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WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR HYPERBOLIC EQUATIONS WITH NON-LIPSCHITZ COEFFICIENTS

Abstract

In this paper we prove the well-posedness of the Cauchy problem for hyperbolic equations with anisotropic elliptic part and some non-Lipschitz coefficients.

Let's consider the Cauchy problem for a second order hyperbolic equation:

$$u_{tt}-\sum_{i,j=1}^{n}a_{ij}\left(t\right)u_{x_ix_j}+\sum_{j=1}^{n}b_j\left(t\right)u_{x_j}+c\left(t\right)u=0,\ \ (t,x)\in[0,T]\times R^n, \tag{1}$$

with initial conditions

$$u\left(0,x\right)=u_0\left(x\right),\ u_t\left(0,x\right)=u_1\left(x\right),\ x\in R^n, \tag{2}$$

where

$$a\left(t,\xi\right)\equiv\sum_{i,j=1}^{n}a_{ij}\left(t\right)\frac{\xi_i\xi_j}{\left|\xi\right|^2}\geq\lambda_0>0. \tag{3}$$

It is known that if $a\left(t,\xi\right)$ satisfies the Lipschitz condition and $b_j\left(t\right),\ c\left(t\right)\in L_\infty\left(0,T\right), j=1,2...,n,$ then for any $u_0\in H^s\left(R_n\right),\ u_1\in H^{s-1}\left(R_n\right)$ the problem (1), (2) has a unique solution

$$u\left(\cdot\right)\in C\left(\left[0,T\right];H^s\left(R^n\right)\right)\cap C^1\left(\left[0,T\right],H^{s-1}\left(R^n\right)\right),$$

where $s\geq1$ (see [1], c.3). From here follows that the problem (1), (2) is well-posed in $C^\infty\left(R^n\right)$.

If we reject the Lipschitz condition then this result generally speaking, stop to be valid (see [2]).

In the paper [3] it is proved that if $a\left(t,\xi\right)\in LL_\omega\left(0,T\right)$, i.e. if $a\left(t,\xi\right)$ satisfies the logarithmic Lipschitz condition:

$$\left|a\left(t+\tau,\xi\right)-\alpha\left(t,\xi\right)\right|\leq c\left|\tau\right|\cdot\left|\log\right|\left|\tau\right|\cdot\omega\left(\left|\tau\right|\right),$$

where $\omega\left(\left|\tau\right|\right)$ monotonically decreasing tends to zero, and $\left|\log\right|\left|\tau\right|\cdot\omega\left(\left|\tau\right|\right)$ tends to infinity, then the solution loses some reqularity then there exists $\delta>0$ such that, for all $u_0\in H^s\left(R_n\right)$ $u_1\in H^{s-1}\left(R_n\right)$ the problem (1),(2) has a unique solution $u\in C\left(\left[0,T\right],H^{s-\delta}_{\left(R_n\right)}\right)\cap C^1\left(\left[0,T\right],H^{s-1-\delta}_{\left(R_n\right)}\right)$(this behavior goes under the name of loss of derivatives). But in this case the problem (1), (2) is well-posed in $C^\infty\left(R^n\right)$.

In the paper [4] it is considered the case when $a_{i,j}\left(t\right)=0,\ i\neq j$ and one a part of coefficients belongs to the class $LL_\omega\left(0,T\right)$ and other part of coefficients satisfies the Lipschitz condition. It is proved that the loss of derivatives occurs in those variables x_k for which appropriate coefficient $a_{kk}\left(t\right)$ belongs to the class $LL_\omega\left(0,T\right)$.

It is interesting to investigate the Cauchy problem for the equation (1) with singular coefficients. Many interesting results have been obtained in this direction. For example in the paper [4] it is supposed that for each $\xi \in R^n |\{0\}$ $a(t,\xi) \in C^1(0,T]$ and

$$t^q \left|a'(t,\xi)\right| \le c,\ t \in (0,T], \tag{4}$$

where $q \ge 1$. It is proved that if $q = 1$, the problem (1), (2) is well-posed in $C^\infty(R^n)$. If $q > 1$ and

$$t^p \left|a(t,\xi)\right| \le c_1,\ t \in (0,T], \tag{5}$$

where $p \in [0,1) \cap [0,q-1)$,then the problem (1), (2) is well-posed in the Geverey class $\gamma^{(s)}(R^n),\, s < \dfrac{q-p}{q-1}$ (see [4]).

