Elshad Kh. EYVAZOV

ON DEFECT INDICES OF ONE-CENTRE DIFFERENTIAL OPERATORS

Abstract

We obtain a formula for calculating defect numbers of one-center differential operators of higher order.

In references on physics (see f.e. [1]) devoted to the problems of atomic and nuclear physics and solid body physics the $\delta-$interactions model of the form

$$-\Delta+\varepsilon\delta\left(x\right), \tag{1}$$

where Δ is Laplacian, ε is a relation constant, $\delta\left(x\right)-\delta$ is a Dirac function were widely discussed F.A. Berezin and L.D. Faddeev first in the paper [2] mathematically motivated that there exists one-parametric property of self-adjoint operators responding to euristic expression (1). After appearance of the monograph [3] the number of papers on spectral theory of singular differential operators significatly increased. Notice that in the mentioned monograph [3] only the classes of solvable models of quantum mechanics with pointwise interaction in the spaces of dimension no more than three, are touched.

The goal of the paper is to give strict sense to the formal operator

$$l\left(D\right)+\varepsilon\delta\left(x\right), \tag{2}$$

where ε is a real number, $x=\left(x_1,x_2,...,x_n\right)\in R_n$,

$$l\left(D\right)=\sum_{|\alpha|\le m}a_\alpha D^\alpha\left(p_0\left(\xi\right)=\sum_{|\alpha|\le m}\left(-i\right)^{|\alpha|}a_\alpha\xi^\alpha\in R_1\quad\text{for}\quad\forall\xi\in R_n\right)$$

is an elliptic differential expression of m order with constant coefficients.

In the space $L_2\left(R_n\right)$ let's consider the operator

$$\tilde{H}:=\left.l\left(D\right)\right|_{C_0^\infty\left(R_n\setminus\{0\}\right)},$$

and denote by $H_0=\overset{\approx}{H}_0$its closure in $L_2\left(R_n\right)$ i.e.

$$D\left(H_0\right)=\overset{\circ}{W}{}_2^{\,m}\left(R_n\setminus\{0\}\right)\quad\text{(Sobolev's space) and}$$

$$H_0\psi\left(x\right)=l\left(D\right)\psi\left(x\right)\quad\text{for}\quad\forall\psi\left(x\right)\in D\left(H_0\right)$$

Using the theory of self-adjoint extensions of symmetric operators (fon Neumann theory [5] and Berezin and Faddeev method [2] on overnormalization of the relation constant ε standing before $\delta-$ function in the expression (2) we calculate the defect numbers of the operator H_0 .

The following conjecture is decisive in the proof of the theorem on the defect index of the operator H_0 .

Conjecture. (see [6. p. 154]). Let $\varepsilon_{n,m}(x)$ be a fundamental solution for an elliptic operator $l(D)$. Then the following estimations are valid in the vicinity of the point $x=0$:

$$|D^{\alpha}\varepsilon_{n,m}(x)| \leq C_0 + C_1 |x|^{m-n-|\alpha|} \ (|\alpha| = 0,1,2,...,m-1; \quad |\alpha| \neq m-n),$$

$$|D^{\alpha}\varepsilon_{n,m}(x)| \leq C_0 + C_1 \ln\left(\frac{1}{|x|}\right), \quad |\alpha| = m-n.$$

In the following theorem we investigate dependence of defect numbers of the operator H_0 on the dimension of the space and order of elliptic differential expression $l(D)$.

Theorem 1. *a) If* $n \geq 2m$, *the operator* H_0 *has defect indices* $(0,0)$; *b) if* $n = 2m-2p+j \quad (p=1,2,...,m; \quad j=0,1)$, *the defect indices of the operator* H_0 *are* (r_p, r_p) *where*

$$r_p = \begin{cases} C_{n+p-1}^{p-1}, & \text{if } p = 2,3,...,m \\ 1, & \text{if } p = 1. \end{cases}$$

