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SOME SPECTRAL PROPERTIES OF A FOURTH ORDER STURM-LIOUVILLE OPERATOR WITH SPECTRAL PARAMETERS IN THE BOUNDARY CONDITION IN THE DISFOCAL CASE

Abstract

In the paper the fourth order Sturm-Liouville problem with spectral parameter in the boundary condition in the disfosal case is considered. The oscillation properties of eigenfunctions are studied, the basicity in the $L_p(0,l)$, $1<p<\infty$, *of the system of eigenfunctions of this problem with a single chosen eigenfunction is proved.*

Consider the following fourth order Sturm-Liouville problem

$$y^{(4)}(x)-\left(q(x)\,y'(x)\right)'=\lambda y(x)\,,\ 0<x<l,\ \ ':=\frac{d}{dx}, \tag{1}$$

$$y'(0)=0, \tag{2.a}$$

$$y(0)\cos\beta+Ty(0)\sin\beta=0, \tag{2.b}$$

$$y'(l)\cos\gamma+y''(l)\sin\gamma=0, \tag{2.c}$$

$$(a\lambda+b)\,y(l)-(c\lambda+d)\,Ty(l)=0, \tag{2.d}$$

where

$$Ty=y'''-qy',$$

and λ is a spectral parameter, q is a absolutely continuous function on $[0,l]$, β, γ, a, b, c, d are real constants, moreover $0\le\beta,\gamma\le\pi/2$, $\sigma=bc-ad>0$ and the equation

$$y''-qy=0 \tag{3}$$

is difocal in $[0,l]$, i.e., there is no nontrivial solution of equation (3) such that $y(a)=0=y'(b)$ for any disting pair of points a and b in $[0,l]$.

The oscillation properties of eigenfunctions and the basis properties in the space $L_p(0,l)$, $1<p<\infty$, of the eigenfunction system of the problem (1), (2) with $q\ge 0$ is considered in [1].

The subject of the present paper in the study of the oscillation properties of eigenfunctions and the basis property in the spaces $L_p(0,l)$, $1<p<\infty$, of the system of eigenfunctions of the boundary value problem (1), (2).

As in [2-4], for the analysis of the oscillation properties of the spectral problem (1), (2) we shall use a Prüfer transformation of the following form:

$$\left\{\begin{array}{r}y(x)=r(x)\sin\psi(x)\cos\theta(x)\,,\\ y'(x)=r(x)\cos\psi(x)\sin\varphi(x)\,,\\ y''(x)=r(x)\cos\psi(x)\cos\varphi(x)\,,\\ Ty(x)=r(x)\sin\psi(x)\sin\theta(x)\,.\end{array}\right. \tag{4}$$

Equation (1) has an equivalent formulation in the matrix form:

$$\upsilon' = M\upsilon$$

where

$$\upsilon = \begin{pmatrix} y \\ y' \\ y'' \\ Ty \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & q & 0 & 0 \\ \lambda & 0 & 0 & 0 \end{pmatrix}.$$

Consider the boundary condition

$$y(l)\cos\delta - Ty(l)\sin\delta = 0, \tag{2d^*}$$

where $\delta \in [0, \pi)$.

Also we need the following results which is basic in the sequel.

Lemma 1. *All the eigenvalues of problem* (1), $(2.a, b, c, d^*)$ *for* $\delta \in \left[0, \frac{\pi}{2}\right)$ *or* $\delta = \frac{\pi}{2}$, $\beta \in \left[0, \frac{\pi}{2}\right)$ *are positive.*

Proof. Let u be a solution of (3) which satisfies the initial conditions $u(0) = 0$, $u'(0) = 1$. The disfocal condition of equation (3) implies that $u'(x) > 0$ in $[0, l]$. Therefore, if h denotes the solution of (3) satisfying the initial conditions $u(0) = c$, $u'(0) = 1$, where c is a sufficiently small constant, then we have also $h'(x) > 0$ on $[0, l]$. Thus $h(x) > 0$ in $[0, l]$.

