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# ON FOURIER TRANSFORMATION OF SOME CLASSES OF SQUARE SUMMABLE FUNCTIONS ON A HILBERT SPACE WITH GAUSS MEASURE

**Abstract**

*In the paper we distinguish three classes of functions possessing n-th order derivatives with respect to finite and denumerable number of directions with square summable different expressions.*

*Necessary and sufficient conditions are imposed on an analytic function of a Hilbert space so that it be a Fourier transformation with respect to Gauss measure of the indicated classes of functions.*

**Introduction.** Let $X$ be a Hilbert space with a scalar product $(x,y)$, $x,y\in X$, $F-\sigma$ be algebra of Borel sets from $X,\mu$ be a Gauss measure on F given by the characteristic functional $\varphi_0(z)=\exp\left\{-\frac{1}{2}(Bz,z)\right\}$, where $B$ is positive kernel operator. By $L_2(X,\mu)$ we denote a space of square summable functions on $X$. The function $f(x)\in L_2(X,\mu)$ determined by the formula $\varphi(z)=\int e^{i(z,x)}f(x)\mu(dx)$ is said to be a Fourier transformation of the function $\varphi(x)$.

It is easy to establish that $\varphi(z)$ is extendable on complex extension of the space $X$ and $\varphi(x+\lambda y)$ is an entire analytic function with respect to a complex variable $\lambda$ for any fixed $x,y\in X$, at each point of $x\in X$ has a Frechet derivative $\varphi^{(k)}(x;y_1,y_2,...,y_k)$ that is a bounded $k$ variable form. In [1] the following inverse problem is solved: under which conditions the entire analytic functions are the Fourier transformations of the functions from $L_2(X,\mu)$ and of some narrow subclasses. In the present paper we find necessary and sufficient conditions on an entire analytic function $\varphi(z)$ for it to be transformation of the following classes:

**1.** Of functions $f(x)\in L_2(X,\mu)$ for which

$$\sum_{m_1,m_2,...,m_r=1}^{\infty}\int\left[f^{(r)}(x;e_1^{m_1},e_2^{m_2},...,e_r^{m_r})\right]^2\mu(dx)<\infty$$

where $\{e_1^{m_1}\},\{e_2^{m_2}\}$,...,$\{e_r^{m_r}\}$ are some systems of vectors in $X,m_1,m_2,...,m_r=1,2,3...$

**2.** Of functions of the form $f(x)\left\|x\right\|^m$, $f(x)\in L_2(X,\mu)$

**3.** Of the functions $f(x)\in L_2(X,\mu)$ for which

$$\left(Sp\left[f^{(r)}(x;\cdot)^2\right]\right)^{\frac{1}{2}}\left\|x\right\|^q\in L_2(X,\mu)$$

where $Sp\left[f^{(r)}(x;\cdot)^2\right]=\sum_{i_1,i_2,...,i_r=1}^{\infty}\left[f^{(r)}(x;h_1,h_2,...,h_{i_r}=1)\right]^2$ and $\{h_i\}$ is some orthonormed basis in $X$.

**1.** To each polynomial function

$$P_n(x) = \sum_{k=1}^{n} \sum_{i_1,i_2,...,i_r=1}^{n} c_{i_1 i_2 ... i_k}(x, e_{i_1})(x, e_{i_2})...(x, e_{i_k}),$$

where $n \geq 1$, $c_{i_1 i_2 ... i_k}$ are the numbers $e_{i_1}\ , e_{i_2}, ..., e_{i_k} \in X$ we associate a differential operator:

$$P_n\left(\frac{1}{i}\frac{d}{dx}\right)\varphi(z) = \sum_{k=0}^{n} \sum_{i_1,i_2,...,i_r=1}^{n} c_{i_1 i_2 ... i_k}\varphi^{(k)}(z; e_{i_1}, e_{i_2}, ..., e_{i_k})$$

**Theorem 1.** *In order the analytic function $\varphi(z)$ on $X$ be a Fourier transformation of the function $f(x) \in L_2(X, \mu)$ for which*

$$\sum_{m_1,m_2,...,m_r=1}^{\infty} \int \left[f^{(r)}(x; e_1^{m_1}, e_2^{m_2}, ..., e_r^{m_r})\right]^2 \mu(dx) < \infty \tag{1}$$

