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ON BEHAVIOR OF SOLUTIONS OF DEGENERATED NONLINEAR PARABOLIC EQUATIONS

Abstract

The aim of this work is studding the behavior of solutions of initial boundary problem for degenerated nonlinear parabolic equation of the second order, conditions of existence and non-existence in whole by time solutions, is establish.

1. The exists and nonexists of solutions. Let's consider the equation

$$\frac{\partial u}{\partial t}=\sum_{i,j=1}^{u}\frac{\partial}{\partial x_j}\left(\omega\left(x\right)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)+f\left(x,t,u\right). \tag{1}$$

In bounded domain $\Omega\subset R^n$, $n\geq 2$ with nonsmooth boundary, namely the boundary $\partial\Omega$ contains the conic points with mortar of the corner $\omega\in(0,\pi)$. Denote by $\Pi_{a,b}=\{(x,t):x\in\Omega,\ a<t<b\}$, $\Gamma_{a,b}=\{(x,t):x\in\partial\Omega,\ a<t<b\}$, $\Pi_a=\Pi_{a,\infty}$, $\Gamma_a=\Gamma_{a,\infty}$. The functions $f\left(x,t,u\right)$, $\dfrac{\partial f\left(x,t,u\right)}{\partial u}$ are continuous by u uniformly in $\overline{\Pi}_0\times\{u:|u|\leq M\}$ at any $M<\infty$, $f\left(x,t,0\right)\equiv 0$, $\left.\dfrac{\partial f}{\partial u}\right|_{u=0}\equiv 0$. Besides the function f is measurable on whole arguments and not decrease by u. Let's consider the Dirichlet boundary condition

$$u=0, x\in\partial\Omega \tag{2}$$

and the initial condition

$$u|_{t=0}=\varphi\left(x\right) \tag{3}$$

in some domain $\Pi_{0,a}$, where $\varphi\left(x\right)$ is a smooth function. Further we'll weak this condition.

Solution of problem $(1)-(3)$ either exist in Π_0 or

$$\lim_{t\to T-0}\max_{\Omega}\left|u\left(x,t\right)\right|=+\infty \tag{4}$$

at some $T=const$.

Assuming that $\omega\left(x\right)$ is measurable non-negative function satisfying the conditions: $\omega\in L_{1,loc}\left(\Omega\right)$ and for any $r>0$ and some fixed $\theta>1$

$$\int_{B_r}\omega^{-1/(\theta-1)}dx<\infty,\ \underset{x\in B_r}{ess\sup}\,\omega\leq c_1 r^{n(\theta-1)}\left(\int_{B_r}\omega^{-1/(\theta-1)}dx\right)^{1-\theta}, \tag{5}$$

here $B_r=\{x\in\Omega:|x|<r\}$.

From condition (5) it follows that

$$\underset{x\in\Omega_r}{ess\sup}\omega\left(x\right)\leq c_1 r^{-n}\int\limits_{B_r}\omega dx \tag{6}$$

and $\omega\in A_\theta$ i.e.

$$\int\limits_{B_r}\omega dx\left[\int\limits_{B_r}\omega^{-1/(\theta-1)}dx\right]^{1-\theta}\leq cr^{n\theta} \tag{7}$$

Condition $(6)-\theta$ is Makenkhoupt's condition (see [3]).

Besides, analogously to [1] we'll assume that $\omega\in D_\mu,\mu<1+p/n$, i.e.

$$\frac{\omega\left(B_s\right)}{\omega\left(B_h\right)}\leq c_1\left(\frac{s}{h}\right)^{n\mu} \tag{8}$$

for any $S\geq h>0$, where $\omega\left(B_s\right)=\int\limits_{B_s}\omega\left(x\right)dx$.

Introduce the Sobole's weight space $W_p^1,W_{p,\omega}^1\left(\Omega\right)$ with finite norm

$$\|u\|_{W_{p,\omega}^1(\Omega)}=\left(\int\limits_{\Omega}\omega\left(x\right)\left(|u|^p+|\nabla u|^p\right)dx\right)^{1/p}.$$

The generalized solution of problem $(1)-(3)$ in $\Pi_{0,a'}$ we'll call the function $u\left(x,t\right)\in W_{p,\omega}^1\left(\Pi_{a,b}\right)$, such that

$$\int\limits_{\Pi_{a,b}}\psi\frac{\partial u}{\partial t}dxdt+\sum_{i,j=1}^n\int\limits_{\Pi_{a,b}}\omega\left(x\right)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\frac{\partial\psi}{\partial x_j}dxdt=$$

$$=\int\limits_{\Pi_{a,b}}f\left(x,t,u\right)\psi\left(x,t\right)dxdt, \tag{9}$$

where $\psi\left(x,t\right)$ is an arbitarary function from $W_{p,\omega}^1\left(\Pi_{a,b}\right),\ \psi|_{\Gamma_{a,b}}=0$, $0<a<b$ are any numbers.