The following substitution [7, theorem 12.1]

$$t = t(x) = l\omega^{-1}\int_0^x h(s)\,ds, \quad \omega = \int_0^l h(s)\,ds, \tag{5}$$

transform $[0, l]$ into the interval $[0, l]$, and equation (1) into

$$(\widehat{p}\ddot{y})^{\cdot\cdot} = \lambda\widehat{r}y, \tag{6}$$

where $\widehat{p} = \left(l\omega^{-1}h\right)^3$, $\widehat{r} = l^{-1}\omega h^{-1}$; $h(x), y(x)$ are taken as functions of t and $\cdot := \dfrac{d}{dt}$. Furtheremore, the following relations are useful in the sequel:

$$\dot{y} = l^{-1}\omega h^{-1}y', \quad l^2\omega^{-2}h^3\ddot{y} = hy'' - h'y', \quad \widehat{T}y = \left(\left(l\omega^{-1}h\right)^3\ddot{y}\right)^{\cdot} = Ty. \tag{7}$$

It is clear from the second relation (7), that the sign of y'' is not necessarily preserved after the transformation (7).

In this case the transformed problem is determined by equation (6) and the boundary condition

$$\dot{y}(0) = 0, \tag{2.a'}$$

$$y(0)\cos\beta + \widehat{T}y(0)\sin\beta = 0, \tag{2.b'}$$

$$\dot{y}(l)\cos\gamma^* + \widehat{p}(l)\,\ddot{y}(l)\sin\gamma^* = 0, \tag{2.c'}$$

$$y(l)\cos\delta - \widehat{T}y(l)\sin\delta = 0, \tag{2d'}$$

where $\gamma^* = arctgl^{-2}\omega^2h^{-1}(l)\left(h(l)\cos\gamma + h'(l)\sin\gamma\right)^{-1} \in [0, \pi/2)$.

It is know that the eigenvalues of (6), $(2.a', b', c', d')$ are given by the max-min principle [3, p.220-221] using the Rayleigh quotient

$$R[y] = \frac{\int\limits_0^l \widehat{p}\ddot{y}^2 dt + N[y]}{\int\limits_0^l \widehat{r}y^2 dt},$$

where

$$N[y] = (y(0))^2 \, ctg\beta + \left(\dot{y}(l)\right)^2 ctg\gamma^* + (y(l))^2 \, ctg\delta.$$

It follows by inspection of the numerator R that zero is an eigenvalue only in the case: $\beta = \delta = \dfrac{\pi}{2}$. Hence, all the eigenvalues of problem $(6), (2.a', b', c', d')$ for $\delta \in \left[0, \frac{\pi}{2}\right)$ *or* $\delta = \frac{\pi}{2}$, $\beta \in \left[0, \frac{\pi}{2}\right)$ *are positive.*

Lemma 1 is proved.

Lemma 2. *Let E be the space of solutions of the problem* $(1), (2.a, b, c)$. *Then* $\dim E = 1$.

The proof is similar to that of [2, lemma 2] using lemma 1 (see also [6, lemma 2.2]).

Lemma 3 [3, lemma 2.2]. *Let* $\lambda > 0$ and u *be a solution of the differential equation* (1) *for* $q \equiv 0$ *which satisfies the boundary conditions* $(2.a, c)$. *If a is a zero of u and u'' in the open interval $(0, l)$, then $u'(x)\,Tu(x) < 0$ in a neighbourhood of a. If a is a zero of u' or Tu in $(0, l)$ then $u(x)\,u''(x) < 0$ in a neighborhood of a.*

Theorem 1. *Let u be a nontrivial solution of the problem* (1), $(2.a, c)$ *for* $\lambda > 0$. *Then the Jacobian* $J[u] = r^3 \cos\psi \sin\psi$ *of the transformation* (4) *does not vanish in* $(0, l)$.

