x_n^*(x_n)$, where $D \subset X_1^* \times ... \times X_n^*$, we call canonical. Consider a subclass of n-sublinear functions of kind $f(x_1, ..., x_n) = \sup\limits_{x_1^* \in D_1, ..., x_n^* \in D_n} x_1^*(x_1) \cdots x_n^*(x_n)$, where $D_1 \subset X_1^*, ..., D_n \subset X_n^*$.

Lemma 6. *If $\varphi_1 : X_1 \to \overline{R}, ..., \varphi_n : X_n \to \overline{R}$ are sublinear even lower semicontinuous functions, then $\varphi_1(x_1) ... \varphi_n(x_n) = \sup\limits_{x_1^* \in \partial\varphi_1, ..., x_n^* \in \partial\varphi_n} x_1^*(x_1) \cdots x_n^*(x_n)$ for $x_1 \in X_1, ..., x_n \in X_n$.*

Proof. From the Hormander theorem (see. [8]) it follows that $\varphi_1(x_1) = \sup\limits_{x_1^* \in \partial\varphi_1} x_1^*(x_1), ..., \varphi_n(x_n) = \sup\limits_{x_n^* \in \partial\varphi_n} x_n^*(x_n)$. As $\varphi_1, ..., \varphi_n$ are even functions, then from $x_1^* \in \partial\varphi_1, ..., x_n^* \in \partial\varphi_n$ it follows that $-x_1^* \in \partial\varphi_1, ..., -x_n^* \in \partial\varphi_n$. Therefore

$$\varphi_1(x_1) \cdots \varphi_n(x_n) = \sup_{x_1^* \in \partial\varphi_1} |x_1^*(x_1)| \cdots \sup_{x_n^* \in \partial\varphi_n} |x_n^*(x_n)| =$$

$$= \sup_{x_1^* \in \partial\varphi_1, ..., x_n^* \in \partial\varphi_n} |x_1^*(x_1)| \cdots |x_n^*(x_n)| = \sup_{x_1^* \in \partial\varphi_1, ..., x_n^* \in \partial\varphi_n} x_1^*(x_1) \cdots x_n^*(x_n).$$

The lemma is proved.

Note that if is $x_1^* \in D_1, ..., x_n^* \in D_n$ it follows $-x_1^* \in D_1, ..., -x_n^* \in D_n$, then $\sup\limits_{x_1^* \in D_1, ..., x_n^* \in D_n} x_1^*(x_1) \cdots x_n^*(x_n) = \sup\limits_{x_1^* \in D_1} x_1^*(x_1) \sup\limits_{x_n^* \in D_n} x_n^*(x_n)$.

Corollary 5. *Let $\varphi_1 : X_1 \to R, ..., \varphi_n : X_n \to R$ be sublinear continuos even functions, $X_1^0, ..., X_n^0$, be subspaces of the sets $X_1, ..., X_n$, respectively, linear functions $\overline{x}_1^* : X_1^0 \to R, ..., \overline{x}_n^* : X_n^0 \to R$ be such that $|\overline{x}_1^*(x_1)| \leq \varphi_1(x_1), ..., |\overline{x}_n^*(x_n)| \leq \varphi_n(x_n)$ for $x_1 \in X_1^0, ..., x_n \in X_n^0$. Then there exist linear continuos functions $x_1^* : X_1 \to R, ..., x_n^* : X_n \to R$, such that $x_1^*(x_1) = \overline{x}_1^*(x_1), ..., x_n^*(x_n) = \overline{x}_n^*(x_n)$ for $x_1 \in X_1^0, ..., x_n \in X_n^0$ and $|x_1^*(x_1) \cdots x_n^*(x_n)| \leq \varphi_1(x_1) \cdots \varphi_n(x_n)$ for $x_1 \in X_1, ..., x_n \in X_n$.*