Let's formulate some auxillary result's from [3],[4]. For this we'll determine $p-$harmonic operator $L_p u=\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right),p>1$.

Lemma 1. ([1]). *There exists positive eigenvalue of spectral problem for operator L_p that corresponds the positive in Ω eigenfunction.*

Lemma 2. ([2]). *Let $u,v\in W_p^1\left(\Omega\right),u\leq v$ on $\partial\Omega$ and*

$$\int\limits_{\Omega}L_p\left(u\right)\eta_{xi}dx\leq\int\limits_{\Omega}L_p\left(\vartheta\right)\eta_{x_i}dx$$

for any $\eta\in\overset{\circ}{W}{}_p^1\left(\Omega\right)$ with $\eta\geq 0$. Then $u\leq\vartheta$ on all domain Ω.

Let $u_0(x) > 0$ be an eigenfunction of spectral problem for the operator L_p corresponding $\lambda = \lambda_1 > 0$, $\int\limits_{\Omega} u_0(x)\,dx = 1$.

Let's assume that the condition:

$$I = \int\limits_{\Omega} \omega(x) \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} - \left| \frac{\partial u_0}{\partial x_i} \right|^{p-2} \frac{\partial u_0}{\partial x_i} \right) \frac{\partial (u_0 \omega)}{\partial x_i} dx \geq 0 \qquad (*)$$

be fulfilled.

Theorem 1. *Let $f(x,t,u) \geq \alpha_0 |u|^{\sigma-1} u$ at $(x,t) \in \Pi_0, u \geq 0$, where $\sigma = const > 1, \alpha_0 = const > 0$. There exists $k = const > 0$ such that if $u(x,0) \geq 0$, $\int\limits_{\Omega} u(x,0)\,u_0(x)\,dx \geq k$, and condition (*) be fulfilled, then*

$$\lim_{t \to T-0} \max_{\Omega} \left(\omega(x)\,u_0(x)\,u(x,t)\right) = \infty,$$

where $T = const > 0$.

Proof. Let's assume the opposite. Then $u(x,t)$ is a solution of equation (1) in Π_0 and condition (2) on Γ_0 be fulfilled. By means of lemma 2 $u(x,t) > 0$ in Π_0. Substituts in (8) $\Psi = \varepsilon^{-1} u_0(x)\,\omega(x)$, $b = a + \varepsilon$, $a > 0$, $\varepsilon > 0$, where $u_0(x) > 0$ in Ω is eigenfunction of spectral problem for the operator L_p corresponding to eigenvalue $\lambda_1 > 0$. Such eigenvalue exists by virtue of lemma 1.

As a result we'll obtain

$$\varepsilon^{-1} \left[\int\limits_{\Omega} \omega(x)\,u_0(x)\,u(x,a+\varepsilon)\,dx - \int\limits_{\Omega} \omega(x)\,u_0(x)\,u(x,a)\,dx \right] +$$

$$+\varepsilon^{-1} \int\limits_{\Pi_{a,a+\varepsilon}} \omega(x) \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial \psi}{\partial x_j} dxdt = \varepsilon^{-1} \int\limits_{\Pi_{a,a+\varepsilon}} u_0 \omega f(x,t,u)\,dxdt. \qquad (10)$$

Let's make same transformations. Let's add and substract to left hand (10)

$$\varepsilon^{-1} \int\limits_{\Pi_{a,a+\varepsilon}} \omega(x) \left| \frac{\partial u_0}{\partial x_i} \right|^{p-2} \frac{\partial u_0}{\partial x_i} \frac{\partial \psi}{\partial x_j} dxdt,$$

and taking into account that $u_0(x)$ the egenfunction of the operator L_p corresponds to $\lambda_1 > 0$ and ε vanich we'll obtain that at all $t > 0$

$$\frac{\partial}{\partial t} \int\limits_{\Omega} u_0(x)\,\omega(x)\,u(x,t)\,dx =$$

$$= -\lambda_1 \int\limits_{\Omega} u_0(x)\,\omega(x)\,u(x,t)\,dx + \int\limits_{\Omega} u_0 \omega(x)\,f(x,t,u)\,dx + I.$$

