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ON REQULARIZED TRACE OF ONE STURM-LIOUVILLE PROBLEM WITH EIGENVALUE DEPENDENT BOUNDARY CONDITION

Abstract

In the paper formula for the trace of Sturm-Liouville operator with eigenvalue parameter in one of the boundary conditions is obtained

Let H be separable Hilbert space. Let, else $L_2 = L_2(H,(0,\pi)) \oplus H$, where $L_2(H,(0,\pi))$ is a hilbert space of vector-functions $y(t)\,(t\in(0,\pi))$, for which $\int\limits_0^\pi \|y(t)\|_H^2\,dt<\infty$. Scalar product $Y,Z\in L_2$ $(Y=\{y(t),y(\pi)\}, Z=\{z(t),z(\pi)\})$ is defined as

$$(Y,Z)=\int\limits_0^\pi (y(t),z(t))_H\,dt+(y(\pi),z(\pi))_H\ .$$

Let's consider the problem

$$l[y]=-y''+Ay+q(y)\,y=\lambda y \tag{1}$$

$$y(0)=0 \tag{2}$$

$$y'(\pi)-\lambda y(\pi)=0. \tag{3}$$

where A is selfadjoint, positively defined operator in H $(A\geq E, E$ is identity operator in $H)$ and has completely continous inverse in H, $q(t)$ is selfadjoint and bounded, for each t, operator in H.

Assume also that operator-function $q(t)$ is weakly measurable, $\|q(t)\|$ is bounded on $[0,\pi]$ as a function of t and satisfies the following conditions:

1. $q(t)$ has the second weak derivative on segment $[0,\pi]$ and $q^{(l)}(t)$, $l=0,1,2$ are selfadjoint kernel operators in H for each $t\in[0,\pi]$: $q^{(l)}(t)\in\sigma_1$, $\left[q^{(l)}(t)\right]^*=q^{(l)}(t)$
2. functions $\left\|q^{(l)}(t)\right\|_{\sigma_1}$, $l=0,1,2$ are bounded on segment $[0,\pi]$;
3. $q'(0)=q'(\pi)=0$
4. $\int\limits_0^\pi (q(t),f,f)\,dt=0$ for each $f\in H$.

For $q(t)\equiv 0$ equation (1) will take on the form

$$l_0[y]=-y''+Ay=\lambda y \tag{1'}$$

It is possible to associate with problem $(1)',(2),(3)$ and $(1)-(3)$ the self-adjoint operators L_0 and $L=L_0+Q$ respectivly, in L_2 , where

$$L_0:\{y(t),y(\pi)\}\to\{l_0[y],y'(\pi)\},\ \ Q:\{y(t),y(\pi)\}\to\{q(t)\,y(t),0\} \tag{4}$$

$$d(L_0)=D(L)=\{Y=\{\{y(t),y(\pi)\}:Ay,y''\in L_2(H,(0,\pi))\}$$

As it is shown in [1], operators L_0 and L have discrete spectr. Let $\mu_1 \leq \mu_2 \leq ...$ be eigenvalues, and $\psi_1(t), \psi_2(t), ...$ corresponding orthonormal eigenvectors of operator L_0, and $\lambda_1 \leq \lambda_2 \leq ...$ are eigenvalues of operator L.

Denote by $\gamma_1 \leq \gamma_2 \leq ...$ eigenvalues, and $\varphi_1, \varphi_2, ...$ orthnormal eigen-vectors of the operaotr A in H, respectivly.

As it is known (see [2]) if for

$$j \to \infty, \ \gamma_j \sim a j^{\alpha} \left(0 < a, \ a > 2\right), \tag{5}$$

then

$$\lambda_n(L) \sim \mu_n(L_0) \sim d n^{\delta}, \tag{6}$$

where $\delta = \dfrac{2\alpha}{2+\alpha}$.