Proof. By the Hahn-Banach theorem there exists linear functions $x_1^* : X_1 \to R, ..., x_n^* : X_n \to R$, such that $x_1^*(x_1) = \overline{x}_1^*(x_1), ..., x_n^*(x_n) = \overline{x}_n^*(x_n)$ for $x_1 \in X_1^0, ..., x_n \in X_n^0$ and $|\overline{x}_1^*(x_1)| \leq \varphi_1(x_1), ..., |\overline{x}_n^*(x_n)| \leq \varphi_n(x_n)$ for $x_1 \in X_1, ..., x_n \in X_n$. From continuity of $\varphi_1, ..., \varphi_n$ follows that $x_1^* : X_1 \to R, ..., x_n^* : X_n \to R$ is continuos. It is clear that $|x_1^*(x_1) \cdots x_n^*(x_n)| \leq \varphi_1(x_1) \cdots \varphi_n(x_n)$ for $(x_1, ..., x_n) \in X_1 \times ... \times X_n$. The corollary is proved.

Let $f : X_1 \times ... \times X_n \to \overline{R}$. Put

$$n - epf = \{((x_1, \alpha_1), ..., (x_n, \alpha_n)) \in$$

$\in (X_1 \times R) \times ... \times (X_n \times R) : f(x_1, ..., x_n) \le \alpha_1 \cdots \alpha_n\}$

The set $C \subset (X_1 \times R) \times ... \times (X_n \times R)$ is called B-convex, if for any $((\overline{x}_1, \overline{\alpha}_1), ..., (\overline{x}_n, \overline{\alpha}_n)) \notin C$ there exists $x^* \in B(X_1 \times ... \times X_n, R)$ and $b \in R$ such, that $x^*(x_1, ..., x_n) - b\alpha_1 \cdots \alpha_n < x^*(\overline{x}_1, ..., \overline{x}_n) - b\overline{\alpha}_1 \cdots \overline{\alpha}_n$ for $((x_1, \alpha_1), ..., x_n, \alpha_n) \in C$.

The set $C \subset (X_1 \times R) \times ... \times (X_n \times R)$ is called strongly B-convex, if for any $((\overline{x}_1, \overline{\alpha}_1), ..., (\overline{x}_n, \overline{\alpha}_n)) \notin C$, there exists $x^* \in B(X_1 \times ... \times X_n, R)$, $b \in R$ and $\varepsilon > 0$ such, that $x^*(x_1, ..., x_n) - b\alpha_1 \cdots \alpha_n \le x^*(\overline{x}_1, ..., \overline{x}_n) - b\overline{\alpha}_1 \cdots \overline{\alpha}_n - \varepsilon$ for $((x_1, \alpha_1), ..., (x_n \alpha_n)) \in C$.

Theorem 2. *If $n - epf$ is B-convex and $dom f = X_1 \times ... \times X_n$, then $f^{**} = f$.*

Proof. Let $((\overline{x}_1, \overline{\alpha}_1), ..., (\overline{x}_n, \overline{\alpha}_n)) \notin n - epf$. Then by definition of B-convex sets there exist $x^* \in B(X_1 \times ... \times X_n, R)$ and $b \in R$, that $x^*(x_1, ..., x_n) - b\alpha_1 \cdots \alpha_n < x^*(\overline{x}_1, ..., \overline{x}_n) - b\overline{\alpha}_1, ..., \overline{\alpha}_n$ for $((x_1, \alpha_1), ..., (x_n, \alpha_n)) \in n - epf$. Putting $\alpha_1 \cdots \alpha_n = f(\overline{x}_1, ..., \overline{x}_n)$ we obtain that $x^*(\overline{x}_1, ..., \overline{x}_n) - bf(\overline{x}_1, ..., \overline{x}_n) < x^*(\overline{x}_1, ..., \overline{x}_n) - b\overline{\alpha}_1 \cdots \overline{\alpha}_n$, i.e. $b(f(\overline{x}_1, ..., \overline{x}_n) - \overline{\alpha}_1 \cdots \overline{\alpha}_n) > 0$. Hence it follows that $b > 0$. Then we get that $\frac{1}{b}x^*(x_1, ..., x_n) - \alpha_1 \cdots \alpha_n < \frac{1}{b}x^*(\overline{x}_1, ..., \overline{x}_n) - \overline{\alpha}_1 \cdots \overline{\alpha}_n$ for $((x_1, \alpha_1), ..., (x_n, \alpha_n)) \in n - epf$. Putting $\alpha_1 \cdots \alpha_n = f(x_1, ..., x_n)$, $\overline{x}^*(x_1, ..., x_n) = \frac{1}{b}x^*(x_1, ..., x_n)$. It is clear that $\overline{x}^*(x_1, ..., x_n) - f(x_1, ..., x_n) < \overline{x}^*(\overline{x}_1, ..., \overline{x}_n) - \overline{\alpha}_1 \cdots \overline{\alpha}_n$ for $(x_1, ..., x_n) \in X_1 \times ... \times X_n$.

