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ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF AN AXISYMMETRIC PROBLEM OF ELASTICITY THEORY FOR A RADIALLY-INHOMOGENEOUS TRANSVERSALLY-ISOTROPIC CYLINDER OF SMALL THICKNESS

Abstract

An axisymmetric problem of elasticity theory is studied by the method of asymptotic integration of equations of elasticity theory [1] for a radially-inhomogeneous transversally-isotropic cylinder of small thickness when mixed boundary conditions are given on lateral surfaces.

Inhomogeneous and homogeneous solutions are constructed. It is shown that when lateral surfaces are simply supported, some penetrating solution corresponds to the first asymptotic process. The stressed state determined by this solution is equivalent to the principal vector of forces applied on arbitrary section $\xi = const$. It is obtained that deflected mode in the cylinder is composed of penetrating deflected mode and edge effect similar to Saint-Venant'l edge effect in the theory of transversally-isotropic inhomogeneous plates.

1. Let's consider an axisymmetric problem of elasticity theory for a radially-inhomogeneous transversally-isotropic hollow cylinder of small thickness. Refer the cylinder to the cylindrical system of coordinates r, φ, z :

$$r_1 \le r \le r_2\ ,\ 0 \le \varphi \le 2\pi,\ -L \le z \le L$$

In the axisymmetric case the equilibrium equations are of the form [2] :

$$\begin{cases} \dfrac{\partial \sigma_{rr}}{\partial r} + \dfrac{\partial \sigma_{rz}}{\partial z} + \dfrac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} = 0 \\ \dfrac{\partial \sigma_{rz}}{\partial r} + \dfrac{\partial \sigma_{zz}}{\partial z} + \dfrac{\sigma_{rz}}{r} = 0 \end{cases} \tag{1.1}$$

Here $\sigma_{rr}, \sigma_{rz}, \sigma_{zz}, \sigma_{\varphi\varphi}$ are the stress tensor components that are expressed by displacement vectors $u_r = u_r(r, z)$, $u_z = u_z(r, z)$ in the following form [3] :

$$\sigma_{rr} = A_{11}\frac{\partial u_r}{\partial r} + A_{12}\frac{u_r}{r} + A_{13}\frac{\partial u_z}{\partial z},\quad \sigma_{rz} = A_{44}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right),$$

$$\sigma_{zz} = A_{13}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) + A_{33}\frac{\partial u_z}{\partial z}, \tag{1.2}$$

$$\sigma_{\varphi\varphi} = A_{12}\frac{\partial u_r}{\partial r} + A_{11}\frac{u_r}{r} + A_{33}\frac{\partial u_z}{\partial z}.$$

After substitution of (1.2) into (1.1) we get an equilibrium equation is displacements

$$\left(L_0+\partial_1 L_1+\partial_1^2 L_1\right)\overline{u}=\overline{0}. \tag{1.3}$$

Here L_k are matrix differential operators of the form

$$L_0=\left\|\begin{matrix}\partial\left(e^{-\varepsilon\rho}\left(b_{11}\partial+\varepsilon b_{12}\right)\right)+\varepsilon\left(b_{11}-b_{12}\right)e^{-\varepsilon\rho}\left(\partial-\varepsilon\right) & 0\\ 0 & \partial\left(e^{-\varepsilon\rho}b_{44}\partial\right)+\varepsilon b_{44}e^{-\varepsilon\rho}\partial\end{matrix}\right\|$$

$$L_1=\left\|\begin{matrix}0 & \varepsilon\left(\partial\left(b_{13}\right)+b_{44}\partial\right)\\ \varepsilon b_{13}\partial+\varepsilon^2\left(b_{13}+b_{44}\right)+\varepsilon\partial\left(b_{44}\right) & 0\end{matrix}\right\|$$

$$L_2=\left\|\begin{matrix}\varepsilon^2 b_{44}e^{\varepsilon\rho} & 0\\ 0 & \varepsilon^2 b_{33}e^{\varepsilon\rho}\end{matrix}\right\|$$

