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INVERSE EIGENVALUE PROBLEM FOR DIFFUSION EQUATION WITH NONSEPARATED BOUNDARY CONDITIONS

Abstract

The inverse problem of spectral analysis for a quadratic pencil of Sturm-Liouville operators on a finite segment is considered; sufficient condition for the solvability of this inverse problem is obtained. Under some additional condition on one of the coefficient functions these are also necessary conditions.

1. Introduction

The solution of partial differential equations by the Fourier method reduces to the problem of spectral analysis of ordinary differential operators. In particular, the solution of the wave equation with some initial and boundary conditions is reduced to study of the quadratic pencil of ordinary differential operators (see e.g. [17, 19]).

In this paper we consider inverse eigenvalue problem for diffusion equation

$$y''(x) + [\lambda^2 - 2\lambda p(x) - q(x)]y(x) = 0, \quad x \in [0, \pi] \tag{1.1}$$

with the boundary conditions

$$y(0) + i\omega y(\pi) = 0 \ , \tag{1.2}$$

$$-i\omega y'(0) + \alpha y(\pi) + y'(0) = 0; \tag{1.3}$$

where $p(x) \in W_2^1[0, \pi], q(x) \in L_2[0, \pi]$ are real-valued functions and $\omega, \alpha \in \mathbb{R}$. By $W_2^1[0, \pi]$ we denote the Sobolev space of functions on $[0, \pi]$ that are absolutely continious and whose first derivative is square integrable on the segment $[0, \pi]$. Let us denote the problem (1.1)-(1.3) by $L(p(x), q(x), \omega, \alpha)$.

The direct and inverse problems of spectral analysis have been most comprehensively investigated for the Sturm-Liouville operator (see [7] and the bibliography therein). Inverse so-called similar boundary-value problems (that is, for the case of problems with characteristic functions differing by constant) for Sturm-Liouville equation ($p(x) \equiv 0$) with the boundary conditions (1.2), (1.3) have been investigated in [15, 16].

Necessary and sufficient conditions of recovering of problems $L(0, q(x), \omega, \alpha_j)$, $j = 1, 2$ from two spectra and some sequence of signs is obtained in [6]. Some versions of inverse problems have been analyzed for eq. (1.1), which is a natural generalization of the Sturm-Liouville equation [3, 4, 5, 8, 9, 12, 13, 14].

In this paper we give sufficient conditions for solvability of inverse problem of recovering of problems $L(p(x), q(x), \omega, \alpha_j), j = 1, 2$ where $|\omega| \neq 0, 1$ from two spectra. In the case $|\omega| = 1$ this problem has been considered in [13].

2. Preliminaries

Let $c(x,\lambda)$ and $s(x,\lambda)$ be the solutions of (1.1) satisfying the initial conditions

$$c(0,\lambda)=s'(0,\lambda)=1, c'(0,\lambda)=s(0,\lambda)=0.$$

λ is called an eigenvalue of the problem $L(p(x),q(x),\omega,\alpha)$ if there exist a non-trivial solution $y(x,\lambda)$ of (1.1) satisfying (1.2) and (1.3).

It is easy to show that eigenvalues of the problem (1.1)–(1.3) coincide with the zeroes of the characteristic function

$$\Delta(\lambda)=\omega^2 c(\pi,\lambda)+\alpha s(\pi,\lambda)+s'(\pi,\lambda).$$

Lemma 2.1. *The following formulas hold*

$$c(\pi,\lambda)=\cos\pi(\lambda-a)-$$

$$-\frac{a_1\cos\pi(\lambda-a)+\pi c_1\sin\pi(\lambda-a)+\int\limits_{-\pi}^{\pi}\Psi_1(t)e^{i\lambda t}dt}{\lambda},$$

$$s'(\pi,\lambda)=\cos\pi(\lambda-a)$$

$$+\frac{a_1\cos\pi(\lambda-a)+\pi c_1\sin\pi(\lambda-a)+\int\limits_{-\pi}^{\pi}\Psi_2(t)e^{i\lambda t}dt}{\lambda},$$

where

$$a=\frac{1}{\pi}\int\limits_0^\pi p(t)dt, c_1=\frac{1}{2\pi}\int\limits_0^\pi [q(t)+p^2(t)]dt,$$

$$a_1=\frac{1}{2}[p(0)-p(\pi)], \Psi_j(t)\in L_2[-\pi,\pi], j=1,2.$$

This lemma is a generalization of representations in [11, p.38] and has been proved in [5].

