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OPTIMAL CONTROL OF SYSTEMS DESCRIBED BY HEAT CONDUCTIVITY EQUATIONS

Abstract

In the given paper, the optimal control problem, which is described by the one-dimensional linear equation of heat conductivity with non-classical boundary conditions, is considered. Using ℓ-problems of the moments the problem of optimal control with the minimal energy is solved.

For the systems described by the ordinary differential equations, N.N.

Krasovsky [1;2] used the ℓ-method of a problem of the moments for the solution of an optimum control problem. After that, the method was used for solving more complicated problems connected with infinitely dimensional problem of moments and with of optimal control problems of systems with distributed parameters [3;4;5]. We shall note that, for infinitely dimensional systems, this problem in some cases is not trivial and it is not completely investigated yet.

In the earlier considered problems it has been investigated a case, when boundary conditions are adjoint. In the given paper the problem with nonadjoint, nonclassical boundary conditions are considered. In addition, it demands to apply this problem a meyhod different from previous ones.

Let a controlled process be describged by the function $y(x,t)$, which inside the domain $D = \{0 \le x \le 1, \quad 0 \le t \le T\}$ satisfies the equation

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + f(x,t) \tag{1}$$

with the initial condition

$$y(x,0) = \varphi(x), \ 0 \le x \le 1, \tag{2}$$

and boundary conditions

$$y(0,t) = \mu(t), \quad y_x(1,t) - y_x(0,t) = \eta(t), \tag{3}$$

where $f(x,t), \varphi(x), \mu(t), \eta(t)$ are the given functions.

Let $f(x,t) = \psi(t) \ p(x)$, where $\psi(t)$ is the given function from $L_2(0,T)$, $p(x)$ allowable controls are the functions from $L_2[0,1]$.

For the given functions $\varphi(x) \in L_2(0,1)$, $\mu(t) \in L_2(0,T)$, $\eta(t) \in L_2(0,T)$, $p(x) \in L_2[0,1]$, $\psi(t) \in L_2(0,T)$ the generalized solution of a problem (1)-(3) we shall name the function $y(x,t) \in W_2^{1,0}(D)$, that satisfies the integral identity.

$$\int_0^1 y(x,t)\,\Phi(x,t)\,dx - \int_0^1 \varphi(x)\,\Phi(x,0)\,dx +$$

$$+\int_0^t\int_0^1 \left[y_x(x,\tau)\,\Phi(x,\tau)-y(x,\tau)\,\Phi_\tau(x,\tau)+p(x)\,\psi(\tau)\,\Phi(x,\tau)\right]dxd\tau-$$

$$-\int_0^t \eta(\tau)\,\Phi(0,\tau)\,d\tau=0 \tag{4}$$

at $t\in[0,T]$ and any function $\Phi(x,t)\in W_2^1(D)\,,\quad \Phi(0,t)=\Phi(1,t)$.

For reception of the generalized solution of problem (1)-(3) we apply the Fourier method.

Therefore, we shall consider the eigen value problem [6]:

$$X''+\lambda X=0$$

$$X(0)=0,\quad X'(0)=X'(1)\,. \tag{5}$$

This problem is not self-adjoint. Adjoint problem to it there will be a problem

$$Y''+\overline{\lambda}Y=0$$

$$Y'(1)=0,\ Y(0)=Y(1)\,. \tag{6}$$

Eigen functions of these problems

$$X_0(x)=x,\quad X_{2k-1}(x)=x\cos(2\pi kx)\,,\quad X_{2k}(x)=\sin(2\pi kx)\,,$$

$$Y_0(x)=2,\quad Y_{2k-1}(x)=4\cos(2\pi kx)\,,\quad Y_{2k}(x)=4(1-x)\sin(2\pi kx)\,,$$

$$k=1,2,3,..$$

form the basis in space of functions $L_2(0,1)$ and are biortogonal.

$$(X_i,Y_j)=\int_0^1 X_i(x)\,Y_j(x)\,dx=\delta_{i,j}, \tag{7}$$

where $\delta_{i,j}$ is a Kronecker symbol.