$\partial_1=\dfrac{\partial}{\partial\xi}$; $\partial_1^2=\dfrac{\partial^2}{\partial\xi^2}$; $\partial=\dfrac{\partial}{\partial\rho}$; $\overline{u}=\left(u_\rho;u_\xi\right)^T$; $u_\rho\left(\rho,\xi\right)$, $u_\xi\left(\rho,\xi\right)$ are displacement vector components; $\rho=\dfrac{1}{\varepsilon}\ln\left(\dfrac{r}{r_0}\right)$, $\xi=\dfrac{z}{r_0}$ are new dimensionless variables; $\varepsilon=\dfrac{1}{2}\ln\left(\dfrac{r_2}{r_1}\right)$ is a small parameter defining the cylinder thickness; $r_0=\sqrt{r_1r_2}$, $\rho\in[-1;1]$, $\xi\in[-l;l]$ $\left(l=\dfrac{L}{r_0}\right)$; $b_{ij}=b_{ij}\left(\rho\right)$ are elastic characteristics considered as arbitrary piecewice-continuous function of variable ρ.

Assume that on lateral surfaces of the cylinder the following mixed boundary conditions are given:

$$\left.\overline{\sigma}\right|_{\rho=\pm1}=\left(M_0+\varepsilon\partial_1 M_1\right)\left.\overline{u}\right|_{\rho=\pm1}=\overline{q}^{\pm}\left(\xi\right) \tag{1.4}$$

where $\overline{\sigma}=\left(u_\rho,\sigma_{\rho\xi}\right)^T$, $\overline{q}^{\pm}\left(\xi\right)=\left(h^{\pm}\left(\xi\right);f^{\pm}\left(\xi\right)\right)^T$,

$$M_0=\left\|\begin{matrix}1 & 0\\ 0 & \varepsilon^{-1}b_{44}e^{-\varepsilon\rho}\partial\end{matrix}\right\|,\quad M_1=\left\|\begin{matrix}0 & 0\\ b_{44} & 0\end{matrix}\right\|$$

Assume that $h^{\pm}\left(\xi\right)$; $f^{\pm}\left(\xi\right)$ are sufficiently smooth functions and with respect to ε they are of order $O\left(1\right)$.

Assume that arbitrary boundary conditions leaving the cylinder in equilibrium state are given at the end faces of the cylinder.

2. Let's consider construction of special solutions of equations (1.3) that satisfy boundary conditions (1.4), i.e. inhomogeneous solutions.

Assuming that the quantity ε is sufficiently small, for constructing inhomogeneuos solutions we use the asymptotic method [1].

We'll look for the solution of (1.3) (1.4) in the form:

$$u_\rho=u_{\rho0}+\varepsilon u_{\rho1}+...;\quad u_\xi=\varepsilon^{-1}\left(u_{\xi0}+u_{\xi1}+...\right). \tag{2.1}$$

Substitution of (2.1) into $(1.3),(1.4)$ reduces to the system whose successive integration with respect to ρ gives relation for the coefficients of the expansion u_ρ, u_ξ:

$$u_{\rho 0}=h^{-}(\xi)+\frac{h(\xi)}{a_0}\int_{-1}^{\rho}\frac{1}{b_{11}}dx+c_1'(\xi)\left[\frac{d_0}{a_0}\int_{-1}^{\rho}\frac{1}{b_{11}}dx-\int_{-1}^{\rho}\frac{b_{13}}{b_{11}}dx\right],$$

$$u_{\xi 0}=c_1(\xi),\quad u_{\xi 1}=c_2(\xi), \tag{2.2}$$

$$c_1''(\xi)=\frac{d_0h'(\xi)+a_0f(\xi)}{a_0q_0-d_0^2}$$

where

$$f(\xi)=f^{+}(\xi)-f^{-}(\xi);\ h(\xi)=h^{+}(\xi)-h^{-}(\xi);\ d_k=\int_{-1}^{\rho}\frac{b_{13}}{b_{11}}\rho^k d\rho;$$

$$a_k=\int_{-1}^{\rho}\frac{\rho^k}{b_{11}}d\rho;\quad q_k=\int_{-1}^{\rho}\frac{\left(b_{13}^2-b_{11}b_{33}\right)}{b_{11}}\rho^k d\rho$$

Analysis of stress state shows that stresses $\sigma_{\rho\rho},\sigma_{\varphi\varphi},\sigma_{\xi\xi}$ with respect to ε have order ε^{-1}, but $\sigma_{\rho\xi}$ have a unit order.