Lemma 2.2. *For the functions $P(\lambda)$ and $R(\lambda)$ to be represented in the form*

$$P(\lambda)=\sin\pi(\lambda-a)+A\pi\frac{4(\lambda-a)}{4(\lambda-a)^2-1}\cos\pi(\lambda-a)+\frac{f(\lambda-a)}{\lambda-a},$$

$$R(\lambda)=\cos\pi(\lambda-a)+B\pi\frac{\sin\pi(\lambda-a)}{\lambda-a}+\frac{g(\lambda-a)}{\lambda-a},$$

where

$$f(\lambda)=d_0\sin\pi\lambda+\int\limits_{-\pi}^{\pi}\widetilde{f}(t)e^{i\lambda t}dt, \quad \widetilde{f}(t)\in L_2[-\pi,\pi], f(0)=f'(0)=0,$$

$$g(\lambda)=d_1\cos\pi\lambda+\int\limits_{-\pi}^{\pi}\widetilde{g}(t)e^{i\lambda t}dt, \quad \widetilde{g}(t)\in L_2[-\pi,\pi], g(0)=0$$

it is necessary and sufficient to have the form

$$P(\lambda)=\pi(\lambda-a)\prod_{k=-\infty,k\neq 0}^{\infty}\frac{u_k-\lambda}{k},\quad u_k=k+a-\frac{A}{k}+\frac{\delta_k}{k},$$

$$R(\lambda)=\prod_{k=-\infty,k\neq 0}^{\infty}\frac{\nu_k-\lambda}{k-\frac{1}{2}\operatorname{sign}k},\quad \nu_k=k-\frac{1}{2}\operatorname{sign}k+a+\frac{B}{k}+\frac{\widetilde{\delta}_k}{k},$$

where a,d_0,d_1,A,B *are some scalars and* $\sum\limits_{k=-\infty,k\neq 0}^{\infty}\left\{|\delta_k|^2+|\widetilde{\delta}_k|^2\right\}<\infty.$

This lemma is a generalization of lemma 3.4.2 from [11] and has been formulated in [5] (see also [13]).

3. Statement and proof of the main theorem

Theorem. *Let two sequences of real numbers* $\{\beta_{1,k}\},\{\beta_{2,k}\}$ *satisfy the following conditions*

$$1)\ \beta_{j,k}=k-\frac{1}{2}\operatorname{sign}k+a+\frac{A+B_j}{k}+\frac{\tau_{j,k}}{k}, \tag{3.1}$$

where $B_1<B_2$, $\sum\limits_{k=-\infty,k\neq 0}^{\infty}\tau_{j,k}^2<\infty(j=1,2)$;

$$\begin{aligned}&2)\ 0<\beta_{1,1}<\beta_{2,1}<\beta_{1,2}<\beta_{2,2}<...,\\ &0>\beta_{1,-1}>\beta_{2,-1}>\beta_{1,-2}>\beta_{2,-2}>...;\end{aligned} \tag{3.2}$$

$$3)\ \Delta_1(k+a)-\Delta_2(k+a)=\frac{(-1)^k\widetilde{C}+\theta_k}{k^2}, \tag{3.3}$$

where $\widetilde{C}=\pi^2(1+\omega^2)A(B_2-B_1)$ *and*

$$\Delta_j(\lambda)=(1+\omega^2)\prod_{k=-\infty,k\neq 0}^{\infty}\frac{\beta_{j,k}-\lambda}{k-\frac{1}{2}\operatorname{sign}k}; \tag{3.4}$$