In this paper we consider the Cauchy problem for a higher order hyperbolic equation with anisotropic elliptic part

$$u_{tt} - \sum_{k=1}^{n} (-1)^{l_k} a_k(t) D_{x_k}^{2l_k} u + \sum_{\left|\frac{a}{l}\right| \le 1} b_a(t) D_x^\alpha u = 0,\ \ (t,x) \in [0,T] \times R^n, \tag{6}$$

with initial conditions

$$u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ x \in R^n, \tag{7}$$

where $l_k \in N,\ \alpha_k \in N \cup \{0\},\ k = 1,2,...,n,\ \left|\dfrac{\alpha}{l}\right| = \dfrac{\alpha_1}{l_1} + ... + \dfrac{\alpha_n}{l_n},$

$$a_k(t) \ge a > 0,\ t \in [0,T], \tag{8}$$

$$t^{q_k} \left|a'_k(t)\right| \le c,\ t \in (0,T],\ k = 1,2...,n, \tag{9}$$

$$b_\alpha(t) \in L_\infty(0,T). \tag{10}$$

In order to formulate the basic results we introduce some denotation. By $W_2^{\lambda l}(R^n)$ we'll denote a functional space with the norm

$$\|u\|_{W_2^{\lambda l}(R^n)} = \left[\int_{R^n} \left(1 + \sum_{k=1}^{n} \xi_k^{2l_k}\right)^\lambda |\hat{u}|^2\, dx\right]^{\frac{1}{2}},$$

where $\lambda \ge 0,\ \hat{u} = F[u]$,$F$ is a Fourier transformation.

Let H be some Hilbert space. For $s \ge 1$ we'll denote by $\gamma^{(s)}(R^n;H)$ the Geverey space, i.e. $u \in \gamma^{(s)}(R^n;H)$ if $u \in C^\infty(R^n;H)$, and for all compact $K \subset R^n$ there exists c,$M > 0$ such that

$$\left\|\frac{\partial^\alpha u(x)}{\partial x^\alpha}\right\|_H \le cM^{|\alpha|}(\alpha!)^s.$$

We introduce the following denotations:

$$x' = (x_1,...,x_{n_1}),\ \ x'' = (x_{n_1+1},...,x_n),\xi' = \left(\xi_1,...\xi_{n_1}\right),\xi'' = \left(\xi_{n_1+1},...,\xi_n\right)$$

$$l' = (l_1,...,l_{n_1}),\ \ l'' = (l_{n_1+1},...,l_n).$$

The main results of this paper are the following theorems.

Theorem 1. *Let the conditions (8)-(10) be satisfied, where*

$$q_k \in [0,1)\,,\ for\ k = 1,2,...,n_1, \tag{11}$$

$$q_k = 1,\quad for\quad k = n_1+1,...,n. \tag{12}$$

Then for any $u_0 \in C^\infty\left(R^{n_2}_{x''}; W_2^{\lambda l'}\left(R^{n_1}_{x'}\right)\right)$, $u_1 \in C^\infty\left(R^{n_2}_{x''}; W_2^{(\lambda-1)l'}\left(R^{n_1}_{x'}\right)\right)$, $\lambda \geq 1$ *the problem (6), (7) has a unique solution*

$$u \in C\left([0,T]\,; C^\infty\left(R^{n_2}_{x''}; W_2^{\lambda l'}\left(R^{n_1}_{x'}\right)\right)\right) \cap C^1\left([0,T]\,; C^\infty\left(R^{n_2}_{x''}; W_2^{(\lambda-1)l'}\left(R^{n_1}_{x'}\right)\right)\right).$$