Proof. Let u be a nontrivial solution of (1) which satisfies the boundary conditions $(2.a, c)$. Assume first that the corresponding angle ψ satisfies $\psi(x_0) = n\pi$ for some integer n and for some $x_0 \in (0, l)$. Then the transformation (4) implies that $u(x_0) = Tu(x_0) = 0$. Using the transformation (7), the solution u of (6) also satisfies $u(t_0) = \widehat{T}y(t_0) = 0$, where $t_0 = l^{-1}\omega \int\limits_0^{x_0} h(s)\,ds \in (0, l)$. However, this is incompartible with the conclusion of lemma 3.

Now assume for the solution u, the corresponding angle ψ satisfies $\psi(x_0) = m\pi/2$ for some integer m and for some $x_0 \in (0, l)$. Then the transformation (4) implies that $u'(x_0) = u''(x_0) = 0$. Using the transformation (7), the solution u of (6) also satisfies $\dot{u}(t_0) = \ddot{u}(t_0) = 0$, where $t_0 = l^{-1}\omega \int\limits_0^{x_0} h(s)\,ds \in (0, l)$. Lemma 3 with these conditions yields a contradiction.

Theorem 1 is proved.

Let $y(x, \lambda)$ be a nontrivial solution of the problem (1), $(2.a, b, c)$ for $\lambda > 0$ and $\theta(x, \lambda), \varphi(x, \lambda)$ the corresponding functions in (4). Without loss of generality we can define the initial values of these functions as follows (see [4, theorem 3.3]):

$$\theta(0, \lambda) = \beta - \frac{\pi}{2}, \ \varphi(0, \lambda) = 0.$$

With obvious modifications, the results stated in [4] are true for solution of the system (1), $(2.a, b, c, d')$. In particular we have the following results.

Theorem 2. *$\theta(l,\lambda)$ is a strictly increasing continuous function of λ.*

Theorem 3. *The eigenvalues of the boundary value problem* $(1), (2.a, b, c, d^*)$ *for $\delta \in \left[0, \frac{\pi}{2}\right]$ (except the case $\beta = \delta = \pi/2$) form an infinite increasing sequence $\{\lambda_n(\delta)\}_{n=1}^{\infty}$ such that $0 < \lambda_1(\delta) < \lambda_2(\delta) < ... < \lambda_n(\delta) < ...$, and in addition*

$$\theta(l, \lambda_n(\delta)) = (2n-1)\pi/2 - \delta, \ n \in \mathbb{N}.$$

Moreover, the eigenfunction $\upsilon_n^{(\delta)}(x)$ corresponding to the eigenvalue $\lambda_n(\delta)$ has $n-1$ simple zeros in the interval $(0,l)$.

Remark 1. In the case $\beta = \delta = \frac{\pi}{2}$ the first eigenvalue of boundary value problem $(1), (2.a, b, c, d^*)$ is equal to zero and the corresponding eigenfunction is constant; the statement of theorem 3 is true at $n \geq 2$.

Obviously, the eigenvalues $\lambda_n(\delta)$ problem $(1), (2.a, b, c, d^*)$, $\delta \in \left[0, \frac{\pi}{2}\right]$ are zeros of the entire function $y(l,\lambda)\cos\delta - Ty(l,\lambda)\sin\delta$. We set $\mu_n = \mu_n(0)$ and $\nu_n = \mu_n\left(\frac{\pi}{2}\right)$, $n \in \mathbb{N}$. Note that the function $F(\lambda) = Ty(l,\lambda)/y(l,\lambda)$ is defined for $\lambda \in D = (\mathbb{C}\backslash\mathbb{R}) \cup \left(\bigcup\limits_{n=1}^{\infty} (\mu_{n-1}, \mu_n)\right)$, where $\mu_0 = -\infty$.

Lemma 4. (see [2, lemma 5]). *Let $\lambda \in D$. Then following relation holds:*

$$\frac{dF(\lambda)}{d\lambda} = \frac{1}{y^2(l,\lambda)} \int\limits_0^l y^2(x,\lambda)\,dx > 0.$$

In equation (1) we set $\lambda = \rho^4$. As is known (see [8], ch II, §4.5, theorem 1), in each subdomain T of the complex ρ-plane equation (1) has four linearly independent solutions $z_k(x,\rho)$, $k = \overline{1,4}$, regular in ρ (for sufficiently ρ) and satisfying the relations

$$z_k^{(s)}(x,\rho) = (\rho\omega_k)^s e^{\rho\omega_k x}\left[1 + O\left(\frac{1}{\rho}\right)\right], \ k = \overline{1,4}, \ s = \overline{0,3}, \tag{8}$$

where ω_k, $k = \overline{1,4}$, are the distinct 4th roots of unity.