Using this asymptote in similar manner as in [3] one can prove that there exists a sequence of natural number $\{n_m\}_{m=1}^{\infty}$ such that

$$\mu_k - \mu_{n_m} \geq d\left(k^{\frac{2\alpha}{2+\alpha}} - n_m^{\frac{2\alpha}{2+\alpha}}\right), \quad k = n_m, n_m+1, ... \tag{7}$$

Let

$$\mu^{(j)} = \sum_{k=n_{j-1}+1}^{n_j} \mu_k \ , \quad \lambda^{(j)} = \sum_{k=n_{j-1}+1}^{n_j} \lambda_k \ , \quad j = 1, 2...,$$

where $n_0 = 0$.

Our main aim in this paper is to calculate the sum $\sum\limits_{j=1}^{\infty}\left(\lambda^{(j)} - \mu^{(j)}\right)$, which is called regularized trace of operator L_0, since as it will be shown further, doesn't depend on choice of a sequence $n_1, n_2, ...$, which satisfies the inequality (6). Regularized traces for operator-differential equations were studied, for instrance, in papers [3, 4, 5] and in the works of many other authors.

Let R_{λ}^{0} and R_{λ} be resolvents of operators L_0 and L. Taking into account the asymptote (6) and the inequality (7) we can prove the following theorem.

Theorem 1. *Let $\|q(t)\|$ be bounded on the segment $[0, \pi]$ and let the condition (5) be fulfilled. Then for large m the following equality is true*

$$\sum_{n=1}^{n_m}(\lambda_n - \mu_n) = -\frac{1}{2\pi i}\int\limits_{|\lambda|=l_m} Sp\left(QR_{\lambda}^{0}\right)d\lambda,$$

where $l_m = \dfrac{1}{2}\left(\mu_{n_m+1} + \mu_{n_m}\right), \ \ \mu_{n_m}, \ m = 1, 2, 3, ..$ *is a subsequence satisfying* (7).

As QR_{λ}^{0} is a kernel operator and eigen-vectors $\psi_1(x), \psi_2(x), ...$ of operator L_0 form an orthonormal basis in L_2, then for large values of m

$$\sum_{j=1}^{m}\left(\lambda^{(j)} - \mu^{(j)}\right) = \sum_{n-1}^{n_m}(\lambda_n - \mu_n) =$$

$$= -\frac{1}{2\pi i}\int\limits_{|\lambda|=l_m} Sp\left(QR_{\lambda}^{0}\right)d\lambda = -\frac{1}{2\pi i}\int\limits_{|\lambda|=l_m}\sum_{n=1}^{\infty}\left(QR_{\lambda}^{0}\psi_n, \psi_n\right)d\lambda =$$

$$= \sum_{n=1}^{\infty}\left[(Q\psi_n,\psi_n)\cdot\frac{1}{2\pi i}\int\limits_{|\lambda|=l_m}\frac{d\lambda}{\lambda-\mu_n}\right]=\sum_{n-1}^{n_m}(Q\psi_n,\psi_n)\,.$$

Scalar product is considered in L_2.

Orthonormal eigen-vectors of operator L_0 are of the form

$$\sqrt{\frac{4x_{j,k}}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}}\{\sin(x_{j,k}t)\,\varphi_j,\ \sin(x_{j,k}\pi)\,\varphi_j\}$$
$$\left(k=\overline{0,\infty},\ j=\overline{1,\infty}\right), \tag{8}$$

where $x_{j,k}$ are the roots of equation (see [2])

$$ctgx\pi=\frac{\gamma_j+x^2}{x} \tag{9}$$

It is known that the eigencalues of operator L_0 fall into two series; $\mu_{j,0}\sim\sqrt{\gamma_j}$, for $j\to\infty$ corresponding to imaginary roots of equation (9) and $\mu_{j,k}=\gamma_j+x_{j,k}^2=\gamma_j+\eta_k$, where $\eta_k\sim k^2$, corresponding to real roots of equation (9).