Theorem 2. *Let the conditions (8)- (10) be satisfied, where*

$$q_k \in [0,1)\,,\ for\ k = 1,2,...,n_1, \tag{13}$$

$$q_k > 1,\ for\ k = n_1+1,...,n. \tag{14}$$

Additionally, let the conditions

$$t^{p_k}\left|a_k(t)\right| \leq c,\ t \in [0,T]\,,\ for\ k = n_1+1,...,n. \tag{15}$$

be satisfied, where $p \in [0,1) \cap [0, q-1)$. *Then for any* $u_0 \in \gamma^{(s)}\left(R^{n_2}_{x''}; W_2^{\lambda l'}\left(R^{n_1}_{x'}\right)\right)$, *and*

$$u_1 \in \gamma^{(s)}\left(R^{n_2}_{x''}; W_2^{(\lambda-1)l'}\left(R^{n_1}_{x'}\right)\right),\ 1 \leq s < \frac{q-p}{q-1}$$

the problem (6), (7) has a unique solution

$$u \in C\left([0,T]\,; \gamma^{(s)}\left(R^{n_2}_{x''}; W_2^{\lambda l'}\left(R^{n_1}_{x'}\right)\right)\right) \cap C^1\left([0,T]\,; \gamma^{(s)}\left(R^{n_2}_{x''}; W_2^{(\lambda-1)l'}\left(R^{n_1}_{x'}\right)\right)\right).$$

Scheme of the proof. The proof is carried out by the standard regularization method that is based on some energetic estimations.

We denote $v(t,\xi) = F[u](t,\xi)$, where F is a Fourier transformation, let's define the weighted energetic function in the following way::

$$E(t) = \int\limits_{R^n}\left[\left|v'(t,\xi)\right|^2 + \left(1 + \left|\xi'\right|_{l'} + d\left(t,\xi''\right)\right)\left|v(t,\xi)\right|^2\right]\cdot\left(1+\left|\xi'\right|_{l'}\right)^{\lambda'} \times$$

$$\times\left(1+\left|\xi''\right|_{l''}\right)^{\lambda''}\exp\left[-\int\limits_0^t \alpha\left(\tau,\xi'\right)d\tau + \beta\left|\xi''\right|_{l''}^{\frac{q-1}{r}}\right]d\xi,$$

where $\lambda' \geq 0, \lambda'' \geq 0$, $\left|\xi'\right|_{l'} = \sum\limits_{k=1}^{n_1}\xi_k^{2l_k}, \left|\xi''\right|_{l''} = \sum\limits_{k=n_1+1}^{n}\xi_k^{2l_k}$, $\beta \geq 0$ is a sufficiently large number,

$$r = \begin{cases} s(q-1), & for\quad q > 1 \\ 1, & for\quad q = 1 \end{cases},$$

$$d\left(t,\xi''\right)=\begin{cases}\sum\limits_{k=n_1+1}^{n}a_k\left(T\right)\xi_k^{2l_k}, & for \quad T^r\left|\xi''\right|_{l''}\le 1,\\ \sum\limits_{k=n_1+1}^{n}a_k\left(\left|\xi''\right|_{l''}^{-\frac{1}{r}}\right)\xi_k^{2l_k}, & for \quad T^r\left|\xi''\right|_{l''}>1, t^r\left|\xi''\right|_{l''}\le 1,\\ \sum\limits_{k=n_1+1}^{n}a_k\left(t\right)\xi_k^{2l_k}, & for \quad t^r\left|\xi''\right|_{l''}>1,\end{cases}$$

and

$$\alpha\left(t,\xi''\right)=\begin{cases}d\left(t,\xi''\right)-\sum\limits_{k=n_1+1}^{n}a_k\left(t\right)\xi_k^{2l_k}, & for \quad t^r\left|\xi''\right|_{l''}\le 1,\\ \dfrac{\left|\sum\limits_{k=n_1+1}^{n}\acute{a_{k_t}}(t)\xi_k^{2l_k}\right|}{\sum\limits_{k=n_1+1}^{n}a_k(t)\xi_k^{2l_k}}, & for \quad t^r\left|\xi''\right|_{l''}>1.\end{cases}$$

At first we prove some auxiliary lemmas. The first of them is obvious.

Lemma 1. *There exists such $c_0>0$, that*

$$a_k\left(t\right)\le a_0,\ k=1,2,...,n_1.$$

Lemma 2. *If $q_k=1,\ k=n_1+1,...,n,$, then there exists such $c_1>0,\ c_2>0$, that*

$$d\left(t,\xi''\right)\le\left[c_1+c_2\ \ell n\ \left(1+\left|\xi''\right|_{l''}\right)\right]\left|\xi''\right|_{l''}.$$