$$4)\ |u(\lambda_k)|\geq 2|\omega|, \tag{3.5}$$

$$\lim_{k\to\infty}k(u(2k+1)-1-\omega^2)=0, \tag{3.6}$$

where

$$u(\lambda)=\frac{B_2\Delta_1(\lambda)-B_1\Delta_2(\lambda)}{B_2-B_1} \tag{3.7}$$

and $\lambda_k(k=\pm 1,\pm 2,...)$ *are zeros of* $\Delta_1(\lambda)-\Delta_2(\lambda)$*;*

$$5)\ u(0)+2\omega^2 s(0)\sum_{k=-\infty,k\neq 0}^{\infty}\frac{g(\lambda_k)}{\lambda_k s'(\lambda_k)}>0, \tag{3.8}$$

where

$$s(0)=\frac{\Delta_1(0)-\Delta_2(0)}{\pi(1+\omega^2)(B_1-B_2)},$$

$$g(\lambda_k)=\frac{1-\omega^2}{4\omega^2}u(\lambda_k)+\frac{1+\omega^2}{4\omega^2}(-1)^k\operatorname{sign}(\omega^2-1)\sigma_k\sqrt{u^2(\lambda_k)-4\omega^2},$$

where $\sigma_k=0$ *if* $|u(\lambda_k)|=2|\omega|$, $\sigma_k=\pm1$ *otherwise and* $\sigma_k=1$ *for sufficiently large* k. *Then there exist problems* $L(p(x),q(x),\omega,\alpha_j)$, $j=1,2$ *with* $\alpha_1<\alpha_2$ *and* $p(0)=p(\pi)$ *which eigenvalues coinside with* $\{\beta_{1,k}\}$ *and* $\{\beta_{2,k}\}$, *respectively.*

Proof. From lemma 2.2 we obtain that $\Delta_j(\lambda), j=1,2$ constructed by (3.4) are of the form

$$\Delta_j(\lambda)=(1+\omega^2)\left\{\cos\pi(\lambda-a)+(B_j+A)\pi\frac{\sin\pi(\lambda-a)}{\lambda-a}+\frac{P_j(\lambda-a)}{\lambda-a}\right\},\qquad(3.9)$$

where

$$P_j(\lambda)=E_j\cos\pi\lambda+\int_{-\pi}^{\pi}\widetilde{P}_j(t)e^{it\lambda}dt,\quad \widetilde{P}_j(t)\in L_2[-\pi,\pi].$$

It is clear from (3.9) that

$$\Delta_1(k+a)-\Delta_2(k+a)=(1+\omega^2)\frac{P_1(k)-P_2(k)}{k}.$$

Now using (3.3) we obtain

$$P_1(k)-P_2(k)=\frac{1}{1+\omega^2}\left\{\frac{(-1)^k\widetilde{C}}{k}+\frac{\theta_k}{k}\right\}.$$

Taking into account the representation of $P_j(\lambda)$, we can write

$$(E_1-E_2)(-1)^k+\int_{-\pi}^{\pi}[\widetilde{P}_1(t)-\widetilde{P}_2(t)]e^{ikt}dt=\frac{1}{1+\omega^2}\left\{\frac{(-1)^k\widetilde{C}+\theta_k}{k}\right\}.$$

From this equality, using the Rieman-Lebesque lemma, we find $E_1=E_2$.

Lemma 3.1. $\widetilde{P}_1(t)-\widetilde{P}_2(t)$ *has a square integrable derivative on* $(-\pi,\pi)$.