Taking into account (4) and (8) we get from theorem 1

$$\sum_{n=1}^{n_m}(Q\psi_n,\psi_n)=$$

$$=\sum_{n=1}^{n_m}\int\limits_0^{\pi}\frac{4x_{j_n,k_n}}{2x_{j_n,k_n}\pi-\sin 2x_{j_n,k_n}\pi+4x_{j_n,k_n}\sin^2 x_{j_n,k_n}\pi}\sin^2 x_{j_n,k_n}t\left(q(t)\,\varphi_{j_n},\varphi_{j_n}\right)dt$$

Denote $f_j(t)=\left(q(t)\,\varphi_j,\varphi_j\right)$. From the condition $\int\limits_0^{\pi}\left(q(t)\,\varphi_j,\varphi_j\right)dt=0$ we have

$$\sum_{n=1}^{n_m}(Q\psi_n,\psi_n)=$$

$$=-\sum_{n=1}^{n_m}\int\limits_0^{\pi}\frac{2x_{j_n,k_n}}{2x_{j_n,k_n}\pi-\sin 2x_{j_n,k_n}\pi+4x_{j_n,k_n}\sin^2 x_{j_n,k_n}\pi}\cos 2x_{j_n,k_n}t\left(q(t)\,\varphi_{j_n},\varphi_{j_n}\right)dt$$

The following theorem is true.

Theorem 2. *Let the condition 3 be fulfilled. If operator function $q(t)$ satisfies the conditions 1-4, then the formula*

$$\lim_{m\to\infty}\sum_{j=1}^{m}\left(\lambda^{(j)}-\mu^{(j)}\right)=-\frac{Spq(\pi)+Spq(0)}{4}$$

is valid.

Let's prove firstly the following lemma.

Lemma. *If operator-function $q(t)$ satisfies the hypothesis of theorem 1, then*

$$\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\left|\frac{2x_{j,k}}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}\int\limits_0^{\pi}\cos(2x_{j,k}t)\,f_j(t)\,dt\right|<\infty \tag{9'}$$

Proof. As $x_{j,k} \sim k + \dfrac{1}{\mu_j + k}$, then integrating by parts twise the integral $\int\limits_0^\pi \cos(2x_{j,k}t)\, f_j(t)\, dt$ and using conditions 2,3 which hold for operator function $q(t)$, we get

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|\frac{2x_{j,k}}{2x_{j,k}\pi - \sin 2x_{j,k}\pi + 4x_{j,k}\sin^2 x_{j,k}\pi}\int\limits_0^\pi \cos(2x_{j,k}t)\, f_j(t)\, dt\right| = const\sum_{k=0}^{\infty}\times$$

$$\times\sum_{j=1}^{\infty}\left|\left(1+O\left(\frac{1}{k}\right)\right)\left(O\left(\frac{1}{k^2}\right)f_j(\pi) + \int\limits_0^\pi \frac{1}{(2x_{j,k}t)^2}\cos(2x_{j,k}t)\, f_j''(t)\, dt\right)\right| \leq$$

$$\leq const\sum_{k=0}^{\infty}\left(\left|\left(q(\pi)\varphi_j,\varphi_j\right)\right| + \int\limits_0^\pi \left|\left(q(t)''\varphi_j,\varphi_j\right)\right| dt\right).$$

According to condition 2, $\left\|q^{(l)}(t)\right\|_{\sigma_1} < const\,(l = 0, 1, 2)$. Then

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|\frac{2x_{j,k}}{2x_{j,k}\pi - \sin 2x_{j,k}\pi + 4x_{j,k}\sin^2 x_{j,k}\pi}\int\limits_0^\pi \cos(2x_{j,k}t)\, f_j(t)\, dt\right| < \infty. \tag{10}$$

Consider now the inner series in $(9')$ at $k = 0$,

$$\sum_{j=1}^{\infty}\int\limits_0^\pi \frac{2x_{j,0}}{2x_{j,0}\pi - \sin 2x_{j,0}\pi + 4x_{j,0}\sin^2 x_{j,0}\pi}\cos(2x_{j,0}t)\, f_j(t)\, dt \tag{11}$$

which corresponds to imaginary root of equation (9).