3. Let's consider a matter on construction of homogeneous solutions. Assume $\overline{q}^{\pm}(\xi)=\overline{0}$ in (1.4). Searching the solutions of homogeneous systems in the form

$$u_\rho(\rho,\xi)=u(\rho)e^{\alpha\xi};\quad u_\xi(\rho,\xi)=w(\rho)e^{\alpha\xi}$$

we get the following not self-adjoint spectral problem

$$\begin{cases}\left(L_0+\alpha L_1+\alpha^2L_2\right)\overline{a}=\overline{0}\\ \left(M_0+\alpha M_1\right)\overline{a}\big|_{\rho=\pm1}=\overline{0}\end{cases} \tag{3.1}$$

where $\overline{a}=(u(\rho),w(\rho))^T$.

For solving (3.1) we use the asymptotic method $[1]$, based on three iteration processes.

We can get homogeneous solutions corresponding to the first iteration process from (2.2) if we put therein $\overline{q}^{\pm}(\xi)=\overline{0}$. We have

$$u_\rho^{(1)}=\varepsilon D\left\langle\frac{d_0}{a_0}\int_{-1}^{\rho}\frac{1}{b_{11}}dx-\int_{-1}^{\rho}\frac{b_{13}}{b_{11}}dx+\varepsilon\left\{-\int_{-1}^{\rho}\frac{1}{b_{11}}\left(\int_{-1}^{y}\frac{b_{13}(b_{12}-b_{11})}{b_{11}}dx\right)dy-\right.\right.$$

$$-\int_{-1}^{\rho}\frac{b_{13}}{b_{11}}xdx+\int_{-1}^{\rho}\frac{b_{12}}{b_{11}}\left(\int_{-1}^{y}\frac{b_{13}}{b_{11}}dx\right)dy+\frac{d_0}{a_0}\left[\int_{-1}^{\rho}\frac{1}{b_{11}}\left(\int_{-1}^{y}\frac{(b_{12}-b_{11})}{b_{11}}dx\right)dy+\right.$$

$$+\int\limits_{-1}^{\rho}\frac{x}{b_{11}}dx-\int\limits_{-1}^{\rho}\frac{b_{12}}{b_{11}}\left(\int\limits_{-1}^{y}\frac{1}{b_{11}}dx\right)dy\Bigg]+\int\limits_{-1}^{\rho}\frac{1}{b_{11}}dx\left[\frac{d_1}{a_0}-\right. \tag{3.2}$$

$$-\frac{1}{a_0}\int\limits_{-1}^{1}\frac{1}{b_{11}}\left(\int\limits_{-1}^{\rho}\frac{b_{13}\left(b_{11}-b_{12}\right)}{b_{11}}dx\right)d\rho-\frac{1}{a_0}\int\limits_{-1}^{1}\frac{b_{12}}{b_{11}}\left(\int\limits_{-1}^{\rho}\frac{b_{13}}{b_{11}}dx\right)d\rho+$$

$$\left.\left.+\frac{d_0}{a_0^2}\left(-a_1-\int\limits_{-1}^{1}\frac{1}{b_{11}}\left(\int\limits_{-1}^{\rho}\frac{\left(b_{12}-b_{11}\right)}{b_{11}}dx\right)d\rho+\left(\int\limits_{-1}^{1}\frac{b_{12}}{b_{11}}\int\limits_{-1}^{\rho}\frac{1}{b_{11}}dx\right)d\rho\right)\right]\right\}+O\left(\varepsilon^2\right)\Big\rangle$$

$$u_{\xi}^{(1)}=E+D\xi.$$

In (3.2) the constant E corresponds to displacement of the cylinder as absolute solid. Therefore we can equale E to zero: $E=0$

Double eigen values $\alpha_0=0$ correspond to these solutions. Appropriate stresses take the form:

$$\sigma_{\rho\rho}^{(1)}=D\left\langle\frac{d_0}{a_0}+\varepsilon\left\{\frac{d_1}{a_0}-\int\limits_{-1}^{\rho}\frac{b_{13}\left(b_{12}-b_{11}\right)}{b_{11}}dx+\right.\right.$$

$$+\frac{1}{a_0}\int\limits_{-1}^{1}\frac{1}{b_{11}}\left(\int\limits_{-1}^{\rho}\frac{b_{13}\left(b_{12}-b_{11}\right)}{b_{11}}dx\right)d\rho-\frac{1}{a_0}\int\limits_{-1}^{1}\frac{b_{12}}{b_{11}}\left(\int\limits_{-1}^{\rho}\frac{b_{13}}{b_{11}}dx\right)d\rho+$$