Proof. Let λ be an arbitrary non-real complex number. Denote by $\Delta_{\lambda}(H)$ and $\Delta_{\overline{\lambda}}(H_0)$ the domains of values of the operators $H_0 - \lambda E$ and $H_0 - \overline{\lambda}E$, respectively, and their orthogonal completions by M_{λ} and $M_{\overline{\lambda}}$. It is known that (see [5. p. 165]) the dimensions of the subspaces M_{λ} and $M_{\overline{\lambda}}$ are the same. Therefore, it suffices to study the structure of the subspace M_{λ}. Denote a fundamental solution of differential operator $l(D) - \lambda$ by $\varepsilon_{n,m}(x,\lambda)$. Show that a defect subspace M_{λ} is a linear span of those functions $D^{\alpha}\varepsilon_{n,m}(x,\lambda)$ that belong to $L_2(R_n)$. Since the function $\varepsilon_{n,m}(x,\lambda)$ and its any order derivatives are square-summable near ∞ (see [7. p. 287]), consequently, the number of functions $D^{\alpha}\varepsilon_{n,m}(x,\lambda)$ that belong $L_2(R_n)$ depends on the amount of the functions $D^{\alpha}\varepsilon_{n,m}(x,\lambda)$ that lie in L_2 near zero. It follows from conjecture 1 that for $n \geq 2m$ none of the functions $D^{\alpha}\varepsilon_{n,m}(x,\lambda)$ is square-nonintegrable near zero and all the functions $D^{\alpha}\varepsilon_{n,m}(x,\lambda)$ up to order $m = \dfrac{n+2p-j}{2} \quad (p=1,2,...,m; \ j=0,1)$ are square-integrable near zero and thereby, all the functions

$$D^{\alpha}\varepsilon_{n,m}(x,\lambda), \ |\alpha| \leq \frac{n+2p-j}{2}, \quad p=1,2,...,m; \ j=0,1$$

belong to $L_2(R_n)$.

At first we prove, that

$$L\left(\{D^{\alpha}\varepsilon_{n,m}(x,\lambda)\}_{|\alpha| \leq \frac{n+2p-j}{2}, \quad p=1,2,...,m; \ j=0,1}\right) \subset M_{\lambda}. \tag{3}$$

Let $g(x) \in (H_0 - \lambda E) D(H_0)$ then there exists such a function $\psi(x) \in D(H_0)$, that

$$H_0\psi(x) - \lambda\psi(x) = g(x).$$

Hence

$$\psi(x) = \int\limits_{R_n} \varepsilon_{n,m}(x-y,\lambda) g(y) dy \ \ (\text{for } \operatorname{Im}\lambda \neq 0).$$

Since

$$D^{\alpha}\psi(x) = \int\limits_{R_n} D^{\alpha}\varepsilon_{n,m}(x-y,\lambda)\, g(y)\, dy, \tag{4}$$

and $(D^{\alpha}\psi)(0) = 0$ for $|\alpha| \le m$ then (4) implies

$$D^{\alpha}\varepsilon_{n,m}(x,\lambda) \in M_{\lambda}, \text{ if } |\alpha| \le \frac{n+2p-j}{2}, \quad p = 1,2,...,m; \ j = 0,1.$$

Thus, (3) is proved. Show that, indeed

$$M_{\lambda} = L\left(\{D^{\alpha}\varepsilon_{n,m}(x,\lambda)\}_{|\alpha| \le \frac{n+2p-j}{2}, \ \ p=1,2,...,m; \ j=0,1}\right). \tag{5}$$

Let there exist such a function $f(x) \in L_2(R_n)$ that $(f,\varphi) = 0$ for any

$$\varphi(x) \in [(H_0 - \lambda E)\, D(H_0)] \oplus L\left(\{D^{\alpha}\varepsilon_{n,m}(x,\lambda)\}_{|\alpha| \le \frac{n+2p-j}{2}, \ \ p=1,2,...,m; \ j=0,1}\right).$$

Since the operator H with domain of definition $W_2^m(R_n)$ acting by the formula

$$(H\psi)(x) = l(D)\,\psi(x), \ \psi(x) \in W_2^m(R_n),$$

is self-adjoint in $L_2(R_n)$, then there exists the function $\psi(x) \in W_2^m(R_n)$ such that $H\psi(x) - \lambda\psi(x) = f(x)$.

Since $f(x)$ is orthogonal to the subspace

$$L\left(\{D^{\alpha}\varepsilon_{n,m}(x,\lambda)\}_{|\alpha| \le \frac{n+2p-j}{2}, \ \ p=1,2,...,m; \ j=0,1}\right),$$

then

$$(D^{\alpha}\psi)(0) = 0, \ |\alpha| \le \frac{n+2p-j}{2}, \quad p = 1,2,...,m; \ j = 0,1$$

i.e. $\psi(x) \in D(H_0)$, consequently $f(x) \in (H_0 - \lambda E)\, D(H_0)$.