From asymptote $x_{j,0} \sim \sqrt{\gamma_j} - \dfrac{1}{2}$ $(j \to \infty)$ we have (by condition (5) $\gamma_j \sim aj^\alpha,\ \alpha > 2$)

$$\frac{2x_{j,0}}{2x_{j,0}\pi - \sin 2x_{j,0}\pi + 4x_{j,0}\sin^2 x_{j,0}\pi} = \frac{1}{1 - \dfrac{\sin 2x_{j,0}\pi}{2x_{j,0}} + 2\sin^2 x_{j,0}\pi} <$$

$$< \frac{1}{1 - \dfrac{\sin 2x_{j,0}\pi}{2x_{j,0}}} < 1 + O\left(\frac{1}{x_{j,0}}\right). \tag{12}$$

Since $q(t) \in \sigma_1$, then

$$\sum_{j=1}^{\infty}\frac{1}{\mu_j^{\frac{\alpha}{2}}}\int\limits_0^\pi |f_j(t)|\, dt < \left(\sum_{j=1}^{\infty}\frac{1}{\mu_j^{\alpha}}\right)^{1/2}\left(\sum_{j=1}^{\infty}\left(\int\limits_0^\pi |f_j(t)|\, dt\right)^2\right)^{1/2} <$$

$$< \left(\sum_{j=1}^{\infty}\frac{1}{\mu_j^{\alpha}}\right)^{1/2}\left(\left(\sum_{j=1}^{\infty}\left(\int\limits_0^\pi |f_j(t)|\, dt\right)\right)^2\right)^{1/2} < \infty. \tag{13}$$

Taking into accoun $(12), (13)$ in (11) we will get

$$\left|\sum_{j=1}^{\infty}\int_{0}^{\pi}\frac{2x_{j,0}}{2x_{j,0}\pi-\sin 2x_{j,0}\pi+4x_{j,0}\sin^2 x_{j,0}\pi}\cos 2x_{j,0}tf_j(t)\,dt\right|<\infty. \tag{14}$$

It follows from (10) and (14) that (9) is true.

Let's turn back to the proof of theorem 1.

Earlier it was obtained that

$$\sum_{j=1}^{\infty}\left(\lambda^{(j)}-\mu^{(j)}\right)=$$

$$=\lim_{m\to\infty}\sum_{n=1}^{n_m}\int_{0}^{\pi}\frac{4x_{j_n,k_n}\sin^2 x_{j_n,k_n}t\left(q(t)\varphi_{j_n},\varphi_{j_n}\right)dt}{2x_{j_n,k_n}\pi-\sin 2x_{j_n,k_n}\pi+4x_{j_n,k_n}\sin^2 x_{j_n,k_n}\pi}=$$

$$=\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\int_{0}^{\pi}\frac{4x_{j,k}}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}\sin^2 x_{j,k}tf_j(t)\,dt \tag{15}$$

Let's calculate the double series on the right hand side of equality (15). First calculate the sum

$$\sum_{k=0}^{\infty}\frac{4x_{j,k}}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}\int_{0}^{\pi}\sin^2 x_{j,k}tf_j(t)\,dt=$$

$$=-\lim_{m\to\infty}\sum_{k=0}^{N}\frac{2x_{j,k}}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}\int_{0}^{\pi}\cos 2x_{j,k}tf_j(t)\,dt \tag{16}$$

Let's investigate the asymptotic behavior of function

$$T_N(t)=\sum_{k=0}^{N}\frac{2x_{j,k}}{2x_{j,k}\pi-\sin 2x_{j,k}\pi+4x_{j,k}\sin^2 x_{j,k}\pi}\cos 2x_{j,k}t.$$

for each fixed j and $N\to\infty$.