$$+\frac{d_0}{a_0}\int\limits_{-1}^{\rho}\frac{\left(b_{12}-b_{11}\right)}{b_{11}}dx+\frac{d_0}{a_0^2}\left(\int\limits_{-1}^{1}\frac{b_{12}}{b_{11}}\int\limits_{-1}^{\rho}\frac{1}{b_{11}}dx\right)d\rho-$$

$$\left.\left.-\int\limits_{-1}^{1}\frac{1}{b_{11}}\left(\int\limits_{-1}^{\rho}\frac{\left(b_{12}-b_{11}\right)}{b_{11}}dx\right)d\rho-\frac{d_0a_1}{a_0^2}\right]\right\}+O\left(\varepsilon^2\right)\Big\rangle$$

$$\sigma_{\varphi\varphi}^{(1)}=D\left\langle\frac{b_{13}\left(b_{11}-b_{12}\right)}{b_{11}}+\frac{d_0}{a_0}\frac{b_{12}}{b_{11}}+O\left(\varepsilon^2\right)\right\rangle$$

$$\sigma_{\xi\xi}^{(1)}=D\left\langle\frac{\left(b_{11}b_{33}-b_{13}^2\right)}{b_{11}}+\frac{d_0}{a_0}\frac{b_{13}}{b_{11}}+O\left(\varepsilon\right)\right\rangle \tag{3.3}$$

$$\sigma_{\rho\xi}^{(1)}=0.$$

There are no solutions having the edge effect character corresponding to the second asymptotic process.

By the third asymptotic process, we look for the solution of (3.1) as follows:

$$\begin{gathered}u^{(3)}=\varepsilon\left(u_0+\varepsilon u_1+...\right),\quad w^{(3)}=\varepsilon\left(w_0+\varepsilon w_1+...\right),\\ \alpha=\varepsilon^{-1}\left(\beta_0+\varepsilon\beta_1+...\right).\end{gathered} \tag{3.4}$$

After substitution of (3.4) into (3.1) for the first terms of expansion we get a spectral problem describing potential solution of transversally-isotropic plate inhomogeneous in thickness:

$$B(\beta_0)\overline{f}_0 = \overline{0} \tag{3.5}$$

where

$$B(\beta_0)\overline{f}_0 = \left\{ t(\beta_0)\overline{f}_0, \ \ C(\beta_0)\overline{f}_0\big|_{\pm 1} = \overline{0} \right\}, \quad t(\beta_0)\overline{f}_0 = \left(B_0 + \beta_0 B_1 + \beta_0^2 B_2\right)\overline{f}_0,$$

$$C(\beta_0)\overline{f}_0 = (C_0 + \beta_0 M_1)\overline{f}_0, \ \overline{f}_0 = (u_0; w_0)^T,$$

$$B_0 = \begin{Vmatrix} \partial(b_{11}\partial) & 0 \\ 0 & \partial(b_{44}\partial) \end{Vmatrix}; \quad B_1 = \begin{Vmatrix} 0 & \partial(b_{13}) + b_{44}\partial \\ \partial(b_{44}) + b_{13}\partial & 0 \end{Vmatrix};$$

$$B_2 = \begin{Vmatrix} b_{44} & 0 \\ 0 & b_{33} \end{Vmatrix}; \quad C_0 = \begin{Vmatrix} 1 & 0 \\ 0 & b_{44}\partial \end{Vmatrix}.$$

Unlike the isotropic case [4, 5] for transversally isotropic plate of inhomogeneous in thickness, β_{0k} may accept pure imaginary values as well.

By the substitution

$$u_0 = -\beta_0^{-3}\left(p_0\psi''\right)' + \beta_0^{-1}p_2\psi' + \beta_0^{-1}(p_1\psi)', \quad w_0 = \beta_0^{-2}p_0\psi'' - p_1\psi \tag{3.6}$$

spectral problem (3.5) is reduced to the following one:

$$\begin{cases} \left(p_0\psi''\right)'' - \beta_0^2\left[(p_1\psi)'' + p_1\psi'' + \left(p_2\psi'\right)'\right] + \beta_0^4 p_3\psi = 0 \\ \left(-\beta_0^{-3}\left(p_0\psi''\right)' + \beta_0^{-1}p_2\psi' + \beta_0^{-1}(p_1\psi)'\right)\Big|_{\rho=\pm 1} = 0 \\ \psi'\big|_{\rho=\pm 1} = 0 \end{cases} \tag{3.7}$$

where

$$p_0 = b_{11}\theta, \quad p_1 = b_{13}\theta, \quad p_2 = b_{44}^{-1}, \quad p_3 = b_{33}\theta, \quad \theta = \left(b_{13}^2 - b_{11}b_{33}\right)^{-1}$$

(3.7) is a generalization of P.F. Papkovich spectral problem [4, 5] for inhomogeneous transversally-isotropic case.