By brevity, we introduce the notation $s(\delta_1, \delta_2) \equiv sgn\delta_1 + sgn\delta_2$.

Using relations (8) and taking account of boundary conditions $(2.a, b, c)$ we obtain

$$y(x,\lambda) = \begin{cases} \left[\sin\left(\rho x + \frac{\pi}{2} sgn\beta\right) - \cos\left(\rho x + \frac{\pi}{2} s(\beta,\gamma)\right) e^{\rho(x-l)}\right] \times \\ \times \left(1 + O\left(\frac{1}{\rho}\right)\right), \text{ if } \beta \in \left(0, \frac{\pi}{2}\right], \\ \left[\sin\left(\rho x - \frac{\pi}{4}\right) - e^{-\rho x} + (-1)^{1-sgn\gamma} \times \right. \\ \left. \times \sqrt{2}\sin\left(\rho l + \frac{\pi}{4}(-1)^{sgn\gamma}\right) e^{\rho(x-l)}\right] \left(1 + O\left(\frac{1}{\rho}\right)\right), \text{ if } \beta = 0, \end{cases} \tag{9}$$

$$Ty(x,\lambda) = \begin{cases} -\rho^3\left[\cos\left(\rho x + \frac{\pi}{2} sgn\beta\right) + \cos\left(\rho l + \frac{\pi}{2} s(\beta,\gamma)\right) e^{\rho(x-l)}\right] \\ \left(1 + O\left(\frac{1}{\rho}\right)\right), \text{ if } \beta \in \left(0, \frac{\pi}{2}\right], \\ -\rho^3\left[\sin\left(\rho x + \frac{\pi}{4}\right) - e^{-\rho x} - (-1)^{1-sgn\gamma} \times \right. \\ \left. \times \sqrt{2}\sin\left(\rho l + \frac{\pi}{4}(-1)^{sgn\gamma}\right) e^{\rho(x-l)}\right] \left(1 + O\left(\frac{1}{\rho}\right)\right), \\ \text{if } \beta = 0. \end{cases} \tag{10}$$

Remark 2. As an immediate consequence of (9), we obtain the number of zeros in the interval $(0, l)$ of the function $y(x, \lambda)$ tends to ∞ as $\lambda \to \pm\infty$.

By taking into account relation (9) and (10), we obtain the asymptotic formulas

$$F(\lambda) = \begin{cases} \left(\sqrt{2}\right)^{1-2sgn\gamma} \rho^3 \dfrac{\cos\left(\rho l + \frac{\pi}{2} sgn\beta + \frac{\pi}{4} sgn\gamma\right)}{\cos\left(\rho l + \frac{\pi}{2} sgn\beta + \frac{\pi}{4}\left(1 + sgn\gamma\right)\right)} \times \\ \left(1 + O\left(\frac{1}{\rho}\right)\right), \text{ if } \beta \in \left(0, \frac{\pi}{2}\right] \\ \left(\sqrt{2}\right)^{1-2sgn\gamma} \rho^3 \dfrac{\cos\left(\rho l + \frac{\pi}{4}\left(sgn\gamma - 1\right)\right)}{\cos\left(\rho l + \frac{\pi}{4} sgn\gamma\right)} \times \\ \left(1 + O\left(\frac{1}{\rho}\right)\right), \text{ if } \beta = 0. \end{cases} \tag{11}$$

Furthermore, we have

$$F(\lambda) = -\left(\sqrt{2}\right)^{1-2sgn\gamma} \sqrt[4]{|\lambda|^3}\left(1 + O\left(\frac{1}{\sqrt[4]{|\lambda|}}\right)\right), \text{ as } \lambda \to -\infty. \tag{12}$$