For deriving formula for $T_N(t)$ let's express m-th term of the sum $T_N(t)$ as a residue at point $x_{j,k}$ of some function of complex variable z having poles at points $x_{j,0}, x_{j,1}, ..., x_{j,N}$ Consider the following function of complex variable

$$g(z)=\frac{z\cos 2zx}{\left(zctgz\pi-z^2-\gamma_j\right)\sin^2 z\pi}. \tag{17}$$

This function has poles at point $x_{j,k}$ and k. Residue at point $x_{j,k}$ is

$$\underset{z=x_{j,k}}{res}\ g(z)=\frac{x_{j,k}\cos 2x_{j,k}t}{\left(ctgx_{j,k}\pi-\dfrac{x_{j,k}\pi}{\sin^2 x_{j,k}\pi}-2x_{j,k}\right)\sin^2 x_{j,k}\pi}=$$

$$=\frac{x_{j,k}\cos 2x_{j,k}t}{\dfrac{1}{2}\sin 2x_{j,k}\pi-x_{j,k}\pi-2x_{j,k}\sin^2 x_{j,k}\pi}=\frac{2x_{j,k}\cos 2x_{j,k}t}{\sin 2x_{j,k}\pi-2x_{j,k}\pi-4x_{j,k}\sin^2 x_{j,k}\pi},$$

and at point k

$$\underset{z=k}{res}\, g(z) = \frac{k\cos 2kt}{k(-1)^k \pi (-1)^k} = \frac{\cos 2kt}{\pi}.$$

Take as a contour of integration the rectangle with vertices at points $\pm iB, A_N \pm iB$, which pass the point $ix_{j,0}$ on the right hand side, and points $-ix_{j,0}$ and 0 on the left hand side over the semicircle. For each fixed j, $B > x_{j,0}$. Then B will tend to infinity, and $A_N = N + \frac{1}{2}$. For such a choice of A_N,

$$x_{j,N} < A_N < x_{j,N+1}\ .$$

Function in (17) is an odd function of z, therefore integral along the part of contour placed on imaginary axis, also along semicircles centered at points $\pm x_{j,0}$ vanishes.

If $z = u + iv$, then for large v and for $u \geq 0$ (17) will be of order $O\left(\frac{1}{e^{2|v|(\pi - x)}|v|}\right)$ and for the given value of A_N the integrals taken along upper and lower sides of contour approach zero, when $B \to \infty$.

Therefore, we get the following formula

$$T_N(t) = -S_N(t) + \frac{1}{2\pi i}\lim_{B\to\infty}\int_{A_N - iB}^{A_N + iB} \frac{z\cos 2zt}{\left(zctgz\pi - z^2 - \gamma_j\right)\sin^2 z\pi}dz +$$

$$+\frac{1}{2\pi i}\lim_{r\to 0}\int_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}} \frac{z\cos 2zt}{\left(zctgz\pi - z^2 - \gamma_j\right)\sin^2 z\pi}dz.$$

For $N \to \infty$

$$\frac{1}{2\pi i}\lim_{B\to\infty}\int_{A_N - iB}^{A_N + iB} \frac{z\cos 2zt}{\left(zctgz\pi - z^2 - \gamma_j\right)\sin^2 z\pi} \sim$$

$$\sim \frac{1}{\pi i}\lim_{B\to\infty}\int\limits_{A_N-i\infty}^{A_N+i\infty}\frac{\cos 2zt}{\sin 2z\pi-2z\ \sin^2 z\pi}dz=$$

$$=\frac{1}{\pi i}\int\limits_{-\infty}^{+\infty}\frac{\cos\left(2N+1\right)tch2tv-i\sin\left(2N+1\right)t}{-ish2v\pi-2\left(A_N+iv\right)\left(1+ch2v\pi\right)}idv=$$

$$=\frac{1}{\pi}\cos\left(2N+1\right)t\int\limits_{-\infty}^{+\infty}\frac{ch2tv}{-ish2v\pi-2\left(A_N+iv\right)\left(1+ch2v\pi\right)}dv+$$

$$+\frac{1}{i\pi}\sin\left(2N+1\right)\int\limits_{-\infty}^{+\infty}\frac{ch2tv}{-ish2v\pi-2\left(A_N+iv\right)\left(1+ch2v\pi\right)}dv$$