So, the solutions corresponding to the third iteration process are of the form:

$$\begin{aligned} u_\rho^{(3)}(\rho;\xi) &= \varepsilon\sum_{k=1}^{\infty} F_k\left[-\beta_{0k}^{-3}\left(p_0\psi_k''\right)' + \beta_{0k}^{-1}p_1\psi_k' + \beta_{0k}^{-1}(p_2\psi_k)' + O(\varepsilon)\right]\exp\left(\frac{\beta_{0k}\xi}{\varepsilon}\right) \\ u_\xi^{(3)}(\rho;\xi) &= \varepsilon\sum_{k=1}^{\infty} F_k\left[\beta_{0k}^{-2}p_0\psi_k'' - p_2\psi_k + O(\varepsilon)\right]\exp\left(\frac{\beta_{0k}\xi}{\varepsilon}\right) \end{aligned} \tag{3.8}$$

For stresses we have

$$\sigma_{\rho\rho}^{(3)} = \sum_{k=1}^{\infty} F_k\left(-\beta_{0k}\psi_k + O(\varepsilon)\right)\exp\left(\frac{\beta_{0k}\xi}{\varepsilon}\right)$$

$$\sigma_{\rho\xi}^{(3)} = \sum_{k=1}^{\infty} F_k \left(\psi_k' + O\left(\varepsilon\right)\right) \exp\left(\frac{\beta_{0k}\xi}{\varepsilon}\right)$$

$$\sigma_{\xi\xi}^{(3)} = \sum_{k=1}^{\infty} F_k \left(-\beta_{0k}^{-1}\psi_k'' + O\left(\varepsilon\right)\right) \exp\left(\frac{\beta_{0k}\xi}{\varepsilon}\right)$$

$$\sigma_{\varphi\varphi}^{(3)} = \sum_{k=1}^{\infty} F_k \left(p_1\left(b_{11}-b_{12}\right)\beta_{0k}^{-1}\psi_k'' + \left(b_{33}b_{12}-b_{13}^2\right)\theta\beta_{0k}\psi_k + O\left(\varepsilon\right)\right) \exp\left(\frac{\beta_{0k}\xi}{\varepsilon}\right)$$

4. Let's analyze deflected mode corresponding to different groups of solutions. We represent the replacements in the form.

$$\begin{aligned} u_\rho\left(\rho,\xi\right) &= u_\rho^{(1)}\left(\rho,\xi\right) + \sum_{k=1}^{\infty} F_k u_k\left(\rho\right) e^{\alpha_k\xi} \\ u_\xi\left(\rho,\xi\right) &= u_\xi^{(1)}\left(\rho,\xi\right) + \sum_{k=1}^{\infty} F_k w_k\left(\rho\right) e^{\alpha_k\xi} \end{aligned} \qquad (4.1)$$

The second term contains displacements defined by the thirs group of solutions.

For stresses we have:

$$\sigma_{\rho\xi} = \sigma_{\rho\xi}^{(1)} + \sum_{k=1}^{\infty} F_k\sigma_{1k}\left(\rho\right) e^{\alpha_k\xi}, \quad \sigma_{\xi\xi} = \sigma_{\xi\xi}^{(1)} + \sum_{k=1}^{\infty} F_k\sigma_{2k}\left(\rho\right) e^{\alpha_k\xi} \qquad (4.2)$$

where

$$\sigma_{1k}\left(\rho\right) = b_{44}\left(e^{-\varepsilon\rho}\varepsilon^{-1}w_k'\left(\rho\right) + \alpha_k u_k\left(\rho\right)\right),$$

$$\sigma_{2k}\left(\rho\right) = b_{13}e^{-\varepsilon\rho}\left(\varepsilon^{-1}u_k'\left(\rho\right) + u_k\left(\rho\right)\right) + \alpha_k b_{33} w_k\left(\rho\right).$$