From here denoting

$$g(t) = \int_{\Omega} u_0(x)\,\omega(x)\,u(x,t)\,dx$$

We have

$$g'(t) = \lambda_1 \int_{\Omega} u_0(x)\,\omega(x)\,u(x,t)\,dx + I + \int_{\Omega} u_0 \omega f(x,t,u)$$

Further, taking into account condition (A) and condition on $f(x,t,u)$ we have

$$g'(t) \geq -\lambda_1 \int_{\Omega} u_0 \omega(x)\,u(x,t)\,dx + a_0 \int_{\Omega} u_0 \omega\,|u|^{\sigma}\,dx \tag{11}$$

So, from (10) we'll obtain

$$g'(t) \geq -\lambda_1 \int_{\Omega} \omega u u_0 dx + a_0 \int_{\Omega} u_0 \omega u^{\sigma} dx \tag{12}$$

By virtue inequality Holder we have

$$\left(\int_{\Omega} u u_0 \omega dx\right)^{\sigma} \leq \left[\left(\int_{\Omega} u^{\sigma} u_0 \omega dx\right)^{1/\sigma} \left(\int_{\Omega} \omega u_0 dx\right)^{\sigma - 1/\sigma}\right]^{\sigma} \leq C_1 \int_{\Omega} u^{\sigma} u_0 \omega dx.$$

In results

$$g'(t) \geq -\lambda_1 g(t) + C g^{\sigma}(t), \quad C = const > 0 \tag{13}$$

If

$$g(0) > c_2 = \left(\frac{\lambda_1}{c}\right)^{1/\sigma}$$

then from (13) we'll obtain $\lim\limits_{t \to T-0} g(t) = +\infty$. This means that

$$\lim_{t \to T-0} \max_{\Omega} \left(\omega(x)\,u_0(x)\,u(x,t)\right) = \infty$$

Theorem is proved.

So equation (1) hasn't solutions in satisfying the boundary condition (2) if $u(x,0) \geq 0$ isn't much small. Now we'll show that at small $|u(x,0)|$ solution of problem (1),(2) exists on whole domain Π_0.

Theorem 2. *We'll assume that* $|f(x,t,u)| \leq (C_3 + C_4 t^m)\,|u|^{\sigma}, \quad \sigma > 1,\ m > 1.$ *There exists* $\delta > 0$ *such that if* $|\varphi(x)| \leq \delta$ *then solution of problem (1),(3) exists in* Π_0 *and* $|u(x,t)| \leq C_5 e^{-\alpha_, t}, \alpha = const > 0$ *not depend at* n.

Proof. Let $\overline{\Omega} \subset B_R$, where $B_R = \{x : |x| \leq R\}$. Let $\vartheta > 0$ in B_R be eigenfunction corresponding to positive eigenvalue λ_1 of the boundary problem

$$L_p u + \lambda u = 0, x \in \Omega, u = 0, x \in \partial\Omega \tag{14}$$

Let's consider the function $V(x,t)=\varepsilon\cdot e^{-\lambda_1 t/2}\cdot\vartheta(x)$. We have

$$V_t-L_pV-f(x,t,V)=\tfrac{1}{2}\varepsilon\lambda_1 e^{-\lambda_1 t/2}\cdot\vartheta(x)-$$

$$-(c_3+c_4t^m)\,\varepsilon^\sigma e^{-\lambda_1 t/2}\cdot\vartheta\geq 0,\,(x,t)\in\Pi_0 \tag{15}$$

$$\text{and } V>0,\,(x,t)\in\Gamma_0,$$

if $\varepsilon>0$ is sufficiently small. Inequality (15) is understood in weak sense (see [4]).

From (15) and lemma 2 follows that $|u|\leq V\leq C_s e^{-\lambda_1 t}$, $|\varphi(x)|\leq\delta=\varepsilon\min\limits_{\Omega}\vartheta(x)$. Let's determine the class of functions K consisting from $g(x,t)$ continuous in $\overline{\Pi}_{-\infty,+\infty}$ equaling to zero at $t\leq T$ and such that $|g(x,t)|\leq Ce^{-ht}$. K is a set of Banach space continuous in $\overline{\Pi}_{-\infty,+\infty}$ functions with norm

$$\|g\|=\sup_{\overline{\Pi}_{-\infty,+\infty}}\left|ge^{ht}\right|.$$

Let $\theta(t)\in C^\infty(R^1)$, $\theta(t)\equiv 0$, $t\leq T$, $\theta(t)=1$, $t>T+1$. Let's determine the operator H on K puthing $Hg=\theta(t)\,z$, $g\in K$, where z is a solution of lineazing problem.