*where $\{e_1^{m_1}\}, \{e_2^{m_2}\}$,...,$\{e_r^{m_r}\}$ are some systems of vectors in $X$, it is sufficient and necessary that*

**1.** *There exist a constant $C > 0$ such that for any polynomial $P_n(x)$*

$$\left| P_n\left(\frac{1}{i}\frac{d}{dx}\right)\varphi(z)\Big|_{z=0}\right|^2 \leq C \int P_n^2(x)\mu(dx)$$

**2.** *There exist constants $c_{m_1 m_2 ... m_k} > 0$ such that*

$$\sum_{m_1,m_2,...,m_r=1}^{\infty} c_{m_1 m_2 ... m_k} < \infty,$$

*and for any polynomial $P_n(x)$ the linear functionals*

$$l_{m_1 m_2 ... m_k}(P_n) = \left[\sum P^{r_1}_{n, e_{i_1}^{m_{i_1}}, ..., e_{r_1}^{m_{r_1}}}\left(\frac{1}{i}\frac{d}{dz}\right)\prod_{v=1}^{r_2} \times\right.$$

$$\left.\times\left(\frac{1}{i}\frac{d}{dz}; B^{-1}e_{j_v}^{m_{j_v}}\right)\prod_{\mu=1}^{r_3}\left(B^{-1}e_{k_{\mu-1}}^{m_{k_{\mu-1}}}, e_{k_\mu}^{m_{k_\mu}}\right)\right] \cdot \varphi(z)\,|_{z=0}$$

*be restricted*

$$|l_{m_1 m_2 ... m_k}(P_n)|^2 \leq c_{m_1 m_2 ... m_k} \int P_n^2(x)\mu(dx)$$

*where summation is taken over all collections*

$$(i, ..., i_{r_1})U(j_1, ..., j_{r_1}) \cup (k_1, ..., k_{r_1}) = (1, 2, 3, ..., r)$$

*and appropriate*

$$(m_{i_1}, m_{i_2} ..., m_{i_{r_1}}) \cup (m_{j_1}, m_{j_2} ..., m_{j_{r_2}})(m_{k_1}, m_{k_2} ..., m_{k_{r_3}}) = (m_1, m_2 ..., m_r)$$

**Proof.** On the proof of the first statement see[1].

Let's prove the second part.

Necessity:

Let for $f(x) \in L_2(X,\mu)$ it hold (1). Then $f^{(r)}(x; e_1^{m_1}, e_2^{m_2}, ..., e_r^{m_r}) \in L_2(X,\mu)$ and the Fourier transformation

$$\psi(z; e_1^{m_1}, e_2^{m_2}, ..., e_r^{m_r}) = \int e^{i(z,x)} f^{(r)}(x; e_1^{m_1}, e_2^{m_2}, ..., e_r^{m_r}) \mu(dx) \tag{2}$$

is determined for it.

Acting by the operator $P_n\left(\frac{1}{i}\frac{d}{dx}\right)$ on both hand sides of (2) and equating to $z=0$ we get

$$\begin{gathered} P_n\left(\frac{1}{i}\frac{d}{dx}\right)\psi(z; e_1^{m_1}, e_2^{m_2}, ..., e_r^{m_r})\,|_{z=0} = \\ = \int f^{(r)}(x; e_1^{m_1}, e_2^{m_2}, ..., e_r^{m_r}) P_n(x)\mu(dx) \end{gathered} \tag{3}$$

Hence

$$\begin{gathered} \left| P_n\left(\frac{1}{i}\frac{d}{dx}\right)\psi(z; e_1^{m_1}, e_2^{m_2}, ..., e_r^{m_r})\right|_{z=0}\Bigg|^2 \le \\ \le \int \left[f^{(r)}(x; e_1^{m_1}, e_2^{m_2}, ..., e_r^{m_r})\right]^2 \mu(dx) \cdot \int P_n^2(x)\mu(dx) \end{gathered} \tag{4}$$