Proof. Let $q_k=1,\ k=n_1+1,...,n$. Then from (9) we have

$$a_k\left(t\right)\le a_k\left(T\right)+\left|a_k\left(t\right)-a_k\left(T\right)\right|\le a_k\left(T\right)+$$

$$+\int\limits_t^T\left|a'_{k_t}\left(s\right)\right|ds\le a_k\left(T\right)+c\ \ell n\ \frac{T}{t}\le c_1+c_2\ \ell n\ \left(1+\frac{1}{t}\right) \tag{16}$$

It follows from (8) and (16), that

$$a_0\left|\xi''\right|_{l''}\le d\left(t,\xi''\right)\le\left[c_3+c_4\ \ell n\ \left(1+\left|\xi''\right|_{l''}\right)\right]\left|\xi''\right|_{l''}.$$

By definition, of $d\left(t,\xi''\right)$ for $\left|\xi''\right|_{l''}\le 1$ we have

$$d\left(t,\xi''\right)=\sum_{k=n_1+1}^{n}a_k\left(T\right)\xi_k^{2l_k}\le c_1\left|\xi''\right|_{l''}. \tag{17}$$

If $T\left|\xi''\right|_{l''}>1$, and $t\left|\xi''\right|_{l''}<1$, then

$$d\left(t,\xi''\right)=\sum_{k=n_1+1}^{n}a_k\left(\left|\xi''\right|_{l''}^{-1}\right)\xi_k^{2l_k}\le\left[c_1+c_2\ \ell n\ \left(1+\frac{1}{\left|\xi''\right|_{l''}^{-1}}\right)\right]\sum_{k=n_1+1}^{n}\xi_k^{2l_k}=$$

$$=\left(c_1+c_2\ \ell n\ \left(1+\left|\xi''\right|_{l''}\right)\right)\left|\xi''\right|_{l''}.$$

If $t\left|\xi''\right|_{l''} > 1$, then

$$d\left(t,\xi''\right) = \sum_{k=n_1+1}^{n} a_k\left(t\right)\xi_k^{2l_k} \le \left[c_1 + c_2\ \ell n\ \left(1+\frac{1}{t}\right)\right]\left|\xi''\right|_{l''} \le$$

$$\le \left[c_1 + c_2\ \ell n\ \left(1+\left|\xi''\right|_{l''}\right)\right]\left|\xi''\right|_{l''}. \tag{18}$$

Statement of the lemma follows from (8), (17) and (18).

The Lemma is proved.

Lemma 3. *If* $q_k = 1,\ k = n_1+1,\ n_2+2,...,n,$ *then there exists such constant* $c_3 > 0, \gamma > 0,$ that

$$\int_0^t \alpha\left(\tau,\xi''\right)d\tau \le c_3 + \gamma\ \ell n\ \ \left(1+\left|\xi''\right|_{l''}\right),$$

If $q_k = q > 1$,$k = n_1+1,\ n_1+2,...,n,$ *then there exists such* $\delta_1 > 0,$ *that*

$$\int_0^t \alpha\left(\tau,\xi''\right)d\tau \le \delta_1\left(\left|\xi''\right|_{l''}^{\frac{q-1}{r}} + 1\right).$$

Proof. Let's consider the case when $q_k = 1,\ k = n_1+1,...,n.$

If $T\left|\xi''\right|_{\ell''} \le 1$ then

$$\int_0^t \alpha\left(\tau,\xi''\right)d\tau \le \int_0^T \alpha\left(\tau,\xi''\right)d\tau \le$$

$$\le \int_0^T \left|\sum_{k=n_1+1}^{n} a_k\left(T\right)\xi_k^{2\ell_k} - \sum_{k=n_1+1}^{n} a_k\left(\tau\right)\xi_k^{2\ell_k}\right|d\tau \le$$

$$\le \sum_{k=n_1+1}^{n}\xi_k^{2\ell_k}\int_0^T\left|a_k\left(T\right)-a_k\left(\tau\right)\right|d\tau \le$$

$$\le T\cdot\max_{k=n_1+1,...,n} a_k\left(T\right)\left|\xi''\right|_{\ell''}\int_0^T a_k\left(\tau\right)d\tau \le a_T,$$

where $a_T = \max\limits_{k=n_1+1,...,n} a_k\left(\tau\right) + \dfrac{1}{T}\max\limits_{k=n_1+1,...,n}\displaystyle\int_0^T a_k\left(\tau\right)d\tau < +\infty.$