We define numbers τ, ν, α_n and β_n, $n \in \mathbf{N}$, and a function $\varphi(x, t)$, $x \in [0, l]$, $t \in \mathbf{R}$, as follows:

$$\tau = \begin{cases} 3\left(1 + s\left(\beta, \delta\right)\right)/4 - 1, \text{ if } \gamma \in \left(0, \frac{\pi}{2}\right], \\ 5/4 - 3/8\left(\left(-1\right)^{sgn\beta} + \left(-1\right)^{sgn\delta}\right) - 1, \text{if } \gamma = 0, \end{cases}$$

$$\nu = \begin{cases} 3\left(1 + s\left(\beta, |c|\right)\right)/4, \quad \text{if } \gamma \in \left(0, \frac{\pi}{2}\right], \\ 5/4 - 3/8\left(\left(-1\right)^{sgn\beta} + \left(-1\right)^{sgn|c|}\right), \text{ if } \gamma = 0, \end{cases}$$

$$\alpha_n = (n - \tau)\frac{\pi}{l}, \; \beta_n = (n - \nu)\frac{\pi}{l},$$

$$\varphi(x, t) = \begin{cases} \sin\left(tx + \frac{\pi}{2} sgn\beta\right) - \cos\left(tl + \frac{\pi}{2} s\left(\beta, \gamma\right)\right) e^{-t(l-x)}, \\ \text{if } \beta \in \left(0, \frac{\pi}{2}\right] \\ \sqrt{2}\sin\left(tx - \frac{\pi}{4}\right) + e^{-tx} + (-1)^{1-sgn\gamma}\sqrt{2}\sin\left(tl + (-1)^{sgn\gamma}\frac{\pi}{4}\right) e^{-t(l-x)}, \\ \text{if } \beta = 0. \end{cases}$$

By virtue [1, theorem 3.1] one has the asymptotic formulas

$$\sqrt[4]{\lambda_n(\delta)} = \alpha_n + O\left(\frac{1}{n}\right), \tag{13}$$

$$\upsilon_n^{(\delta)}(x) = \varphi(x, \alpha_n) + O\left(\frac{1}{n}\right), \tag{14}$$

where relation (14) holds uniformly for $x \in [0, l]$.

By (12) we have

$$\lim_{\lambda \to -\infty} F(\lambda) = -\infty \tag{15}$$

Remark 3. It follows by theorem 3, lemma 4 and relation (15) that if $\lambda < 0$ or $\lambda = 0$ and $\beta \in [0, \pi/2)$, then $F(\lambda) = \dfrac{Ty(l, \lambda)}{y(l, \lambda)} < 0$; besides, if $\lambda = 0$ and $\beta = \pi/2$, then $Ty(l, \lambda) = 0$.

Lemma 5. *If* $\lambda \geq 0$ and $\lambda \in \left(\mu_{n-1}, \mu_n\right]$, $n \in \mathbf{N}$, *then* $m(\lambda) = n - 1$.

The proof is similar to that of [2, theorem 4] using theorem 2 and remark 3.

Theorem 4. *The eigenvalues of spectral problem* (1), $(2.a, b, c, d^*)$ *for* $\delta \in \left(\frac{\pi}{2}, \pi\right)$ *form the infinitely increasing sequence* $\{\mu_n(\delta)\}_{n=1}^{\infty}$, *such that* $\mu_n(\delta) > 0$ *for* $n \geq 2$. *Besides*

a) the eigenfunction $y_n^{(\delta)}(x)$, *corresponding to the eigenvalue* $\mu_n(\delta) \geq 0$ *has exactly* $(n-1)$ *simple zeros in the interval* $(0, l)$;

b) if $\beta \in \left[0, \frac{\pi}{2}\right)$, *then* $\mu_1(\delta) > 0$ *for* $\delta \in \left(\frac{\pi}{2}, \delta_0\right)$; $\mu_1(\delta) = 0$ *for* $\delta = \delta_0$; $\mu_1(\delta) < 0$ *for* $\delta \in (\delta_0, \pi)$, *where* $\delta_0 = arctgTy(l, 0)/y(l, 0)$;

c) if $\beta = \frac{\pi}{2}$, *then* $\lambda_1(\delta) < 0$.