Denote integrals in the right side of the latter relation by I_1, I_2 respectivly. Then

$$\frac{1}{2\pi i}\lim_{B\to\infty}\int\limits_{A_N-iB}^{A_N+iB}\frac{z\cos 2zt}{\left(zctgz\pi-z^2-\gamma_j\right)\sin^2 z\pi}dz=I_1+I_2+\psi\left(A_Nt\right), \tag{17'}$$

where

$$\psi\left(A_Nt\right)=O\left(\lim_{B\to\infty}\int\limits_{A_N-iB}^{A_N+iB}\frac{z\cos 2zt}{\sin^2 z\pi}dz\right) \tag{17''}$$

Let's estimate firstly I_1 :

$$\left|I_1\right|\leq\frac{1}{\pi}\int\limits_{-\infty}^{+\infty}\frac{1}{2\sqrt{A_N^2+v^2}}\left|\frac{ch2tv}{\dfrac{sh2v\pi}{2i\left(A_N+iv\right)}-\left(1+ch2v\pi\right)}\right|dv\leq$$

$$\leq\frac{1}{\pi}\int\limits_{-\infty}^{+\infty}\frac{1}{2\sqrt{A_N^2+v^2}}\frac{ch2tv}{\left|\left|\dfrac{sh2v\pi}{2i\left(A_N+iv\right)}\right|-\left(1+ch2v\pi\right)\right|}dv=$$

$$=\frac{1}{\pi}\int\limits_{-\infty}^{+\infty}\frac{1}{2\sqrt{A_N^2+v^2}}\frac{ch2tv}{\left|\left|\dfrac{sh2v\pi}{2\left(A_N^2+v^2\right)}\right|-\left(1+ch2v\pi\right)\right|}dv\leq$$

$$=\frac{1}{2A_N\pi}\int\limits_{-\infty}^{+\infty}\frac{ch2tv}{\left|\left|\dfrac{sh2v\pi}{2}\right|-\left(1+ch2v\pi\right)\right|}dv\leq$$

$$\leq\frac{1}{2A_N\pi}\int\limits_{-\infty}^{+\infty}\frac{ch2tv}{\left|\left|\dfrac{1+ch2v\pi}{2}\right|-\left(1+ch2v\pi\right)\right|}dv<$$

$$< \frac{1}{A_N\pi}\int\limits_{-\infty}^{+\infty}\frac{ch2tv}{1+ch2v\pi}dv=\frac{2}{A_N\pi}\int\limits_{0}^{+\infty}\frac{ch2tv}{1+ch2v\pi}dv=\frac{const}{A_N\cos\frac{x}{2}} \tag{18}$$

We can obtain the similar estimation for I_2. it is possible also to show, that

$$\lim_{N\to\infty}\psi\left(A_N t\right)=0 \tag{18'}$$

So

$$\int\limits_0^\pi T_N(t)\,f_j(t)=-\int\limits_0^\pi S_N(t)\,f_j(t)\,dt+$$

$$+\frac{1}{2\pi i}\int\limits_0^\pi f_j(t)\int\limits_{A_N-i\infty}^{A_N+\infty}\frac{z\cos 2zt}{\left(zctgz\pi-z^2-\mu_j\right)\sin^2 z\pi}dzdt+$$

$$+\frac{1}{2\pi i}\lim_{r\to 0}\int\limits_0^\pi f_j(t)\int\limits_{\substack{|z|=r\\-\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}\frac{z\cos 2zt}{\left(zctgz\pi-z^2-\mu_j\right)\sin^2 z\pi}dzdt \tag{19}$$