If $T\left|\xi''\right|_{l''} > 1$, then

$$\int_0^t \alpha\left(\tau,\xi''\right)ds \le \int_0^{\left|\xi''\right|_{l''}^{-1}} \alpha\left(s,\xi''\right)d\tau +$$

$$+\int\limits_{|\xi''|_{\ell''}^{-1}}^{T}\alpha\left(s,\xi''\right)ds\leq\int\limits_{0}^{|\xi''|_{l''}^{-1}}\left|d\left(\tau,\xi''\right)-\sum_{k=n_1+1}^{n}a_k\left(\tau\right)\xi_k^{2l_k}\right|d\tau+$$

$$+\int\limits_{|\xi''|_{\ell''}^{-1}}^{T}\frac{\left|\sum\limits_{k=n_1}^{n}a'_{k\tau}\left(\tau\right)\xi_k^{2\ell_k}\right|}{\sum\limits_{k=n_1+1}^{n}a_k\left(\tau\right)\xi_k^{2\ell_k}}d\tau\leq\int\limits_{0}^{|\xi''|_{l''}^{-1}}d\left(\tau,\xi''\right)d\tau+\sum_{k=n_1+1}^{n}\xi_k^{2l_k}\cdot\int\limits_{0}^{|\xi''|_{l''}^{-1}}\alpha_k\left(\tau\right)d\tau+$$

$$+\frac{c}{a}\sum_{k=n_1+1}^{n}\xi_k^{2\ell_k}\int\limits_{|\xi''|_{\ell''}^{-1}}^{T}\frac{d\tau}{\tau}\leq\int\limits_{0}^{|\xi''|_{\ell''}^{-1}}\left[c_1+c_2\ell n\left(1+\left|\xi''\right|_{\ell''}\right)\right]\left|\xi''\right|_{l''}d\tau+$$

$$+\sum_{k=n_1}^{n}\xi_k^{2\ell_k}\cdot c\int\limits_{0}^{|\xi''|_{\ell''}^{-1}}\ell n\frac{T}{\tau}d\tau+\frac{c}{a}\sum_{k=n_1+1}^{n}\int\limits_{|\xi''|_{\ell''}^{-1}}^{T}\frac{d\tau}{\tau}=c_1+c_2\ell n\left(1+\left|\xi''\right|_{\ell''}^{-1}\right)+$$

$$+c\left|\xi''\right|_{\ell''}\cdot\int\limits_{0}^{|\xi''|_{\ell''}^{-1}}\ell n\frac{T}{\tau}d\tau+\frac{c}{a}\int\limits_{|\xi''|_{\ell''}^{-1}}^{T}\frac{d\tau}{\tau}\leq c_3+c_4\ell n\left(1+\left|\xi''\right|_{\ell''}\right).\qquad(19)$$

Now let's consider the case $q_k>1,\ k=n_1+1,...,n$. In this case $r=(q-1)\,s$. If $T^r\left|\xi''\right|_{\ell''}\leq 1$, then

$$\int\limits_0^t\alpha\left(\tau,\xi''\right)d\tau\leq\int\limits_0^T\alpha\left(\tau,\xi''\right)d\tau\sum_{k=n_1+1}^{n}\int\limits_0^T\left|a_k\left(T\right)-a_k\left(\tau\right)\right|\xi_k^{2\ell_k}d\tau\leq$$

$$\leq\max_{k=n_1+1,...,n}a_k\left(T\right)\left|\xi''\right|_{l''}+\int\limits_0^T c\tau^{-p}d\tau\left|\xi''\right|_{\ell''}\leq$$

$$\leq a_T\cdot T^{1-r}+c\cdot\frac{1}{1-p}T^{1-p}\left|\xi''\right|_{\ell''}\leq a_T T^{1-r}+\frac{c}{1-p}T^{1-p-r}.$$