Integrating the first hand side of (2) by parts ([2]) and acting by the operator $P_n\left(\frac{1}{i}\frac{d}{dx}\right)$ on the both hand sides after integration by parts we get

$$P_n\left(\frac{1}{i}\frac{d}{dx}\right)\psi(z; e_1^{m_1}, e_2^{m_2}, ..., e_r^{m_r})\Big|_{z=0} =$$

$$= \left[\sum P^{(r_1)}_{n, e_{i_1}^{m_{i_1}}, ..., e_{r_1}^{m_{r_1}}}\left(\frac{1}{i}\frac{d}{dz}\right)\prod_{v=1}^{r_2}\left(\frac{1}{i}\frac{d}{dz}; B^{-1} e_{j_v}^{m_{j_v}}\right)\times\right.$$

$$\left.\times\prod_{\mu=1}^{r_3}\left(B^{-1} e_{k_{\mu-1}}^{m_{k_{\mu-1}}}, e_{k_\mu}^{m_{k_\mu}}\right)\right]\cdot\varphi(z)\Bigg|_{z=0} = l_{m_1 m_2 ... m_k}(P_n).$$

Taking into account (3) and (4) we get

$$|l_{m_1 m_2 ... m_k}(P_n)|^2 = \left|P_n\left(\frac{1}{i}\frac{d}{dx}\right)\psi(z; e_1^{m_1}, e_2^{m_2}, ..., e_r^{m_r})\right|_{z=0}\Bigg|^2 \le$$

$$= \int \left[f^{(r)}(x; e_1^{m_1}, e_2^{m_2}, ..., e_r^{m_r})\right]^2 \cdot \mu(dx) \cdot \int P_n^2(x)\mu(dx)$$

Having denoted

$$c_{m_1 m_2 ... m_k} = \int \left[f^{(r)}(x; e_1^{m_1}, e_2^{m_2}, ..., e_r^{m_r})\right]^2 \mu(dx)$$

we get $|l_{m_1m_2...m_k}(P_n)|^2 \leq c_{m_1m_2...m_k}\|P_n\|^2$ and $\sum\limits_{m_1,m_2,...,m_r\geq 1}^{n} c_{m_1m_2...m_r} < \infty$

**Sufficiency.**

Let is given $|l_{m_1m_2...m_k}(P_n)|^2 \leq c_{m_1m_2...m_k}\|P_n\|^2$ and

$$\sum_{m_1,m_2,...,m_r\geq 1}^{n} c_{m_1m_2...m_r} < \infty.$$

Then it is known that ([2]) this functional has the representation

$$l_{m_1m_2...m_k}(P_n) = \int f^{(r)}(x; e_1^{m_1}, e_2^{m_2},...,e_r^{m_r})P_n(x)\mu(dx)$$

where $f^{(r)}(x; e_1^{m_1}, e_2^{m_2},...,e_r^{m_r})$ is a generalized derivative of $f(x)$ and is square summable with respect to measure $\mu$. From the general theory on representation of a linear continuous functional it is known that the norm

$$\left\|f^{(r)}(x; e_1^{m_1}, e_2^{m_2},...,e_r^{m_r})\right\|^2 \leq c_{m_1m_2...m_k}$$

Consequently,

$$\sum_{m_1,m_2,...,m_r\geq 1}^{\infty} \int \left[f^{(r)}(x; e_1^{m_1}, e_2^{m_2}, ..., e_r^{m_r})\right]^2 \mu(dx) =$$

$$= \sum \left\|f^{(r)}(x; e_1^{m_1}, e_2^{m_2}, ..., e_r^{m_r})\right\|^2 \leq \sum_{m_1,m_2,...,m_r\geq 1} c_{m_1m_2...m_r} < \infty.$$

**Theorem 2.** *In order an analytic function $\varphi(z)$ be a Fourier transformation of $f(x)$ on $X$ such that $f(x)\|x\|^m \in L_2(X,\mu)$, it is necessary and sufficient that there exist the constants $c_{i_1i_2...i_r}$, $\sum\limits_{i_1,i_2,...,i_m}^{\infty} c_{i_1,i_2,...,i_m} < \infty$ such that*

$$\left|\prod_{s=1}^{m}\left(\frac{1}{i}\frac{d}{dz}; e_{i_s}\right) P_n\left(\frac{1}{i}\frac{d}{dx}\right)\varphi(z)\Big|_{z=0}\right|^2 \leq c_{i_1i_2...i_m}\int P_n^2(x)\mu(dx), \qquad (5)$$