It follows from orthogonality of $f(x)$ to the subspace $(H_0 - \lambda E)\, D(H_0)$ that $f(x) = 0$. Thus, equality (5) is proved. Since the system of functions $\{D^{\alpha}\varepsilon_{n,m}(x,\lambda)\}$ is linear independent, the dimension of the subspace M_{λ} for $n = 2m - 2p + j$, $(p = 1,2,...,m; \ j = 0,1)$ equals r_p, and for $n \ge 2m$ equals zero, so, the theorem is proved.

Remark to theorem 1.

a) For $m = 2$ the statement a) of the theorem is proved in [4. p. 184].

b) Physical meaning of the statement a) is in the fact that for $n \ge 2m$ there are no pointwise interactions.

c) For $n = 1, m = 2$ the operator H_0 has defect indices (2.2). Since the operator H_0 possesses four parametric families (see theorem 2) of self-adjoint extensions in the space $L_2(R_1)$, then besides δ interactions there exist additional types of pointwise interactions (f.e. $\delta'-$ interactions (see [3. p. 121]));

d) Statement b) of the theorem for Schrodinger's one-center operator in three dimensional case was proved in the paper [2], for Schrodinger multi-center operator in [8].

Select some orthonormed bases $\left\{e_k^{+}(x)\right\}_{k=1}^{r_p}$ and $\left\{e_k^{-}(x)\right\}_{k=1}^{r_p}$ in the spaces M_{λ} and $M_{\overline{\lambda}}$, respectively. Let $U = \left(u_{k_j}\right)$ be a unitary matrix of order r_p. We denote self-adjoint extension of the operator H_0 corresponding to a unitary matrix U by H_u. Applying the general theory of exrensions of symmetric operators (see [5]. p. 166]) we arrive at the following theorem.

Theorem 2. *Self-adjoint estensions H_u of the operator H_0 are given by the formula:*

$$D(H_u) = \{\psi(x) = \varphi(x) + \sum_{j=1}^{r_p} d_j \left[e_j^+(x) + \sum_{k=1}^{r_p} u_{kj} e_k^-(x) \right] :$$

$$\varphi(x) \in D(H_0), \quad d_j \in \mathbb{C}, \quad j = 1, 2, ..., r_p\},$$

$$H_u \psi(x) = H_0 \varphi(x) + \overline{\lambda} \sum_{j=1}^{r_p} d_j e_j^+(x) + \lambda \sum_{j=1}^{r_p} \sum_{k=1}^{r_p} u_{kj} e_k^-(x).$$

References

[1]. Demkov Yu. N., Ostrovsky V.N. *Method of zero radius potentials in atom physics* L.: Izd. LGU, 1975 (Russian).

[2]. Berezin F.A., Faddeev L.D. *Remarks on Schrodinger's equation with a singular potential.* DAN SSSR, 1961, Vol. 137, No 5, pp. 1011-1014 (Russian).

[3]. Albeverio S., Geztesy., Heogh Krohn., Holden H. *Solvable models in quantum mechanics.* M., Mir, 1991, 566 p. (Russian).

[4]. Reed M., Saymon B. *Contemparary methods of mathematical physics.* 1987, Vol.2, M.: Mir, 395 p. (Russian).

[5]. Naimark M.A. *Linear differential operators.* M., Nauka, 1969, 504 p. (Russian).

[6]. Misohata S. *Theory of partial equations* M., Mir, 1977, 504 p. (Russian).

[7]. Hormander L. *Analysis of partial linear differential operators.* 1986, Vol.2, M., Mir, 455 p. (Russian).

[8]. Eyvazov E.Kh. *Investigation of spectral properties of some differential operators with generalized coefficients.* Thesis for PhD. Institute of mathematics and mechanics AS Azerb. SSR. Baku, 1979, 112 p.

Elshad Kh. Eyvazov
Baku State University
Applied Mathematics and Cybernetics Faculty
23, Z.I. Khalilov str., AZ 1148, Baku, Azerbaijan.
Tel.: (99412) 568 37 91.

Received September 24, 2008; Revised December 03, 2008