Therefore $\sup\limits_{(x_1,...,x_n) \in X_1 \times ... \times X_n} \{\overline{x}^*(x_1, ..., x_n) - f(x_1, ..., x_n)\} \le \overline{x}^*(x_1, ..., x_n) - \alpha_1 \cdots \alpha_n$, i.e. $f^*(\overline{x}^*) \le \overline{x}^*(\overline{x}_1, ..., \overline{x}_n) - \overline{\alpha}_1 \cdots \overline{\alpha}_n$. Then $\overline{\alpha}_1 \cdots \overline{\alpha}_n \le \overline{x}^*(\overline{x}_1, ..., \overline{x}_n) - f^*(\overline{x}^*) \le f^{**}(\overline{x}_1, ..., \overline{x}_n)$. We get that $\overline{\alpha}_1 \cdots \overline{\alpha}_n \le f^{**}(\overline{x}_1, ..., \overline{x}_n) \le f(\overline{x}_1, ..., \overline{x}_n)$. Therefore $f^{**}(\overline{x}_1, ..., \overline{x}_n) = f(\overline{x}_1, ..., \overline{x}_n)$. The theorem is proved.

Theorem 3. *If $n - epf$ is strongly B-convex and $dom\ f \neq \oslash$, then $f^{**} = f$.*

Proof. If $((\overline{x}_1, \overline{\alpha}_1), ..., (\overline{x}_n, \overline{\alpha}_n)) \notin n - epf$, then by definition of strong B-convex sets there exists $x^* \in B(X_1 \times ... \times X_n, R)$, $b \in R$ and $\varepsilon > 0$ which that

$$x^*(x_1, ..., x_n) - b\alpha_1 \cdots \alpha_n \le x^*(\overline{x}_1, ..., \overline{x}_n) - b\overline{\alpha}_1 \cdots \overline{\alpha}_n - \varepsilon$$

for $((x_1, \alpha_1), ..., (x_n, \alpha_n)) \in n - epf$. Putting $(x_1, ..., x_n) = (\tilde{x}_1, ..., \tilde{x}_n) \in dom\ f$ and $f(\tilde{x}_1, ..., \tilde{x}_n) \le \alpha_1 \cdots \alpha_n$, then for $\alpha_1 \cdots \alpha_n \to +\infty$ we get that $b \ge 0$.

If $b > 0$, then by analogy of the proof of theorem 2 we get that $\overline{x}^* = \frac{1}{b}x^* \in dom\ f^*$ and $f^{**} = f$. Also by analogy of the proof of theorem 2 we get that $b > 0$ for $(\overline{x}_1, ..., \overline{x}_n) \in dom\ f$.