The proof parallels the proof of theorem 4 [2] using theorems 1,2,3 and lemmas 4,5.

The following non-selfadjoint boundary value problem

$$y^{(4)}(x) - \left(q(x)\, y'(x)\right)' = \lambda y(x), \ x \in (0, l),$$

$$y(0) = y'(0) = Ty(0) = y'(l)\cos\gamma + y''(l)\sin\gamma = 0, \tag{2**}$$

has an infinite set of non-positive eigenvalues ρ_n tending to $-\infty$ and satisfying the asymptote

$$\lambda_n = -\left(n - \frac{1}{4}(1 + sgn\gamma)\right)^4 \frac{\pi^4}{l^4} + o\left(n^4\right), \ n \to \infty.$$

Remark 4. The number of zeros of the eigenfunction $y_1^{(\delta)}(x)$ corresponds to an eigenvalue $\mu_1(\delta) < 0$ can be arbitrary. Really, as $\mu_1(\delta) < 0$ varies, new zeros of the corresponding eigenfunction $y_1^{(\delta)}(x)$ enter to the interval $(0, l)$ only through the endpoint $x = 0$, since $y_1^{(\delta)}(1) \neq 0$ by theorem 3, and hence the number $m(\mu_1(\delta))$ of its zeros in $(0, l)$, in the case $\beta \in \left(0, \frac{\pi}{2}\right]$, is asymptotically equivalent to the number of the eigenvalues of problem (1), (2^{**}) which are upper than $\mu_1(\delta)$. In the case $\beta = 0$ see [6, §5, theorem 5.3].

For $c \neq 0$, *we find a positive integer* N *form the inequality* $\mu_{N-1} < -d/c \leq \mu_N$.

Theorem 5. *The eigenvalues of the boundary value problem (1), (2) form an infinitely increasing sequence* $\lambda_1, \lambda_2, ..., \lambda_n, ...$, *moreover,* $\lambda_n > 0$ for $n \geq 3$. *The corresponding eigenfunctions* $y_1(x), y_2(x), ..., y_n(x), ...$ *have the following oscillation properties:*

(a) if $c = 0$, *then* $y_n(x)$, $n \geq 2$ *has exactly* $n-1$ *simple zeros; the number of zeros of the eigenfunction* $y_1(x)$ *is equal to zero in the case* $\lambda_1 \geq 0$, *can be arbitrary in the case* $\lambda_1 < 0$.

(b) if $c \neq 0$, *then* $y_n(x)$ *has exactly* $n-1$ *simple zeros for* $n \leq N$ *and exactly* $n-2$ *simple zeros for* $n > N$ *in the interval* $(0, l)$, *in the case* $\lambda_n \geq 0$; *if* λ_1 *or* λ_1, λ_2 *be negative, then the number of zeros of the eigenfunctions* $y_1(x)$ *or* $y_1(x)$, $y_2(x)$ *can be arbitrary.*

The proof parallels the proof of theorem 2.2 [1] using remark 4.

Theorem 6. [1, theorem 3.1]. *One has the asymptotic formulas*

$$\sqrt[4]{\lambda_n} = \beta_n + O\left(\frac{1}{n}\right), \tag{16}$$

$$y_n(x) = \varphi(x, \beta_n) + O\left(\frac{1}{n}\right), \tag{17}$$

where relation (17) holds uniformly for $x \in [0, l]$.

We denote by $(B.C.)$ the set of separated boundary conditions $(2.a, b, c)$.

The spectral problem (1), (2) reduced to a problem on eigenvalues for the linear operator L in Hilbert space $H = L_2(0, l) \oplus \mathbf{C}$ with scalar product

$$(\hat{y}, \hat{u}) = (\{y, m\}, \{u, s\}) = (y, u)_{L_2} + \sigma^{-1} m\bar{s},$$

where $(y, u)_{L_2} = \int\limits_0^l y\bar{u}dx$,

$$L\hat{y} = L\{y, m\} = \left\{(Ty)'(x), dTy(l) - by(l)\right\},$$

with the domain

$$D(L) = \left\{\hat{y} = \{y, m\} \in H : y(x) \in W_2^4(0, l),\right.$$

$$\left.(Ty)' \in L_2(0, l), y \in (B.C.),\ m = ay(l) - cTy(l)\right\}$$

that is dense in H [9].