Let's consider connection of homogeneous solutions with principal vector P, of stresses acting in the section $\xi = const$. We have:

$$P = 2\pi\varepsilon\int_{-1}^{1}\left(\sigma_{\rho\xi} + \sigma_{\xi\xi}\right)e^{2\varepsilon\rho}d\rho \qquad (4.3)$$

Substituting (4.2) into (4.3) we have:

$$P = 2\pi\varepsilon\omega_0 D + 2\pi\varepsilon\sum_{k=1}^{\infty} F_k\omega_k e^{\alpha_k\xi} \qquad (4.4)$$

where $\omega_0 = \dfrac{d_0^2}{a_0} - q_0 + O\left(\varepsilon\right);\ \omega_k = \displaystyle\int_{-1}^{1}\left(\sigma_{1k}\left(\rho\right) + \sigma_{2k}\left(\rho\right)\right)e^{2\varepsilon\rho}d\rho.$

Show that all $\omega_k = 0\,(k = 1, 2, ...)$. For that we consider the following boundary value problem:

$$\sigma_{\rho\xi}\big|_{\xi=\xi_j} = \sigma_{1k}\left(\rho\right)e^{\alpha_k\xi_j}, \quad \sigma_{\xi\xi}\big|_{\xi=\xi_j} = \sigma_{2k}\left(\rho\right)e^{\alpha_k\xi_j} \qquad (4.5)$$

where $j = 1, 2$.

The principal vector that corresponds to stress state of problem (4.5) in the section $\xi = const$ is reduced to the form

$$P_k = 2\pi\varepsilon\omega_k e^{\alpha_k \xi} \tag{4.6}$$

According to condition of solvability of elasticity theory problem the principal vector P_k must not depend on the variable ξ. However, in relation (4.6) the right hand side depends on ξ.Hence it follows that $P_k = 0$ i.e. $\omega_k = 0$.

For the principal vector from (4.4) we get:

$$P = 2\pi\varepsilon\omega_0 D \tag{4.7}$$

Stress strain corresponding to the third group of solutions is self-balanced at each section $\xi = const$. Solution (3.2) corresponding to the first asymptotic process determines the internal deflected mode of the cylibder. First terms of its expansion in ε determine momentless stress state.

The third asymptotic process is determined by solutions (3.8) that are of boundary layer character. The first terms of (3.8) are equivalent to Saint-Venant's edge effect of inhomogeneous transversally-isotropic plate. For purely imaginary β_{0k} the Saint-Venant's boundary layer damps very slowly and the soloutions (3.8) should be reckoned to penetrating solutions. Therefore, in this case, deflected mode of transversally-isotropic and isotropic shell differ qualitatively. When β_{0k} is not pure imaginary general picture of deflected mode in qualitative respect is similar to appropriate picture for isotropic radially-inhomogeneous cylinders. In quantity respect they differ by rapidity of damping of Saint-Venant's boundary layer.

5. Let's consider a question on stress relief from the end faces of the cylinder. Assume that the following stresses are given at the end faces of the cylinder:

$$\sigma_{\rho\xi}|_{\xi=\pm l} = f_{1s}(\rho), \quad \sigma_{\xi\xi}|_{\xi=\pm l} = f_{2s}(\rho) \tag{5.1}$$

Here $f_{1s}(\rho)$, $f_{2s}(\rho)$ $(s = 1,2)$ are sufficiently smooth functions and they satisfy the equilibrium conditions.

$$2\pi\varepsilon \int_{-1}^{1} (f_{11}(\rho) + f_{21}(\rho))\, e^{2\varepsilon\rho} d\rho = 2\pi\varepsilon \int_{-1}^{1} (f_{12}(\rho) + f_{22}(\rho))\, e^{2\varepsilon\rho} d\rho$$

As it was shown, not self-balanced part of stresses may be relieved by means of penetrating solution (3.2) and relation between the constant D and principal vector P is given by the equation (4.7).

Further we'll assume that $P = 0$. By the accepted supposition $D = 0$.