*where $\{e_i\}$ is some orthonormed basis in $X$.*

**Proof.** Necessity. Let $\varphi(z)$ be a Fourier transformation of the function $f(x)$. Then

$$\varphi(z) = \int e^{i(z,x)} f(x)\mu(dx)$$

and

$$\prod_{s=1}^{m}\left(\frac{1}{i}\frac{d}{dz}; e_{i_s}\right) P_n\left(\frac{1}{i}\frac{d}{dx}\right)\varphi(z) =$$

$$= \int e^{i(z,x)} f(x)\cdot P_n(x)\cdot (x, e_{i_1})(x, e_{i_2})...(x, e_{i_s})\mu(dx).$$

Consequently,

$$\prod_{s=1}^{m}\left(\frac{1}{i}\frac{d}{dz};e_{i_s}\right)P_n\left(\frac{1}{i}\frac{d}{dx}\right)\varphi(z)\,|_{z=0}=$$

$$=\int f(x)\cdot P_n(x)\cdot(x,e_{i_1})(x,e_{i_2})...(x,e_{i_s})\mu(dx).$$

Taking modulus of both hand sides and having applied the Schwartz inequality we get (5), where

$$c_{i_1i_2...i_m}=\int f^2(x)\cdot(x,e_{i_1})^2(x,e_{i_2})^2...(x,e_{i_s})^2\mu(dx).$$

Having summed up with respect to $i_1,i_2,...,i_k=1,2,...$ we find

$$\sum_{i_1,i_2,...,i_m}^{\infty}c_{i_1,i_2,...,i_m}=\int f^2(x)\cdot\sum_{i=1}^{\infty}(x,e_{i_1})^2...\sum_{i_m}^{\infty}(x,e_{i_s})^2\mu(dx)=$$

$$=\int f^2(x)\cdot\|x\|^2\cdot...\cdot\|x\|^2\,\mu(dx)=\int f^2(x)\cdot\|x\|^{2m}\,\mu(dx)<\infty.$$

**Sufficiency.**

Let (1) be fulfilled and $\sum\limits_{i_1,i_2,...,i_m\geq 1}^{\infty}c_{i_1i_2...i_m}<\infty$. The linear bounded functional

$$l_\varphi(P_n)=\prod_{s=1}^{m}\left(\frac{1}{i}\frac{d}{dz};e_{i_s}\right)P_n\left(\frac{1}{i}\frac{d}{dx}\right)\varphi(z)\,|_{z=0}$$

is determined on all polynomials $\{P_n(x)\}$. Consequently, it has continuation on $L_2(X,\mu)$ with the same constants $c_{i_1i_2...i_m}>0$. Then $l_\varphi(P_n)$ has a representation $l_\varphi(P_n)=\int\rho_{i_1...i_m}(x)P_n(x)\mu(dx)$, where $\rho_{i_1...i_m}(x)\in L_2(X,\mu)$ and $\int\rho^2_{i_1...i_m}(x)\mu(dx)=c_{i_1i_2...i_m}$.

Let's consider $\psi(z)=\int e^{i(z,x)}\rho_{i_1...i_m}(x)\mu(dx)$. Then $\psi(z)$ is an analytical function and for any polynomial $P_n(x)$

$$l_\varphi(P_n)=P_n\left(\frac{1}{i}\frac{d}{dx}\right)\psi(z)\bigg|_{z=0}=\int\rho_{i_1...i_m}(x)P_n(x)\mu(dx).$$