By virtue of above obtained estimation H transforms K in K if T is sufficiently big. The operator H is a fully continuous. This follows from the obtained estimation and theorem on Holderness of solutions of parabolic equations in $\Pi_{-a,a}$ at any a ([4]). From Lere-Shauder theorem, consequence that the operator H has fixed point z. This shows the existence of solution.

The theorem is proved.

From theorem 2 it follows that if $u(x,0)\geq 0$, $|u(x,0)|\leq\delta$, then the solution of problem (1)-(3) exists in Π_0 and possitive in Π_0 by virtue of lemma 2.

Let's indicate the sufficient condition, at which all nonnegative solutions of problem (1)-(3) have "blow-up", i.e.

$$\lim_{t\to T-0}\max_{\Omega}\left(\omega(x)\,u_0(x)\,u(x,t)\right)=+\infty, \tag{16}$$

where $T=const>0$.

Theorem 3. *Let $f(x,t,u)\geq C_6e^{\lambda_1\sigma t}u^\sigma$ at $(x,t)\in\Pi_0$, $u\geq 0$, $\sigma=const>1$, λ_1 be positive eigenvalue of problem (14) in Ω that corresponds to the positive in Ω eigenfunction. If $u(x,0)\geq 0$, $u(x,0)\not\equiv 0$, where $u(x,t)$ is solution of problem (1)-(3), then it holds (16).*

Proof. Similarly how it has been established by inequality (13) we'll obtain

$$g'(t)\geq-\lambda_1 g+C_7e^{\lambda_1\sigma t}g^\sigma(t) \tag{17}$$

where

$$g(t)=\int\limits_{\Omega}\omega(x)\,u_0(x)\,u(x,t)\,dx$$

Let $g(t)=\psi(t)e^{\lambda_1 t}$. From (17) if follows that $\psi'\geq C_8\psi^{\sigma}$. Hence $\psi(t)\to+\infty$ at $t\to T-0$. Thus $g(t)$ tends so $+\infty$ at $t\to T-0$. Consequently $\max\limits_{\Omega}(\omega(x)u_0(x)u(x,t))$ is also tends to infinity

Theorem is proved.

From theorem 3 we can obtain the following property of solutions of equation (1)

Corollary: *Let $f(x,t,u)\geq C_8e^{\lambda_1\sigma t}u^{\sigma}$ and at $(x,t)\in\Pi_0$, $u\geq 0$ where $\sigma>1$. Then there isn't positive in Π_0 solutions of equation (1).*

2. The estimation of solutions. We'll obtain the estimations for solutions of problem (1)-(3) in case $f(x,t,u)=0$ in ternus to characterising on infinity of initial and weight functions, without a lower's condition on initial function.

Assume, that $\varphi(x)\in L_1(\Omega)$. Denote by $k=n(p-1-\mu)+p, r>0$ fixed number. Let's consider the following initial characteristics for $u(x,t)$ and $\varphi(x)$

$$\varphi_r(t)=\sup_{\tau\in(0,t)}\sup_{\rho\geq r}\left(\frac{\omega(B_\rho)}{\rho^{n+p}}\right)^{1/(p-2)}\cdot\|u(x,\tau)\|_{L_\infty(B_\rho)},$$

$$|||u(x,\tau)|||_r=\sup_{\rho\geq r}\rho^{-k/(p-2)}\left[\frac{\omega(B_\rho)}{\rho^{n\cdot\mu}}\right]^{1/(p-2)}\int\limits_{B_\rho}u(x,\tau)\,dx,$$

$$|||u(x,0)|||_r=\|\varphi\|_r\,.$$

Let's rewrite the definition of generalized solution (9) in the following form:

$$\int\limits_{\Omega}u(x,t)\psi(x,t)\,dx+\int\limits_0^t\int\limits_{\Omega}\left(-u\psi_t+\omega\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\frac{\partial\psi}{\partial x_j}dxdt\right)=$$

$$=\int\limits_{\Omega}\varphi(x)\psi(x,0)\,dx,\quad\forall\ o<t<T.\tag{18}$$

Lemma 3: *Assume that $u(x,t)\in W^1_{p,\omega}(\Pi_{a,b})$ is a generalized solution of problem (1)-(3) is initial function $\varphi(x)\in C_0^\infty(\Omega)$. Then the following estimation is true*