If $T\left|\xi''\right|_{\ell''}>1$, then

$$\int\limits_0^t\alpha\left(\tau,\xi''\right)d\tau\leq\int\limits_0^{|\xi''|_{\ell''}^{-\frac{1}{r}}}\alpha\left(\tau,\xi''\right)d\tau+\int\limits_{|\xi''|_{\ell''}^{-\frac{1}{r}}}^{T}\alpha\left(\tau,\xi''\right)d\tau\leq$$

$$\leq\int\limits_0^{|\xi''|_{\ell''}^{-\frac{1}{r}}}\left|d\left(\tau,\xi''\right)-\sum_{k=n_1+1}^{n}a_k\left(\tau\right)\xi_k^{2l_k}\right|d\tau+\int\limits_{|\xi''|_{\ell''}^{-\frac{1}{r}}}^{T}\alpha\left(\tau,\xi''\right)d\tau\leq$$

$$\leq \sum_{k=n_1+1}^{n} a_k\left(\left|\xi''\right|_{\ell''}^{-\frac{1}{r}}\right)\xi_k^{2\ell_k}\int_0^{|\xi''|_{l''}^{-\frac{1}{r}}} d\tau+\sum_{k=n_1+1}^{n}\xi_k^{2\ell_k}\int_0^{|\xi''|_{\ell''}^{-\frac{1}{r}}} a_k(\tau)\,d\tau+$$

$$+\int_{|\xi''|_{\ell''}^{-\frac{1}{r}}}^{T}\frac{\left|\sum\limits_{k=n_1+1}^{n} \acute{a}_{k_\tau}(\tau)\,\xi_k^{2\ell_k}\right|}{\sum\limits_{k=n_1+1}^{n} a_k(\tau)\,\xi_k^{2\ell_k}}d\tau\leq\frac{c}{\left(\left|\xi''\right|_{\ell''}^{-\frac{1}{r}}\right)^{p}}\cdot\left|\xi''\right|_{\ell''}\cdot\int_0^{|\xi''|_{\ell''}^{-\frac{1}{r}}} d\tau+$$

$$+c\left|\xi''\right|_{\ell''}\cdot\int_0^{|\xi''|_{\ell''}^{-\frac{1}{r}}}\frac{d\tau}{\tau^p}d\tau+\frac{c}{a}\int_{|\xi''|_{\ell''}^{-\frac{1}{r}}}^{T}\frac{d\tau}{\tau^q}\leq c\left|\xi''\right|_{\ell''}^{\frac{p}{r}+1}\cdot\left|\xi''\right|_{\ell''}^{-\frac{1}{r}}+$$

$$+c\left|\xi''\right|_{\ell''}\cdot\frac{1}{1-p}\left(\left|\xi''\right|_{\ell''}^{-\frac{1}{r}}\right)^{1-p}+\frac{c}{a}\frac{1}{1-q}\left(T^{1-q}-\left(\left|\xi''\right|_{\ell''}^{-\frac{1}{r}}\right)^{1-q}\right)<$$

$$<c\left|\xi''\right|_{\ell''}^{1-\frac{1-p}{r}}+\frac{c}{1-p}\left|\xi''\right|_{\ell''}^{1-\frac{1-p}{r}}+\frac{c}{a(q-1)}\left|\xi''\right|_{\ell''}^{\frac{q-1}{r}}. \tag{20}$$

As $r=(q-1)s$ and $s<\dfrac{q-p}{q-1}$, follows that $1-\dfrac{1-p}{s}<\dfrac{1}{s}$ and $\dfrac{q-1}{r}=\dfrac{1}{s}$. Then according to the Young inequality there exists such $\delta>0$, that

$$\left|\xi''\right|_{l''}^{1-\frac{1-p}{r}}\leq c_{1\delta}+\delta_1\left|\xi''\right|_{l''}^{\frac{1}{s}}. \tag{21}$$

Thus, by (20) and (21) the following inequality is valid

$$\int_0^t \alpha\left(\tau,\xi''\right)d\tau\leq\delta\left|\xi''\right|^{\frac{1}{s}}+c_\delta,$$

where $\delta=\delta_1\dfrac{a(2+p)}{1-p}+\dfrac{c}{a(q-1)};\qquad c_\delta=c_{1\delta}\dfrac{c(2+\delta)}{1-p}$.