Using condition 4 for the third term of right hand side of equality (19) we have

$$\int\limits_0^\pi f_j(t)\int\limits_{\substack{|z|=r\\-\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}\frac{z\cos 2zt}{\left(zctgz\pi-z^2-\mu_j\right)\sin^2 z\pi}dzdt=$$

$$=\int\limits_0^\pi f_j(t)\int\limits_{\substack{|z|=r\\-\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}\frac{z\left(\cos 2zt-1\right)}{\left(zctgz\pi-z^2-\mu_j\right)\sin^2 z\pi}dzdt=$$

$$=\int\limits_0^\pi f_j(t)\int\limits_{\substack{|z|=r\\-\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}\frac{-2z\sin^2 zt}{\left(zctgz\pi-z^2-\mu_j\right)\sin^2 z\pi}dzdt$$

From which for $r\to 0$

$$\int\limits_0^\pi f_j(t)\int\limits_{\substack{|z|=r\\-\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}\frac{-2z\sin^2 zt}{\left(zctgz\pi-z^2-\mu_j\right)\sin^2 z\pi}dzdt\sim\int\limits_0^\pi f_j(t)\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{2re^{i\varphi}\left(re^{i\varphi}t\right)^2}{\gamma_j\left(re^{i\varphi}t\right)^2}d\varphi dt=$$

$$=\int\limits_0^\pi\frac{2t^2}{\gamma_j\pi^2}f_j(t)\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}re^{i\varphi}d\varphi dt\to 0, \tag{20}$$

From (19) and (20) we get

$$\lim_{N\to\infty}\int\limits_0^\pi T_N(t)\,f_j(t)\,dt=-\lim_{N\to\infty}\int\limits_0^\pi S_N(t)\,f_j(t)\,dt+$$

$$+\frac{1}{2\pi i}\lim_{N\to\infty}\int_0^{\pi} f_j(t)\int_{A_N-i\infty}^{A_N+i\infty}\frac{z\cos 2zt}{\left(zctgz\pi - z^2-\mu_j\right)\sin^2 z\pi}dzdt \tag{21}$$

Taking into account estimates for I_1 and I_2 also $(18')$ we have

$$\lim_{N\to\infty}\left|\int_0^{\pi} f_j(t)\int_{A_N-i\infty}^{A_N+i\infty}\frac{z\cos 2zt}{\left(zctgz\pi - z^2-\gamma_j\right)\sin^2 z\pi}dzdt\right| \le$$

$$\le \lim_{N\to\infty}\left|\int_0^{\pi}\frac{const}{A_N}\frac{1}{\cos\frac{t}{2}}f_j(t)\,dt\right| + \lim_{N\to\infty}\left|\int_0^{\pi}\psi(A_N t)\,f_j(t)\,dt\right| \tag{22}$$

Under the condition $\int\limits_{\pi-\delta}^{\pi}\frac{f_j(t)}{\pi-t}<\infty, \quad (\delta>0)$

$$\lim_{N\to\infty}\frac{const}{A_N}\int_0^{\pi}\frac{f_j(t)}{\cos\frac{t}{2}}dt = 0 \tag{23}$$

That is why taking into consideration $(18'),(22),(23)$ in (21)

$$\lim_{N\to\infty}\int_0^{\pi}T_N(t)\,f_j(t) = -\lim_{N\to\infty}\int_0^{\pi}S_N(t)\,f_j(t)\,dt =$$

$$= -\frac{1}{\pi}\sum_{k=0}^{\infty}\int_0^{\pi}f_j(t)\cos 2kt dt = -\frac{f_j(\pi)+f_j(0)}{4} \tag{24}$$

For (15) and (24) we get

$$\sum_{j=1}^{\infty}\left(\lambda^{(j)}-\mu^{(j)}\right) = -\sum_{j=1}^{\infty}\frac{f_j(\pi)+f_j(0)}{4} = -\frac{Spq(\pi)+Spq(0)}{4}$$

The theorem is proved.

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Received April 18, 2007; Revised January17, 2008