On the other hand

$$l_\varphi(P_n)=\prod_{s=1}^{m}\left(\frac{1}{i}\frac{d}{dz};e_{i_s}\right)P_n\left(\frac{1}{i}\frac{d}{dx}\right)\varphi(z)\bigg|_{z=0}=$$

$$=\int f(x)\cdot P_n(x)\cdot(x,e_{i_1})(x,e_{i_2})...(x,e_{i_s})\mu(dx).$$

By the uniqueness theorem, from these equalities we'll have

$$\rho_{i_1...i_m}(x)=f(x)\cdot(x,e_{i_1})(x,e_{i_2})...(x,e_{i_s})\pmod{\mu}$$

and

$$\int f^2(x)\cdot (x,e_{i_1})^2 (x,e_{i_2})^2 ... (x,e_{i_s})^2 \mu(dx) \le c_{i_1 i_2 ... i_m}.$$

Then

$$\int f^2(x)\cdot \|x\|^2 \mu(dx) =$$

$$= \sum \int f^2(x)\cdot (x,e_{i_1})^2 (x,e_{i_2})^2 ... (x,e_{i_s})^2 \mu(dx) =$$

$$= \sum_{i_1,i_2,...,i_m \ge 1}^{\infty} c_{i_1 i_2 ... i_m} < \infty.$$

The $(k+l)-$linear form $f^{(k)}(x;h_1,h_2,...,h_k) \times g^{(l)}(x;h_1^1,h_2^1,...,h_l^1)$ is called a product of two derivatives $f^{(k)}(x;h_1,h_2,...,h_k)$ and $g^{(k)}(x;h_1^1,h_2^1,...,h_l^1)$.

We consider the expression

$$\sum \left[ f^{(k)}(x;h_{i_1},h_{i_2},...,h_{i_k}) \right]^2,$$

where summation is taken with respect to some orthonormal basis $\{h_i\}$ of the space $X$. It this sum is finite for all bases its quantity is independent of the chosen basis, is said to be a trace of $2k$-linear form and denoted by $Sp\left[f^{(k)}(x;\cdot)\right]^2$.

**Theorem 3.** *In order $\varphi(z)$ be a Fourier transformation of the function $f(x)$ such that*

$$\left( Sp\left[ f^{(r)}(x;\cdot) \right]^2 \right)^{\frac{1}{2}} \|x\|^q \in L_2(X,\mu)$$

*it is necessary and sufficient that there exist the constants $c_{m_1,m_2,...,m_r l_1 ... l_q} > 0$ such that $\sum\limits_{\substack{m_1,m_2,...,m_r=1 \\ l_1,l_2,...,l_r=1}}^{\infty} c_{m_1 m_2 ... m_r l_1 l_2 ... l_q} < \infty$, and for any polynomial $P_n(x)$, $n \ge 1$.*

$$\left| \prod_{s=1}^{m} \left( \frac{1}{i}\frac{d}{dz}; e_{l_s} \right) \left[ \sum P^{(r_1)}_{n,a_{i_1}^{m_{i_1}},...,a_{r_1}^{m_{r_1}}} \left( \frac{1}{i}\frac{d}{dz} \right) \prod_{v=1}^{r_2} \left( \frac{1}{i}\frac{d}{dz}; B^{-1} a_{j_v}^{m_{j_v}} \right) \times \right.\right.$$

$$\left.\left. \times \prod_{\mu=1}^{r_3} \left( B^{-1} a_{k_{\mu-1}}^{m_{k_{\mu-1}}}, a_{k_\mu}^{m_{k_\mu}} \right) \right] \cdot \varphi(z) \right|_{z=0} \Bigg|^2 \le c_{m_1 m_2 ... m_r l_1 l_2 ... l_q} \int P_n^2(x)\mu(dx),$$

*where summation is taken over all collections*

$$(i_1,...,i_{r_1}) \cup (j_1,...,j_{r_2}) \cup (k_1,...,k_{r_3}) = (1,2,3,...,r)$$

*and appropriate*

$$(m_{i_1},m_{i_2},...,m_{i_{r_1}}) \cup (m_{j_1},m_{j_2},...,m_{j_{r_2}})(m_{k_1},m_{k_2},...,m_{k_{r_3}}) = (m_1,m_2,...,m_r)$$

$$r_1+r_2+r_3 = r \quad \{e_l\},\ l=1,2,... \quad \{a_1^{m_1}\},\{a_2^{m_2}\},...,\{a_r^{m_r}\}$$