Obviously, the operator L is well defined. By immediate verification we conclude that problem (1), (2) is equivalent to the following spectral problem

$$L\hat{y} = \lambda\hat{y},\ \ \hat{y} \in D(L), \tag{18}$$

i.e., the eigenvalues λ_n of problem (1), (2) and those of problem (18) coincide, moreover, there exists a corresponding between the eigenfunctions

$$\hat{y} = (y(x), m) \longleftrightarrow y(x).$$

The operator L will be self-adjoint, discrete, semibounded from below in H and so possesses by system of eigenvectors $\{y_n(x),\ m_n\}_{n=1}^{\infty}$ that forms orthogonal basis in H, where $y_n(x),\ n \in \mathbf{N}$, are eigenfunctions of problem (1), (2) and $m_n = ay_n(l) - cTy_n(l)$.

The eigenvalues $\nu_n,\ n \in \mathbf{N}$, of the boundary value problem $(1), (2.a, b, c, d^*)$ for $\delta = \frac{\pi}{2}$ are zeros of the function $F(\lambda)$. In similar way (see the proof of theorem 2.2 [1]), one can show that the equation $F(\lambda) = 0$ has the unique solution $\nu_n = \lambda_n\left(\frac{\pi}{2}\right)$ in each interval $\left(\mu_{n-1}, \mu_n\right)$. Consequently,

$$\mu_{n-1} < \nu_n < \mu_n,\ n \in \mathbf{N}. \tag{19}$$

Lemma 6. $m_n = ay_n(l) - cTy_n(l) \neq 0$ for $n \in \mathbf{N}$.

Proof. Let $m_k = 0$, where k be some positive integer. If $c \neq 0$, then $Ty_k(l) = \frac{a}{c}y_k(l)$. In view of $(2.d)$ we have $\frac{\sigma}{c}y_k(l) = 0$. Since $\sigma > 0$, it follows that $y_k(l) = 0$. Hence $Ty_k(l) = 0$. If $c = 0$, then $ad \neq 0$, consequently $y_k(l) = 0$. By (2.d) we obtain $Ty_k(l) = 0$. Hence, $y_k(l) = Ty_k(l) = 0$, which contradicts the relation (19). Lemma 7 is proved.

Let $\delta_n = \left(\|y_n\|_2^2 + m_n^2/\sigma \right)^{\frac{1}{2}}$, where $\|\cdot\|_p$ is the norm in $L_p(0,l)$. Then the system $\{\hat{\upsilon}_n\}_{n=1}^{\infty}$, $\hat{\upsilon}_n = \dfrac{1}{\delta_n}\hat{y}_n$ is a orthonormal basis in the space H. Then for any vector $\hat{f} = \{f, \tau\}$ it holds the expression

$$f = \{f,\tau\} = \sum_{n=1}^{\infty} \left(\hat{f}, \hat{\upsilon}_n \right) \hat{\upsilon}_n = \sum_{n=1}^{\infty} \left(\{f,\tau\}, \{\upsilon_n, s_n\} \right) \{\upsilon_n, s_n\} =$$

$$= \sum_{n=1}^{\infty} \left((f, \upsilon_n)_{L_2} + \sigma^{-1}\tau s_n \right) \{\upsilon_n, s_n\},$$

whence the equalities

$$f = \sum_{n=1}^{\infty} \left((f, \upsilon_n)_{L_2} + \sigma^{-1}\tau s_n \right) \upsilon_n \tag{20}$$

$$\tau = \sum_{n=1}^{\infty} \left((f, \upsilon_n)_{L_2} + \sigma^{-1}\tau s_n \right) s_n \tag{21}$$

follow, where $s_n = \dfrac{m_n}{\delta_n}$, $n \in \mathbf{N}$.