We'll look for the solution in the form (4.2). For determining arbitrary constants $F_k\,(k = 1,2,...)$ whose variations will be considered as independent as in $[6-9]$, we'll use Lagrange's variational principle. Since homogeneous solutions satisfy the equilibrium equation and boundary conditions on lateral surface, the variational principle takes the form:

$$\sum_{s=1}^{\infty}\int_{-1}^{1}\left[\left(\sigma_{\rho\xi}-f_{1s}\left(\rho\right)\right)\delta u_{\rho}+\left(\sigma_{\xi\xi}-f_{2s}\left(\rho\right)\right)\delta u_{\xi}\right]\Big|_{\xi=\pm l}\cdot e^{2\varepsilon\rho}d\rho=0 \qquad (5.2)$$

From (5.2) we get an infinite system of linear algebraic equations

$$\sum_{k=1}^{\infty}Q_{jk}F_{k}=\tau_{j};\quad (j=1,2,...) \qquad (5.3)$$

where

$$Q_{jk}=\int_{-1}^{1}\left(\sigma_{1k}\left(\rho\right)u_{j}\left(\rho\right)+\sigma_{2k}\left(\rho\right)w_{j}\left(\rho\right)\right)e^{2\varepsilon\rho}d\rho\times$$

$$\times\left[\exp\left(-\left(\alpha_{k}+\alpha_{j}\right)l\right)+\exp\left(\left(\alpha_{k}+\alpha_{j}\right)l\right)\right]$$

$$\tau_{j}=\int_{-1}^{1}\left(f_{11}\left(\rho\right)u_{j}\left(\rho\right)+f_{21}\left(\rho\right)w_{j}\left(\rho\right)\right)e^{2\varepsilon\rho}d\rho\exp\left(-\alpha_{j}l\right)+$$

$$+\int_{-1}^{1}\left(f_{12}\left(\rho\right)u_{j}\left(\rho\right)+f_{22}\left(\rho\right)w_{j}\left(\rho\right)\right)e^{2\varepsilon\rho}d\rho\exp\left(\alpha_{j}l\right).$$

Solvability and convergence of the reduction method for the system (6.3) is proved in [10].

We'll look for the constants F_k in the form

$$F_{k}=F_{k0}+\varepsilon F_{k1}+... \qquad (5.4)$$

After substitution of (5.4) into (5.3) we get:

$$\sum_{j=1}^{\infty}M_{nj}F_{j0}=g_{n}\qquad (n=1,2,.....). \qquad (5.5)$$

Here

$$M_{nj}=\int_{-1}^{1}\left\{\psi_{n}'\left[-\beta_{0j}^{-3}\left(p_{0}\psi_{j}''\right)'+\beta_{0j}^{-1}p_{1}\psi_{j}'+\beta_{0j}^{-1}\left(p_{2}\psi_{j}\right)'\right]+\right.$$

$$\left.+\beta_{0n}^{-1}\psi_{n}''\left[p_{2}\psi_{j}-\beta_{0j}^{-2}p_{0}\psi_{j}''\right]\right\}d\rho\left[\exp\left(-\frac{\left(\beta_{0n}+\beta_{0j}\right)l}{\varepsilon}\right)+\exp\left(\frac{\left(\beta_{0n}+\beta_{0j}\right)l}{\varepsilon}\right)\right];$$

$$g_{n}=\int_{-1}^{1}\left\{f_{11}\left[-\beta_{0n}^{-3}\left(p_{0}\psi_{n}''\right)'+\beta_{0n}^{-1}p_{1}\psi_{n}'+\beta_{0n}^{-1}\left(p_{2}\psi_{n}\right)'\right]+f_{21}\left[\beta_{0n}^{-2}p_{0}\psi_{n}''-p_{2}\psi_{n}\right]\right\}d\rho\times$$

$$\times \exp\left(-\frac{\beta_{0n}l}{\varepsilon}\right)+\int\limits_{-1}^{1}\left\{f_{12}\left[-\beta_{0n}^{-3}\left(p_0\psi_n''\right)'+\beta_{0n}^{-1}p_1\psi_n'+\beta_{0n}^{-1}\left(p_2\psi_n\right)'\right]+\right.$$

$$\left.+f_{22}\left[\beta_{0n}^{-2}p_0\psi_n''-p_2\psi_n\right]\right\}d\rho \exp\left(\frac{\beta_{0n}l}{\varepsilon}\right).$$

Definition $F_{cr}\,(p=1,2,...)$ is invariably reduced to the inversion of the same matrices that coincide with matrices of system (5.5).

Notice that when $b_{12}=b_{13}=\lambda;\ \ b_{44}=G;\ \ b_{11}=b_{33}=2G+\lambda$ all the solutions entirely coincide with the solutions for radially inhomogeneous isotropic cylinder [11].

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