Proof. According to Carleson's theorem [2] (see also [1]) concerning convergence of Fourier series almost everywhere

$$\widetilde{P}_1(t)-\widetilde{P}_2(t)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}[P_1(k)-P_2(k)]e^{-ikt},\quad a.e.$$

Therefore, taking into account

$$t=2\sum_{k=1}^{\infty}(-1)^k\frac{\sin kt}{k},\quad -\pi<t<\pi,$$

we find

$$\widetilde{P}_1(t)-\widetilde{P}_2(t)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}[P_1(k)-P_2(k)]e^{-ikt}=\text{const}$$

$$+\frac{\widetilde{C}}{2\pi}\sum_{k=-\infty,k\neq 0}^{\infty}\frac{(-1)^k}{k}e^{-ikt}+\frac{1}{2\pi}\sum_{k=-\infty,k\neq 0}^{\infty}\frac{\theta_k}{k}e^{-ikt}=$$

$$=\mathrm{const}+\frac{\widetilde{C}it}{2\pi}+\frac{1}{2\pi}\sum_{k=-\infty,k\neq 0}^{\infty}\frac{\theta_k}{k}e^{-ikt}.$$

Since $\sum\limits_{k=-\infty,k\neq 0}^{\infty}\theta_k^2<\infty$, from the last formula we obtain that $\widetilde{P}_1(t)-\widetilde{P}_2(t)$ is equivalent to a function that has square integrable derivative on $(-\pi,\pi)$.

Let us denote

$$s(\lambda)=\frac{\Delta_1(\lambda)-\Delta_2(\lambda)}{\alpha_1-\alpha_2},$$

where $\alpha_j=\pi(1+\omega^2)B_j, j=1,2$. Taking into account (3.9), integrating by part (lemma 3.1) and using Paley-Wiener theorem [18, p.101], we find

$$(\lambda-a)s(\lambda)=\sin\pi(\lambda-a)-A\pi\frac{4(\lambda-a)}{4(\lambda-a)^2-1}\cos\pi(\lambda-a)+\frac{\Psi(\lambda-a)}{\lambda-a}, \quad (3.10)$$

where $\Psi(\lambda)=D\sin\pi\lambda+\int\limits_{-\pi}^{\pi}\widetilde{\Psi}(t)e^{it\lambda}dt,\quad \widetilde{\Psi}(t)\in L_2[-\pi,\pi], D\in\mathbb{C}$. Therefore, lemma 2.2 implies that for zeros $\lambda_k(k=\pm1,\pm2,...)$ of the function $s(\lambda)$ the following asymptotic formula hold

$$\lambda_k=k+a+\frac{A}{k}+\frac{\tau_k}{k},\qquad \sum_{k=-\infty,k\neq 0}^{\infty}\tau_k^2<\infty. \quad (3.11)$$

From (3.7) and (3.9) we obtain

$$u(\lambda)=(1+\omega^2)\left\{\cos\pi(\lambda-a)+A\pi\frac{\sin\pi(\lambda-a)}{\lambda-a}+\right.$$

$$\left.+\frac{E_1\cos\pi(\lambda-a)+\int\limits_{-\pi}^{\pi}E(t)e^{it(\lambda-a)}dt}{\lambda-a}\right\}. \quad (3.12)$$

From the last formula and (3.6) we find $E_1=E_2=0$.

Lemma 3.2 *The following inequalities hold*

$$0<\beta_{1,1}<\beta_{2,1}<\lambda_1<\beta_{1,2}<\beta_{2,2}<\lambda_2<...,$$

$$0>\beta_{1,-1}>\beta_{2,-1}>\lambda_{-1}>\beta_{1,-2}>\beta_{2,-2}>\lambda_{-2}....$$

Proof. Putting $\lambda=0$ in (3.4) and taking into account (3.2) we obtain $\Delta_j(0)>0$. Again, taking into account (3.2), we find that in each interval $(\beta_{2,k},\beta_{1,k+1})$ and $(\beta_{2,-k},\beta_{1,-(k+1)})$ the function $s(\lambda)$ possess at least one zero λ_k and λ_{-k}, respectively.