Let $\tau = 0$. Then from (20) and (21) we get, respectively,

$$f = \sum_{n=1}^{\infty} (f, \upsilon_n)_{L_2} \upsilon_n, \tag{22}$$

$$0 = \sum_{n=1}^{\infty} (f, \upsilon_n)_{L_2} s_n. \tag{23}$$

Let r be an arbitrary fixed natural numbers. By lemma 6 we have $s_r \neq 0$. Then in view of (23) we obtain

$$(f, \upsilon_r)_{L_2} = -\frac{1}{s_r} \sum_{\substack{n=1 \\ n \neq r}}^{\infty} (f, \upsilon_n)_{L_2} s_n. \tag{24}$$

Taking into account (24), from (22) we get

$$f = \sum_{\substack{n=1 \\ n \neq r}}^{\infty} (f, \upsilon_n)_{L_2} \left(\upsilon_n - \frac{s_n}{s_r} \upsilon_r \right) = \sum_{\substack{n=1 \\ n \neq r}}^{\infty} (f, y_n)_{L_2} \left(\frac{1}{\delta_n} \upsilon_n - \frac{s_n}{\delta_n s_r} \upsilon_r \right) =$$

$$= \sum_{\substack{n=1 \\ n \neq r}}^{\infty} (f, y_n)_{L_2} \left(\frac{y_n}{\delta_n^2} - \frac{m_n}{\delta_n^2 m_r} y_r \right). \tag{25}$$

We have

$$\left(y_n, \frac{y_k}{\delta_k^2} - \frac{m_k}{\delta_k^2 m_r} y_r \right) = \frac{1}{\delta_k^2} (y_n, y_k) - \frac{m_k}{\delta_k^2 m_r} (y_n, y_r) =$$

$$= \frac{1}{\delta_k^2} \left\{ (y_n, y_k) - \frac{m_k}{m_r} (y_n, y_r) \right\} = \frac{1}{\delta_k^2} \left[\{ (\hat{y}_n, \hat{y}_k) - \sigma^{-1} m_n m_k \} - \right.$$

$$-\frac{m_k}{m_r}\left\{(\hat{y}_n,\hat{y}_r)-\sigma^{-1}m_nm_r\right\}\Big]=\frac{1}{\delta_k^2}\left[\delta_n\delta_k\delta_{n_k}-\sigma^{-1}m_nm_k+\frac{m_k}{m_r}\sigma^{-1}m_nm_r\right]=$$

$$=\frac{\delta_n}{\delta_k}\delta_{nk}=\delta_{nk},$$

where δ_{nk} is the Kronecker delta, i.e., the system

$$\{u_n(x)\}_{n=1,n\neq r}^{\infty},\ u_n(x)=\frac{1}{\delta_n^2}\left\{y_n(x)-\frac{m_n}{m_r}y_r(x)\right\}$$

is conjugate to the system $\{y_n(x)\}_{n=1,n\neq r}^{\infty}$. Hence, the system $\{u_n(x)\}_{n=1,n\neq r}^{\infty}$ is a Riesz basis in $L_2(0,l)$. Then the system $\{y_n(x)\}_{n=1,n\neq r}^{\infty}$ is also Riesz basis in the space $L_2(0,l)$.

Thus, we proved the following

Theorem 7. *Let r be an arbitrary natural number. Then the system $\{y_n(x)\}_{n=1,n\neq r}^{\infty}$ forms a Riesz in $L_2(0,l)$.*

Lemma 7 [1, lemma 4.1]. *One has the asymptotic formula*

$$u_n(x)=l^{-1}y_n(x)+O\left(\frac{1}{n}\right).$$

Theorem 8. *Let r be an arbitrary fixed natural number. Then the system $\{y_n(x)\}_{n=1,n\neq r}^{\infty}$ is a basis in the space $L_p(0,l)$, $1<p<\infty$.*

The proof parallels the proof of theorem 5.1 [1] using theorems 5,6,7 and lemma 7.

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