Let $(\overline{x}_1, ..., \overline{x}_n) \notin dom\ f$. If $b > 0$, then we get that $f^{**} = f$. If $b = 0$, then $x^*(x_1, ..., x_n) \le x^*(\overline{x}_1, ..., \overline{x}_n) - \varepsilon$ for $(x_1, ..., x_n) \in dom\ f$. Let $\overline{x}^* \in dom\ f^*$. Then $f^*(\overline{x}^*) \ge \overline{x}^*(\overline{x}_1, ..., \overline{x}_n) - f(x_1, ..., x_n)$. Therefore

$$\overline{x}^*(x_1, ..., x_n) + \mu x^*(x_1, ..., x_n) - f(x_1, ..., x_n) \le f^*(\overline{x}^*) + \mu x^*(\overline{x}_1, ..., \overline{x}_n) - \mu\varepsilon$$

for $\mu > 0$. Hence it follows that

$$f^*(\overline{x}^* + \mu x^*) \le f^*(\overline{x}^*) + \mu x^*(\overline{x}_1, ..., \overline{x}_n) - \mu\varepsilon,$$

which we can write in this form

$$\overline{x}^*(\overline{x}_1,...,\overline{x}_n)+\mu\varepsilon-f^*(\overline{x}^*)\leq(\overline{x}^*+\mu x^*)(\overline{x}_1,...,\overline{x}_n)-f^*(\overline{x}^*+\mu x^*)\leq f^{**}(\overline{x}_1,...,\overline{x}_n)$$

Putting here $\mu=\dfrac{a+f^*(\overline{x}^*)-\overline{x}^*(\overline{x}_1,...,\overline{x}_n)}{\varepsilon}$, we get that $a\leq f^{**}(\overline{x}_1,...,\overline{x}_n)$ for sufficiently large a. Therefore $f^{**}(\overline{x}_1,...,\overline{x}_n)=+\infty$. The theorem is proved.

If $n-epf$ is B-convex (strongly B-convex) set, then f is called B-convex (strongly B-convex) function. It is easy to verify that sum of finite number of B-convex functions is also B-convex. Therefore , if f is B-convex function and $x^*\in B(X_1\times...\times X_n,R)$, then $f+x^*$ is also a B-convex function.

If $f\in AB_0(X_1\times...\times X_n)$, i.e. there exist the set I that, $f(x_1,...,x_n)=\sup\limits_{\alpha\in I}(x^*_\alpha(x_1,...,x_n)+c_\alpha)$, where $x^*_\alpha\in AB_0(X_1\times...\times X_n,R)$, $c_\alpha\in R$ and $dom\ f\neq\oslash$, then $g(\nu)=\sup\limits_{\alpha\in I}\left(\sum\limits_i x^*_\alpha\left(x^i_1,...,x^i_n\right)+c_\alpha\right)$, $\nu=\sum\limits_i x^i_1\otimes...\otimes x^i_n$, is a proper convex lower semicontinuous function on $X_1\otimes...\otimes X_n$ and $g(x_1\otimes...\otimes x_n)=f(x_1,...,x_n)$ for $(x_1,...,x_n)\in X_1\times...\times X_n$. Therefore from $((\overline{x}_1,\overline{\alpha}_1),...,(\overline{x}_n,\overline{\alpha}_n))\notin n-epf$ it follows that $g(\overline{x}_1\otimes...\otimes\overline{x}_n)>\overline{\alpha}_1\cdots\overline{\alpha}_n$, i.e.

$$(\overline{x}_1\otimes...\otimes\overline{x}_n,\overline{\alpha}_1...\overline{\alpha}_n)\notin epg=\{(\nu,a)\in(X_1\otimes...\otimes X_n)\times R:g(\nu)\leq a\}.$$

Then by theorem 2.9.2 [6] $(\overline{x}_1\otimes...\otimes\overline{x}_n,\overline{\alpha}_1...\overline{\alpha}_n)$ and epg is strong by separable. Hence it follows that $n-epf$ is strongly B-convex, and by theorem 3 $f^{**}=f$. We get that the following Corollary is true.

Corollary 6. *If* $f\in AB_0(X_1\times...\times X_n)$, *then* $f^{**}=f$.

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