We denote $G_\delta=\{\lambda:|\lambda-a-n|\geq\delta, n=0,\pm1,\pm2,...\}$ for some small fixed $\delta>0$ and recall that

$$|\sin\pi(\lambda-a)|\geq C_\delta e^{|\operatorname{Im}\lambda|},\qquad \lambda\in G_\delta,$$

where C_δ does not depend on λ (see e.g [20, p.13])(We remark that it also follows from the fact that $\sin\pi(\lambda-a)$ is a function of sine type [10]).

Taking into account the latter fact, the representation (3.10) and applying Rouche's theorem it can easily be shown that the function $s(\lambda)$ has exactly one zero λ_k (λ_{-k}) in each interval $(\beta_{2,k},\beta_{1,(k+1)})$ $\Big((\beta_{2,-k},\beta_{1,-(k+1)})\Big)$ and $s(\lambda)$ does not possess other zeros.

Put

$$g(\lambda)=s(\lambda)\sum_{k=-\infty,k\neq 0}^{\infty}\frac{g(\lambda_k)}{(\lambda-\lambda_k)s'(\lambda_k)},$$

where

$$g(\lambda_k)=\frac{1-\omega^2}{4\omega^2}u(\lambda_k)+\frac{1+\omega^2}{4\omega^2}(-1)^k\operatorname{sign}(\omega^2-1)\sigma_k\sqrt{u^2(\lambda_k)-4\omega^2}. \tag{3.13}$$

From (3.5) it follows that $u^2(\lambda_k)\geq 4\omega^2$. Consequently, all $g(\lambda_k)$ are real. Using the asymptotic formula (3.11), the representation (3.12) and taking into account the following asymptotic formulas

$$\cos x=1+O(x^2),$$

$$\sin x=O(x),$$

$$\sqrt{1+x}=1+\frac{x}{2}+O(x^2),\quad as\quad x\to 0$$

and the fact that $\sigma_k=1$ for sufficiently large k, it can easily be shown that

$$\sum_{k=-\infty,k\neq 0}^{\infty}(\lambda_k g(\lambda_k))^2<\infty.$$

Then ([5]) the function $g(\lambda)$ is of the form

$$g(\lambda)=\frac{1}{\lambda-a}\int_{-\pi}^{\pi}K(t)e^{it(\lambda-a)}dt, \tag{3.14}$$

where $K(t)\in L_2[-\pi,\pi]$.

Define the function $s_1(\lambda)$ by

$$s_1(\lambda)=\frac{1}{1+\omega^2}u(\lambda)-\frac{2\omega^2}{1+\omega^2}g(\lambda). \tag{3.15}$$

According to (3.12) and (3.14),

$$s_1(\lambda)=\cos\pi(\lambda-a)+A\pi\frac{\sin\pi(\lambda-a)}{\lambda-a}+\frac{1}{\lambda-a}\int_{-\pi}^{\pi}F(t)e^{it(\lambda-a)}dt.$$

Therefore, using lemma 2.2, for zeros ν_k $(k=\pm1,\pm2,...)$ we obtain the following formula

$$\nu_k=k-\frac{1}{2}\operatorname{sign}k+a+\frac{A}{k}+\frac{\xi_k}{k},\qquad \sum_{k=-\infty,k\neq 0}^{\infty}\xi_k^2<\infty. \tag{3.16}$$

Taking $\lambda = \lambda_k$ in (3.15) we find

$$s_1(\lambda_k) = \frac{1}{2}\left[u(\lambda_k) - (-1)^k \operatorname{sign}(\omega^2 - 1)\sigma_k \sqrt{u^2(\lambda_k) - 4\omega^2}\right].$$