*are orthonormed bases.*

**Proof. Sufficiency.** Denote by

$$A(e_{l_1}, e_{l_2}, ..., e_{l_q}, a_1^{m_1}, a_2^{m2},..., a_r^{m_r})$$

a linear operator acting on $\varphi(z)$ by the formula

$$A(e_{l_1}, e_{l_2}, ..., e_{l_q}, a_1^{m_1}, a_2^{m2},..., a_r^{m_r})\varphi(z) =$$

$$= \prod_{s=1}^{q}\left(\frac{1}{i}\frac{d}{dz}; e_{l_s}\right)\left[\sum P^{(r_1)}_{n, a_{i_1}^{m_{i_1}},...,a_{r_1}^{m_{r_1}}}\left(\frac{1}{i}\frac{d}{dz}\right)\prod_{v=1}^{r_2}\left(\frac{1}{i}\frac{d}{dz}; B^{-1}a_{j_v}^{m_{j_v}}\right)\times\right.$$

$$\left.\times\prod_{\mu=1}^{r_3}\left(B^{-1}a_{k_{\mu-1}}^{m_{k_{\mu-1}}}, a_{k_\mu}^{m_{k_\mu}}\right)\right]\varphi(z).$$

Then

$$l_\varphi(P_n) = A(e_{l_1}, e_{l_2},..., e_{l_q}, a_1^{m_1}, a_2^{m2},..., a_r^{m_r})\varphi(z)\mid_{z=0}$$

is a linear bounded functional on a system of all polynomials $\{P_n(x)\}_{n\geq 1}$. It has a unique continuation on $L_2(X,\mu)$ with the same constant $c_{m_1 m_2 ... m_r l_1 l_2 ... l_q} > 0$. By the theorem on representation of a linear bounded functional $l_\varphi(P_n)$ there exists

$$\rho_{m_1 m_2 ... m_r l_1 ... l_q}(x) \in L_2(X,\mu)$$

that

$$l_\varphi(P_n) = \int \rho_{m_1 m_2 ... m_r l_1 ... l_q}(x) P_n(x)\mu(dx).$$

On the other hand, it is known [2] that

$$A(e_{l_1}, e_{l_2}, ..., e_{l_q}, a_1^{m_1}, a_2^{m2},..., a_r^{m_r})\varphi(z) =$$

$$= \int e^{i(z,x)} f^{(r)}(x; a_1^{m_1}, a_2^{m_2},..., a_r^{m_r}) \cdot (x, e_{l_1})(x, e_{l_2})...(x, e_{l_q})\mu(dx).$$

Then by the theorem on uniqueness of the representation

$$\rho(x)_{m_1 m_2 ... m_r l_1 ... l_q} = f^r(x; a_1^{m_1}, a_2^{m_2},..., a_r^{m_r}) \cdot (x, e_{l_1})(x, e_{l_2})...(x, e_{l_q}) \pmod{\mu}.$$

Consequently

$$\int [f^r(x; a_1^{m_1}, a_2^{m_2},..., a_r^{m_r})]^2 \cdot (x, e_{l_1})^2(x, e_{l_2})^2...(x, e_{l_q})^2\mu(dx) \leq c_{m_1 m_2 ... m_r l_1 ... l_q}.$$

Having summed with respect to $m_1, m_2, ..., m_r,\ l_1, l_2, ..., l_q = 1, 2, ...$ we get

$$\int\sum\left[f^{(r)}(x; a_1^{m_1}, a_2^{m_2},..., a_r^{m_r})\right]^2\sum_{l_1=1}^{\infty}(x, e_{l_1})^2 ... \sum_{l_q=1}^{\infty}(x, e_{l_q})^2\mu(dx) \leq$$

$$\leq \sum_{\substack{m_1, m_2, ..., m_r = 1\\ l_1, l_2, ..., l_r = 1}}^{\infty} c_{m_1 m_2 ... m_r l_1 l_2 ... l_q}.$$

Consequently

$$\int Sp\left[f^{(r)}(x; \cdot)\right]^2 \|x\|^{2q}\mu(dx) < \infty.$$

The necessity is proved in [1].

## References

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