$$|u(x,t)|\leq C_9\left[\beta(t)\right]^{(n+p-n(\mu-1))/\lambda}\left[\frac{\rho^{n\mu}}{\omega(B_\rho)}\right]^{n/\lambda}\left[\int\limits_{t/\varphi}^{t}\int\limits_{B_{2\rho}}u^p dxdt\right]^{(p-n(\mu-1))}\tag{19}$$

for $\forall\ o<t<T$, where $\beta(t)=t^{-n(p-2)/k}\cdot\varphi_r^{p-2}(t)+t^{-1}$,

$$\lambda=n(2p-2-p\mu)+p^2.$$

Proof: Let $f(x,t)\in L_\infty(0,T:L_s(B_\rho))\cap L_p\left(0,T:\overset{\circ}{W}{}^1_{p,\omega}(B_\rho)\right)$, $s,p>1$. Using the weigh multiplicate inequality from [3], we obtain the inequality

$$\int\limits_0^T\int\limits_{B_\rho}|f(x,t)|^q\,dxdt\leq$$

$$\leq C_{10}\frac{\rho^{n\cdot\mu}}{\omega(B_\rho)}\left(\operatorname*{ess\,sup}_{0<t<T}\int_{B_\rho}|f|^s\,dx\right)^{(p-n(\mu-1))/n}\int_0^T\int_{B_\rho}\omega\,|\nabla f|^p\,dxdt \tag{20}$$

$q=p+\frac{s}{n}(p-n(\mu-1))$. Let $\rho>0, T>0$ are fixed. Let's consider the sequence $T_k=T/2-T/2^{k+1}$, $\rho_k=\rho+\rho/2^{k+1}$, $\overline{\rho}_k=\frac{1}{2}\left(\rho_k+\rho_{k+1}\right)$, $k=0,1,\ldots.$ Denote by $B_k=B_{\rho_k}$, $\overline{B}_k=B_{\rho_k}$, $\Pi_k\equiv B_k\times(T_k,T)$, $\overline{\Pi}_k\equiv\overline{B}_k\times(T_{k+1},T)$.

Let $\xi_k(x,t)$ be cutting function in Π_k satisfying the conditions $\xi_k=1$, $(x,t)\in\overline{\Pi}_k$, $|\nabla\xi_k|\leq 2^{k+2}/\rho$, $0\leq\frac{\partial\xi_\kappa}{\partial t}\leq 2^{k+2}\cdot T$.

Besides, let $\alpha>0$, $\alpha_k=\alpha-\alpha/2^{k+2}$, $k=0,1,2,\ldots$

Let's substitute $\psi(x,t)=(u-\alpha_k)_t^{p-1}\xi_k^p$ in integral identity (18). Doing transformation, analogously [5] we'll obtain

$$\sup_{T_{k+1}\leq t\leq T}\int_{\overline{B}_k}\upsilon_k^s dx+\iint_{\overline{\Pi}_k}\omega\,|\nabla\vartheta_k|^p\,dxdt\leq C_{11}2^{kp}\beta(t)\iint_{\Pi_k}\vartheta_k^s dxdt \tag{21}$$

where $\vartheta_k=(u-\alpha_k)^{2(p-1)/p}$, $s=p^2/2(p-1)$.

Estimating the right part (21) using (20) and doing some calculations we'll obtain

$$-\iint_{\overline{\Pi}_k}\vartheta_{k+1}^q dxdt\leq\iint_{\overline{\Pi}_k}|\vartheta_{k+1}\xi_k|^q\,dxdt\leq C_{12}\frac{\rho^{n\cdot\mu}}{\omega(B_\rho)}\times$$

$$\times\left\{\iint_{\overline{\Pi}_k}\omega\,|\nabla\vartheta_k|^p\,dxd\tau+\frac{2^{kp}}{\rho^p}\iint_{\overline{\Pi}_k}\omega\vartheta_k^p dxd\tau\right\}\left(\sup_{T_{k+1}\leq t\leq T}\int_{\overline{B_k}}\vartheta_k^s dx\right)^{(p-n(\mu-1))/n}\leq$$

$$\leq C_{12}\frac{\rho^{n\cdot\mu}}{\omega(B_\rho)}\left[\beta(t)\right]^{1+(p-n(\mu-1))/n}\left[\iint_{\overline{\Pi}_k}\vartheta_{k+1}^s dxd\tau\right]^{1+(p-n(\mu-1))/n}. \tag{22}$$