Using the method of separation of variables, generalized solution of equation (1) with conditions (2),(3) may be written in the form:

$$y(x,t)=\int_0^1 G(x,s,t)\,\varphi(s)\,ds+$$

$$+\int_0^t\int_0^1 \{G(x,s,t-\tau)\,f(s)\,\psi(\tau)+$$

$$+\left[G(x,s,t-\tau)-G_t(x,s,t-\tau)\right]\eta(\tau)-G_t(x,s,t-\tau)\,\mu(\tau)\}\,dsd\tau, \tag{8}$$

where

$$G(x,s,t-\tau)=X_0(x)\,Y_0(s)+\sum_{k=1}^{\infty}\{X_{2k}(x)\,Y_{2k}(s)+$$

$$+\left[X_{2k-1}(x)-2\sqrt{\lambda_k}(t-\tau)X_{2k}(x)\right]Y_{2k-1}(s)\Big\}e^{-\lambda_k(t-\tau)}. \tag{9}$$

Let us formulate statement of a minimal energy control problem .

Let be $a(x)$ the given function from $L_2(0,1)$. In the class of allowable controls it is required to find, such a control $p(x)\in L_2[0,1]$ that the solution of a problem (1)-(3), corresponding to it satisfied the condition

$$y(x,T)=a(x). \tag{10}$$

Therewith the functional

$$J=\int\limits_0^1 p^2(x)\,dx \tag{11}$$

accept the least possible value.

Taking into account representation of the solution $y(x,t)$ of problem (1)-(3) from (9), we shall receive

$$\iint\limits_D G(x,s,T-t)\,\psi(t)\,p(s)\,dsdt=q(x), \tag{12}$$

where

$$q(x)=a(x)-\int\limits_0^1 G(x,s,T)\,\varphi(s)\,ds+$$

$$+\iint\limits_D\left[G_t(x,s,T-\tau)\,\mu(\tau)+\left(G_t(x,s,T-\tau)-G(x,s,T-\tau)\right)\eta(\tau)\right]dsd\tau. \tag{13}$$

From equality (12), letting

$$p(x)=p_0X_0(x)+\sum_{k=1}^{\infty}\left(p_{2k}X_{2k}(x)+p_{2k-1}X_{2k-1}(x)\right)$$

$$q(x)=q_0X_0(x)+\sum_{k=1}^{\infty}\left(q_{2k}X_{2k}(x)+q_{2k-1}X_{2k-1}(x)\right) \tag{14}$$

where

$$p_0=\int\limits_0^1 p(x)\,Y_0(x)\,dx,\quad p_{2k}=\int\limits_0^1 p(x)\,Y_{2k}(x)\,dx,\quad p_{2k-1}=\int\limits_0^1 p(x)\,Y_{2k-1}(x)\,dx$$

$$q_0=\int\limits_0^1 q(x)\,Y_{2k}(x)\,dx,\quad q_{2k}=$$

$$=\int\limits_0^1 q(x)\,Y_{2k}(x)\,dx,\quad q_{2k-1}=\int\limits_0^1 q(x)\,Y_{2k-1}(x)\,dx \tag{15}$$

we receive the system of equations

$$p_0 \int_0^T \psi(\tau)\, d\tau = q_0, \quad p_{2k} \int_0^T e^{-\lambda_k (T-\tau)} \psi(\tau)\, d\tau = q_{2k},$$

$$p_{2k-1} \int_0^T e^{-\lambda_k (T-\tau)} \psi(\tau)\, d\tau = q_{2k-1}.$$

Thus, the minimal energy control problem requires to find p_0, p_{2k}, p_{2k-1}, $k = 1, 2, ...$ such that they satisfied the equations (16) and the functional $J = \sum\limits_{k=0}^{\infty} p_k^2$ accepted the least value.

Supposing $\alpha_0 = \int\limits_0^T \psi(t)\, dt$, $\alpha_{2k} = \int\limits_0^T e^{-\lambda_k (T-\tau)} \psi(t)\, dt$, $\alpha_{2k-1} = \int\limits_0^T e^{-\lambda_k (T-\tau)} \psi(t)\, dt$ assume that $\alpha_k \neq 0$, for $q_k \neq 0$ and the series $J = \sum\limits_{k=0}^{\infty} \frac{q_k^2}{\alpha_k^2}$ converges. Under these conditions a minimal energy control problem has the unique solution and optimal control looks like

$$p(x) = \frac{q_0}{\alpha_0} X_0(x) + \sum_{k=1}^{\infty} \left(\frac{q_{2k}}{\alpha_{2k}} X_{2k}(x) + \frac{q_{2k-1}}{\alpha_{2k-1}} X_{2k-1}(x) \right).$$

If even at one simultaneously $q_k \neq 0$ and $\alpha_k = 0$, minimal energy control problem has no solution.

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