This implies

$$u(\lambda_k) = \frac{\omega^2 + s_1^2(\lambda_k)}{s_1(\lambda_k)}.$$

From this equalities we obtain

$$\operatorname{sign} s_1(\lambda_k) = \operatorname{sign} u(\lambda_k).$$

Since $\Delta_1(\lambda_k) = \Delta_2(\lambda_k)$,

$$u(\lambda_k) = \Delta_j(\lambda_k), \quad j = 1, 2.$$

Now, using lemma 3.2 and the fact that $\Delta_j(0) > 0$ $(j = 1, 2)$ it is easy to see that $\operatorname{sign} \Delta_j(\lambda_k) = (-1)^k$. Hence,

$$\operatorname{sign} s_1(\lambda_k) = \operatorname{sign} u(\lambda_k) = \operatorname{sign} \Delta_j(\lambda_k) = (-1)^k$$

and from (3.8) we have $s_1(0) > 0$. Therefore, taking into account (3.16), applying Rouche's theorem, we find that the function $s_1(\lambda)$ has exactly one zero in each interval $...(\lambda_{-2}, \lambda_{-1}), (\lambda_{-1}, 0), (0, \lambda_1), (\lambda_1, \lambda_2)....$ Consequently, zeros of $s_1(\lambda)$ and $s(\lambda)$ interlace in the following meaning

$$... < \lambda_{-2} < \nu_{-2} < \lambda_{-1} < \nu_{-1} < 0 < \nu_1 < \lambda_1 < \nu_2 < \lambda_2 < ... \tag{3.17}$$

Besides, sequences $\{\lambda_k\}$ and $\{\nu_k\}$ behave as (3.11) and (3.16), respectively. Therefore, according to [4] there exist functions $q(x) \in L_2[0, \pi]$ and $p(x) \in W_2^1[0, \pi]$ such that $\{\lambda_k\}$ and $\{\nu_k\}$ are eigenvalues of the problems, associated with the equation

$$y''(x) + [\lambda^2 - 2\lambda p(x) - q(x)]y(x) = 0$$

and boundary conditions

$$y(0) = y(\pi) = 0$$

and

$$y(0) = y'(\pi) = 0,$$

respectively; besides

$$s(\lambda) = s(\pi, \lambda), \tag{3.18}$$

$$s_1(\lambda) = s'(\pi, \lambda), \tag{3.19}$$

where $s(x, \lambda)$ is the solution of the constructed equation satisfying the following initial conditions

$$s(0, \lambda) = s'(0, \lambda) - 1 = 0.$$

Let $c(x, \lambda)$ be the solution of constructed equation satisfying the initial conditions

$$c(0, \lambda) - 1 = c'(0, \lambda) = 0.$$

Lemma 3.3. *The following identity holds*

$$c(\pi,\lambda)=\frac{1}{1+\omega^2}u(\lambda)+\frac{2}{1+\omega^2}g(\lambda).$$

Proof. From identity

$$c(\pi,\lambda)s'(\pi,\lambda)-c'(\pi,\lambda)s(\pi,\lambda)=1$$

and from (3.19) it follows that

$$c(\pi,\lambda_k)=\frac{1}{s'(\pi,\lambda_k)}=\frac{1}{s_1(\lambda_k)}.$$

According to (3.13)

$$\frac{1}{1+\omega^2}u(\lambda_k)+\frac{2}{1+\omega^2}g(\lambda_k)=$$

$$=\frac{1}{2\omega^2}u(\lambda_k)+\frac{1}{2\omega^2}(-1)^k\operatorname{sign}(\omega^2-1)\sigma_k\sqrt{u^2(\lambda_k)-4\omega^2}$$

$$=\frac{2}{u(\lambda_k)-(-1)^k\operatorname{sign}(\omega^2-1)\sigma_k\sqrt{u^2(\lambda_k)-4\omega^2}}.$$

Hence, taking into account (3.13) and (3.15), we find

$$\frac{1}{1+\omega^2}u(\lambda_k)+\frac{2}{1+\omega^2}g(\lambda_k)=\frac{1}{s_1(\lambda_k)}.$$

These imply

$$c(\pi,\lambda_k)=\frac{1}{1+\omega^2}u(\lambda_k)+\frac{2}{1+\omega^2}g(\lambda_k).$$