Further, we'll use the following estimation

$$mesA_{k+1}=mes\left\{(x,t)\in\Pi_{k+1}/u(x,t)>\alpha_{n+1}\right\}\leq k^{-p}2^{-(k+1)p}\iint_{\overline{\Pi}_k}\vartheta_k^s dxd\tau \tag{23}$$

From (20) the Holder inequality and using estimation (23) we have

$$\iint_{\Pi_{k+1}}\vartheta_{k+1}^q dxd\tau\leq\left(\iint_{\Pi_{k+1}}\vartheta_{k+1}^q dxd\tau\right)^{s/q}(mesA_{k+1})^{1-s/q}\leq$$

$$\leq C_{13}\alpha^{-p(1-s/q)}\left[\frac{\rho^{n\cdot\mu}}{\omega(B_\rho)}\right]^{s/q}(B(t))^{((n+p-n(\mu-1)/n)\cdot(s/q))}\times$$

$$\times \left(\iint\limits_{\Pi_k} \vartheta_s^k dx d\tau \right)^{(1+(p-n(\mu-1)/n)\cdot(s/q))} \tag{24}$$

Hence, using [4] denoting

$$M = C_{13} \left[\frac{\rho^{n\cdot\mu}}{\omega(B_\rho)} \right]^{n/\lambda} \cdot (\beta(t))^{(n+p-n(\mu-1))/n} \left(\iint\limits_{\Pi_k} u^p dx d\tau \right)^{(p-n(\mu-1))/\lambda}$$

we'll obtain that $\sup\limits_{\Pi_{a,b}} u(x,t) \le M$.

Lemma 3 is proved.

Denote $\eta(t) = \sup\limits_{\tau\in(0,t)} \eta_r(\tau) = \sup\limits_{\tau\in(0,t)} |||u(x,\tau)|||_r$

Lemma 4. *Let's assume that $u(x,t) \in W^1_{p,\omega}(\Pi_{a,b})$ be generalized solution of problem of (1)-(3), the initial function $\varphi(x) \in C_0^\infty(\Omega)$. Then the estimations*

$$\varphi_r(t) \le C_{14} \int\limits_0^t \tau^{-n(p-2)/k} \varphi_r^{p-1}(\tau)\, d\tau + C_{15} [\eta(t)]^{(p-n(\mu-1))/k} \tag{25}$$

$$\eta(t) \le C_{16} |||\varphi|||_r + C_{17} \left(\int\limits_0^t \tau^{(p-n(\mu-1)/p\alpha)-1} (\varphi_r(\tau))^{(p-2/p)} \eta(\tau)\, d\tau + \right.$$

$$\left. + \int\limits_0^t \tau^{(p-n(\mu-1)/p\alpha)-1} (\varphi_r(\tau))^{(p-2)/p} \eta(\tau)\, d\tau \right) \tag{26}$$

are true.

Proof. Let's estimate the following integrals

$$\left[\frac{\rho^{n\cdot\mu}}{\omega(B_\rho)} \right] \tau^{n/\alpha} \left[\frac{\omega(B_\rho)}{\rho^{n+p}} \right]^{1/(p-2)} \tau^{(-n(p-2)/\alpha)(n+p-n(\mu-1))/\lambda} \cdot \varphi_r^{(p-2)((n+p-n(\mu-1))/\lambda)} \times$$

$$\times \left(\int\limits_{t/4}^t \int\limits_{B_{2\rho}}^t u^p dx d\tau \right)^{(p-n(\mu-1))/\lambda} \le [\varphi_r(t)]^{(p-2)((n+p-n(\mu-1))/\lambda)} \times$$

$$\times \left(\int\limits_0^t \tau^{-n(p-2)/\alpha} \varphi_r^p(\tau)\, d\tau \right)^{(p-n(\mu-1))/\lambda} \le C_{18} \varphi_r(t) + (\eta(t))^{(p-n(\varpi))/\alpha}, \tag{27}$$

$$\left[\frac{\rho^{n\cdot\mu}}{\omega(B_\rho)} \right]^{n/\lambda} \tau^{n/\alpha} \left[\frac{\omega(B_\rho)}{\rho^{n+p}} \right]^{1/(p-2)} \tau^{-(n+p-n(\mu-1))/\lambda} \left(\int\limits_{t/4}^t \int\limits_{B_{2s}}^t u^p dx d\tau \right) \le$$

$$\le C_{19} (\varphi_r(t))^{(p-1)(p-n(\mu-1))/\lambda} + (\eta(t))^{(p-n(\mu-1))/\lambda} \le$$

$$\leq C_{20}\varphi_r(t) + (\eta(t))^{(p-n(\mu-1))/\alpha}. \tag{28}$$

Now multiplying the both parts (19) on $\left[\frac{\omega(B_\rho)}{\rho^{n+p}}\right]^{1/(p-2)} \tau^{n/\alpha}, \tau \in (t/4, t), \forall t > 0$ and allowing for estimations (27), (28) we'll obtain estimation (25).