Lemma 4. *There exists such $M>0$, that*

$$E(t)\leq ME(0),\quad t\in[0,T].$$

Proof. We introduce the following functions:

$$H(t,\xi)=\left(1+\left|\xi'\right|_{\ell'}\right)^{\lambda'}\left(t+\left|\xi''\right|_{\ell''}\right)^{\lambda''}\cdot\exp\left[-\int_0^t\alpha\left(\tau,\xi''\right)d\tau+\beta\left|\xi''\right|_{\ell''}^{\frac{q-1}{r}}\right],$$

$$E_0(t,\xi)=\left[\left|v_t'(t,\xi)\right|^2+\left(1+\left|\xi'\right|_{\ell'}+d\left(t,\xi''\right)\right)\left|v(t,\xi)\right|^2\right]H(t,\xi).$$

If $t^r\left|\xi''\right|_{\ell''}<1$, then by definition of $d\left(t,\xi''\right)$ we have

$$\frac{dE_0(t,\xi)}{dt}=2\,\mathrm{Re}\left[v_{tt}''(t,\xi)\,\overline{v'(t,\xi)}+\left(1+\left|\xi'\right|_{\ell'}+d\left(t,\xi''\right)\right)v(t,\xi)\,\overline{v_t'(t,\xi)}\right]\times$$

$$\times H(t,\xi)+E_0(t,\xi)\cdot\left(-\alpha\left(t,\xi''\right)\right). \tag{22}$$

On other hand

$$v''_{tt}(t,\xi)=-\sum_{k=1}^{n}a_k(t)\,\xi_k^{2\ell_k}v(t,\xi)-\sum_{|\alpha:\ell|\le 1}b_\alpha(t)(i\xi)^\alpha v(t,\xi), \tag{23}$$

From (22) and (23) we obtain

$$\frac{dE_0(t,\xi)}{dt}=2\operatorname{Re}\left[-\sum_{k=1}^{n_1}a_k(t)\,\xi_k^{2\ell_k}+\left(1+\left|\xi'\right|_{\ell'}\right)+\right.$$

$$\left.+\left(d\left(t,\xi''\right)-\sum_{k=n_1+1}^{n}a_k(t)\,\xi_k^{2\ell_k}\right)\right]v(t,\xi)\,\overline{v'_t(t,\xi)}H(t,\xi)-$$

$$-2\operatorname{Re}\sum_{|\alpha:l|\le 1}b_\alpha(t)(i\xi)^\alpha v(t,\xi)\,\overline{v'_t(t,\xi)}H(t,\xi)-\alpha(t,\xi)\,E_0(t,\xi) \tag{24}$$

By our supposition $q_k<1$ for $k=1,2,...,n_1$. Therefore we can easily see that $a\le a_k(t)\le a_{k,T}$ $k=1,2,...,n_1$, where $a_{k,T}$ is some constant for each $k=1,2,...,n_1$ and $T>0$.

Since $t^r\left|\xi\right|_{\ell}<1$, then using the definition of $\alpha(t,\xi)$ and the Cauchy inequality we get

$$2\operatorname{Re}\left(d\left(t,\xi''\right)-\sum_{k=n_1+1}^{n}a_k(t)\,\xi_k^{2\ell_k}\right)\times$$

$$\times v(t,\xi)\,\overline{v'_t(t,\xi)}H(t,\xi)-\alpha\left(t,\xi''\right)E_0(t,\xi)\le 0 \tag{25}$$

The other terms in (24) are estimated from above by $E_0(t,\xi)$. Then from (24), (25) we get that when $t^r\left|\xi''\right|_{\ell''}<1$ there exists such a constant $M_1>0$,that

$$\frac{dE_0(t,\xi)}{dt}\le M_1E_0(t,\xi). \tag{26}$$

If $t^r\left|\xi''\right|_{\ell''}>1$ then

$$\frac{dE_0(t,\xi)}{dt}=2\operatorname{Re}\left[v''_{tt}(t,\xi)\,\overline{v'_t(t,\xi)}+\left(1+\left|\xi'\right|_{\ell'}+d\left(t,\xi''\right)\times\right.\right.$$

$$\left.\times v(t,\xi)\,\overline{v'_t(t,\xi)}\right]H(t,\xi)-\alpha\left(t,\xi''\right)E_0(t,\xi)+d'_t\left(t,\xi''\right)\left|v(t,\xi)\right|\Big]^2H(t,\xi),$$

where

$$d'_t\left(t,\xi''\right)=\sum_{k=n_1+1}^{n}a'_{k_t}(t)\,\xi_k^{2\ell_k}.$$

After simple transformations we get

$$\frac{dE_0(t,\xi)}{dt}=2\operatorname{Re}\left[-\sum_{k=1}^{n}a_k(t)\,\xi_k^{2\ell_k}-\sum_{|\alpha:l|\le 1}b_\alpha(t)(i\xi)^\alpha+\right.$$