Define

$$r(\lambda)=c(\pi,\lambda)-\left[\frac{1}{1+\omega^2}u(\lambda)+\frac{2}{1+\omega^2}g(\lambda)\right]=c(\pi,\lambda)-s'(\pi,\lambda)$$

$$-\left[\frac{1}{1+\omega^2}u(\lambda)+\frac{2}{1+\omega^2}g(\lambda)-s'(\pi,\lambda)\right]=c(\pi,\lambda)-s'(\pi,\lambda)$$

$$-\left[\frac{1}{1+\omega^2}u(\lambda)+\frac{2}{1+\omega^2}g(\lambda)-s_1(\lambda)\right]=c(\pi,\lambda)-s'(\pi,\lambda)-2g(\lambda).$$

Putting $\lambda=k+a+\frac{1}{2}(k\in\mathbb{N})$ in lemma 2.1 and taking into account (3.19) we find $p(0)=p(\pi)$, where $p(x)$ is a coefficient function of a constructed problems. Now from the latter fact we find

$$r(\lambda)=\frac{1}{\lambda}\int\limits_{-\pi}^{\pi}G(t)e^{it\lambda}dt,$$

for some $G(t)\in L_2[-\pi,\pi]$. Therefore

$$r(\lambda)=o\left(\frac{e^{\pi|\operatorname{Im}\lambda|}}{|\lambda|}\right),\quad as\quad |\lambda|\to\infty. \tag{3.20}$$

Let Γ_n be a contour bounding the square $\left\{\lambda : |\operatorname{Re}\lambda - a| \le n + \dfrac{1}{2}, |\operatorname{Im}\lambda| \le n + \dfrac{1}{2}\right\}$. Then due to (3.10), there exist such a constant $c > 0$ that

$$|s(\pi, \lambda)| \ge \frac{c}{|\lambda|} e^{\pi|\operatorname{Im}\lambda|}, \quad \lambda \in \Gamma_n, \tag{3.21}$$

where c does not depend on n. Since $r(\lambda_k) = 0$, $\dfrac{r(\lambda)}{s(\lambda)}$ is an entire function and from (3.20) and (3.21) it is clear that it tends to zero on the Γ_n. According to Liouville's theorem it yields $\dfrac{r(\lambda)}{s(\lambda)} \equiv 0$ which completes the proof of lemma 3.3.

Using lemma 3.3 and (3.18), (3.19) we find

$$\omega^2 c(\pi, \lambda) + s'(\pi, \lambda) = u(\lambda). \tag{3.22}$$

Now, we will show that characteristic functions $\widetilde{\Delta}_1(\lambda), \widetilde{\Delta}_2(\lambda)$ of the constructed problems $L(p(x), q(x), \omega, \alpha_j), j = 1, 2$ coincide with $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$, respectively. Indeed, according to (3.18), (3.19) and (3.22)

$$\widetilde{\Delta}_1(\lambda) = \omega^2 c(\pi, \lambda) + s'(\pi, \lambda) + \alpha_1 s(\pi, \lambda)$$

$$= u(\lambda) + \alpha_1 s(\pi, \lambda) = \frac{\alpha_2 \Delta_1(\lambda) - \alpha_1 \Delta_2(\lambda)}{\alpha_2 - \alpha_1} + \alpha_1 \frac{\Delta_1(\lambda) - \Delta_2(\lambda)}{\alpha_1 - \alpha_2} = \Delta_1(\lambda).$$

Similarly, we can show that

$$\widetilde{\Delta}_2(\lambda) \equiv \Delta_2(\lambda).$$

The proof of the theorem is complete.

Remark. We note that under some restriction on the potential $q(x)$, eigenvalues $\beta_{j,k} (j = 1, 2; k = \pm 1, \pm 2, ...)$ of the problems $L(p(x), q(x), \omega, \alpha_j)$ with $\alpha_1 < \alpha_2$, $p(0) = p(\pi)$ satisfy all conditions of the theorem.

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