For getting estimation (26) we'll substitute in integral identity (18) $\psi(x,t) = \tau^{1/p}u^{1-2/p}\xi^p$. We'll obtain

$$\int_0^t \int_{B_{2\rho}} \omega\tau^{1/p} \cdot |\nabla u|^p u^{-2/p}\xi^p dx d\tau \leq$$

$$\leq C_{21}\rho^{-p} \int_0^t \int_{B_{2\rho}} \omega\tau^{1/p}u^{p-2/p} dx d\tau + C_{22} \int_0^t \int_{B_{2\rho}} \tau^{1/p-1}u^{2(p-1)/p} dx d\tau \tag{29}$$

Let's estimate integral of the right in (29). We have

$$\rho^p \int_0^t \int_{B_{2\rho}} \omega\tau^{1/p}u^{p-2/p} dx d\tau \leq \omega(B_{2\rho})\rho^{-(n+p)} \int_0^t \int_{B_{2\rho}} \tau^{1/p}u^{p-2/p} dx d\tau \leq$$

$$\leq C_{23}\left(\frac{\omega(B_\rho)}{\rho^n}\right)^{-1/p}\left(\frac{\omega(B_\rho)}{\rho^{n\cdot\mu}}\right)^{-1/(p-2)}\rho^{1+\alpha/(p-2)} \times$$

$$\times \int_0^t \tau^{((p+1)/p\alpha)(p-n(\mu-1))-1}(\varphi_r(t))^{(p-2)(p+1)/p}\eta(\tau)d\tau \tag{30}$$

The second integral on the right in (29) we'll estimate by the following way

$$\int_0^t \int_{B_{2\rho}} \tau^{\frac{1}{p}-1}u^{2(p-1)/p} dx d\tau \leq$$

$$\leq \left(\frac{\omega(B_\rho)}{\rho^n}\right)^{-1/p}\left(\frac{\omega(B_\rho)}{\rho^{n\cdot\mu}}\right)^{-1/(p-2)}\rho^{1+\alpha/(p-2)} \times$$

$$\times \int_0^t \tau^{(p-n(\mu-1))/p\alpha-1}(\varphi_r(\tau))^{(p-2)/p}\eta(\tau)d\tau \tag{31}$$

Now, let's substitute in integral identity (18) $\psi(x,t) = \xi^p(x)$. Then we'll obtain

$$\int_{B_{2\rho}} u(x,t)dx \leq \int_{B_{2\rho}} \varphi(x)dx + C_{24}\rho^{-1}\int_0^t \int_{B_{2\rho}} \omega|\nabla u|^{p-1}\xi^{p-1}dx d\tau \tag{32}$$

Let's estimate the secong integral on the right in (32). We have

$$\int_0^t \int_{B_\rho} \omega|\nabla u|^{(p-1)}\xi^{p-1}dx d\tau \leq \left(\int_0^1 \int_{B_{2\rho}} \omega\tau^{1/p} \cdot |\nabla u|^p u^{-2/p}\xi^p dx d\tau\right)^{(p-1)/p} \times$$

$$\times\left(\int_0^t\int_{B_{2\rho}}\omega\tau^{-(p-1)/p}u^{2(p-1)/p}dxd\tau\right)^{1/p} \tag{33}$$

Taking into account the second multiplies in (33)

$$\int_0^t\int_{B_{2\rho}}\omega\tau^{-(p-1)/p}u^{2(p-1)/p}dxd\tau\le C_{25}\frac{\omega\left(B_\rho\right)}{\rho^n}\int_0^t\int_{B_{2\rho}}\tau^{1/p-1}u^{2(p-1)/p}dxd\tau. \tag{34}$$