$$+\left(1+\left|\xi'\right|_{\ell'}\right)\right]v\left(t,\xi\right)\overline{v_t'\left(t,\xi\right)}H\left(t,\xi\right)-\alpha\left(t,\xi''\right)E_0\left(t,\xi\right)+$$

$$+\sum_{k=n_1+1}^{n}a_{kt}'\left(t\right)\left|v\left(t,\xi\right)\right|^2H\left(t,\xi\right)d\xi\le$$

$$\le 2\operatorname{Re}\left[-\sum_{k=1}^{n}a_k\left(t\right)\xi_k^{2\ell_k}-\sum_{|\alpha:l|\le 1}b_\alpha\left(t\right)\left(i\xi\right)^\alpha+\left(1+\left|\xi'\right|_{\ell'}\right)\right]v\left(t,\xi\right)\overline{v_t'\left(t,\xi\right)}H\left(t,\xi\right)+$$

$$+\sum_{k=n_1+1}^{n}a_{k_t}'\left(t\right)\xi_k^{2\ell_k}-\left|\sum_{k=n_1+1}^{n}a_{k_t}'\left(t\right)\xi_k^{2\ell_k}\right|\left|v\left(t,\xi\right)\right|^2\cdot H\left(t,\xi\right)\le M_2E_0\left(t,\xi\right)\qquad(27)$$

Thus, by (26) and (27) it follows that

$$\frac{dE_0\left(t,\xi\right)}{dt}\le M_3E_0\left(t,\xi\right),\qquad(28)$$

where $M=\max\left(M_1,M_2\right)$.

By applying the Gronwall inequality from (28) we get

$$E_0\left(t,\xi\right)\le ME_0\left(t,\xi\right),\quad M=M_3T.$$

So

$$E\left(t\right)=\int_{Rn}E_0\left(t,\xi\right)d\xi\le M\int_{Rn}E_0\left(t,\xi\right)=ME\left(0\right).$$

Lemma 4 is proved.

If $q>1$ by Lemmas 1-3 there exists such constant $0<c_4<c_5$, that for any $\varepsilon>0,\ \delta_2>0,\ \eta\le\lambda'$ the inequality

$$c_4\int_{Rn}\exp\left[\left(\beta-\delta_1\right)\left|\xi''\right|^{\frac{q-1}{r}}\right]\left[\left|v_t'\left(t,\xi\right)\right|^2+\left(1+\left|\xi\right|_l\right)\left|v\left(t,\xi\right)\right|^2\right]\times$$

$$\times\left(1+\left|\xi'\right|_{l'}\right)^{\lambda'}\left(1+\left|\xi''\right|_{l''}\right)^{\eta}d\xi\le$$

$$\le E\left(t,\xi\right)\le c_5\int_{Rn}\exp\left[\left(\beta+\delta_2\right)\left|\xi\right|^{\frac{q-1}{r}}\right]\left[\left|v_t'\left(t,\xi\right)\right|^2+\left(1+\left|\xi\right|_l\right)\left|v\left(t,\xi\right)\right|^2\right]$$

$$\cdot\left(1+\left|\xi'\right|_{l'}\right)^{\lambda'}\left(1+\left|\xi''\right|_{l''}\right)^{\lambda''+\varepsilon}d\xi,\qquad(29)$$

is valid. If $q=1$, the inequality (16) is valid for $\eta=\lambda''-\gamma$.

Further, we consider a regularized problem by replacing the function $a_k\left(t\right)$ with the functions $a_{\varepsilon,k}\left(t\right)=t\varepsilon+a_k\left(t\right),\ \ k=n_1+1,...,n$. The similar estimation is valid for the regularized problem, where c_4 and c_5 are independent from $\varepsilon>0$. Obviously, the appropriate Cauchy problem is well-posed for the regularized equation. Using the inequality (29), we get necessary apriority estimations for $u_\varepsilon\left(t,x\right)$, where $u_\varepsilon\left(t,x\right)$ is a solution of the Cauchy regularized problem. Further, by the standard scheme (see [1], chapter.3) it is proved that the limit of regularized solutions $u_\varepsilon\left(t,x\right)$ in appropriate functional spaces is a solution of the problem (1), (2).

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