Now allowing for estimations (30), (31), (32) in (33) we'll obtain

$$\int_0^t\int_{B_{2\rho}}\omega\left|\nabla u\right|^{p-1}\xi^{p-1}dxd\tau\le C_{25}\left(\frac{\omega\left(B_\rho\right)}{\rho^{n\cdot\mu}}\right)^{-1/(p-2)}\rho^{1+\alpha/(p-2)}\times$$

$$\times\left(\int_0^t\tau^{((p+1)/p\alpha)(p-n(\mu-1))-1}\left(\varphi_r\left(\tau\right)\right)^{(p-2)(p+1)/p}\eta\left(\tau\right)d\tau+\right.$$

$$\left.+\int_0^t\tau^{(p-n(\mu-1))/p\alpha-1}\varphi_r^{(p-2)/2}\left(\tau\right)\eta\left(\tau\right)d\tau^{(p-1)/p}\right)\times$$

$$\times\int_0^t\tau^{(p-n(\mu-1))/p\alpha-1}\left(\varphi_r\left(\tau\right)^{(p-1)/p}\eta\left(\tau\right)d\tau\right)^{1/p}. \tag{35}$$

Multiplying inequality (32) $\rho^{-\alpha/(p-2)}\rho^{-n\cdot\mu/(p-2)}\left(\omega\left(B_\rho\right)\right)^{1/(p-2)}$, using inequality (35), then we'll obtain

$$\eta\left(t\right)\le C_{27}\left|\left|\left|\varphi\right|\right|\right|_r+$$

$$+C_{28}\left(\int_0^t\tau^{((p+1)/p\cdot\alpha)(p-n(\mu-1))}\left(\varphi_r\left(\tau\right)^{(p-2)(p+1)/p}\eta\left(\tau\right)d\tau\right)\right)+$$

$$+\int_0^t\tau^{(p-n(\mu-1))/p\alpha-1}\left(\varphi_r\left(\tau\right)^{(p-2)/2}\eta\left(\tau\right)d\tau\right).$$

Lemma 4 is proved.

Theorem 4. *Let $u\left(x,t\right)\in W^1_{p,\omega}\left(\Pi_{a,b}\right)$ be generalized solution of problem (1)-(3) and $\left|\left|\left|\varphi\right|\right|\right|_r<\infty, r>0$ be fixed. Then if relative $\omega\left(x\right)$ to conditions (4), (7) and $\mu<1+p/n$ fulfilled, then*

$$\left|\left|\left|\varphi\right|\right|\right|_r<C_{29}t^{1/(p-2)} \tag{36}$$

$$\left|\left|\left|u\left(x,t\right)\right|\right|\right|_r<C_{30}t^{1/(p-2)} \tag{37}$$

$$\sup_{B_\rho}\left|u\left(x,t\right)\right|\le C_{31}t^{p(n+1)-n(\mu+1)/k(p-2)}\rho^{n+p}\cdot\omega^{-1}\left(B_\rho\right). \tag{38}$$

Proof: The proof of theorem follows from lemma 4 using the method of paper [5]. Thus for obtaining estimations (37), (38) the estimations are at first obtained

$$|||u(x,t)|||_r < C_{32} \, |||\varphi|||_r \ ,$$

$$\sup_{B_s} |u(x,t)| \le C_{33} \, |||\varphi|||_r^{(\rho-n(\mu-1))/k} \, \rho^{n+p} \cdot \omega^{-1}(B_\rho) \, t^{-n/k}. \tag{39}$$

Further, using these estimations we obtain estimations (37), (38)

Corollary: *Let in theorem 4* $\omega(x) = |x|^\theta$, $0 < \theta < p$. *Then conditions (4), (7)* $\mu = 1 + \theta/n$, *are fulfilled and we have the following estimation*

$$\sup_{B_\rho} |u(x,t)| \le C_{34} \left(\sup_{\rho \ge r} \rho^{-\beta/(p-2)} \int_{B_\rho} \varphi(x)\, dx \right)^{(p-\theta)/\beta} \cdot \rho^{(p-\theta)/(p-2)} \cdot t^{-n/\beta}, \tag{40}$$

where $\beta = n(p-2) + p - \theta$.

Note that estimation (39) is a exactly that proves to be true following class of exact solutions

$$u_\theta(x,t) = \left(1 - \left(\frac{p-2}{p-\theta} \right) \left(\frac{n}{\beta} \right)^{1/(p-1)} \left(\frac{|x|}{t^{1/\beta}} \right)^{(p-\theta)/(p-1)} \right)^{(p-1)/(p-2)} .$$

In case $\alpha = 0$ and considering Cauchy problem estimation (40) is consider with the result of paper [5].

Remark: Estimations of type (38) we can a;so obtain for $\sup_{B_\rho} |\nabla u(x,t)|$

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