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ON NON-LOCAL SOLVABILITY OF NON-LINEAR DIFFERENTIAL OPERATOR EQUATIONS OF PARABOLIC TYPE IN BANACH SPACE

Abstract

In the paper we study the Cauchy problem for non-linear differential-operator equations with unbounded, non densely given, variable operator coefficients in Banach space. New classes of evolution equations for which Cauchy problem is globally solvable, are distinguished.

The Cauchy problem for differential-operator equations of first order was studied in different aspects by many authors (see. e.g. [1-3,6], in which there is a wide bibliography). The Cauchy problem for differential-operator equations of second order in the Banach space was also studied [4-5]. The present paper is devoted to studying quasilinear differential-operator equations of parabolic type with unbounded, non densely defined variable operator coefficients in Banach space.

At first, let's consider the Cauchy problem for quasilinear differential-operator equation of first order

$$u'(t) + A(t, u(t))u(t) = f(t, u(t)) \qquad (0 < t \le T) \tag{1}$$

$$u(0) = u_0, \tag{2}$$

in Banach space E. Here, $A(t, \upsilon)$ is a linear operator given for each $t \in [0, T]$ and $\upsilon \in E$ with domain of definition $D(A(t, \upsilon)) = D(t)$, $f(t, \upsilon)$ is the given continuous function and u_0 is the given element from E.

Definition 1. *The function $u(t)$ continuous on $[0, T]$, continuously differential on $(0, T]$ with values in E, satisfying for each $t \in (0, T]$ equation (1) and initial condition $u(0) = u_0$, for which it holds the inequality*

$$\|A_0^\alpha u(t) - A_0^\alpha u(\tau)\| \le C\omega\left(|t - \tau|\right) \tag{3}$$

where $A_0 = A(0, u_0)$, $\alpha \in [0, 1)$, A_0^α is fractional power of the operator A_0; $\omega(t)$ is a positive, monotonically increasing, defined on $(0, \infty)$ function for some $p \in \left[\frac{1}{2\beta - 1}, \infty\right)$ $\beta \in \left(\frac{1}{2}, 1\right]$ satisfies the condition

$$\int_0^{T_0} \frac{\omega^p(r)}{r^p} \left|\log r\right|^p dr < \infty.$$

will be said to be a solution of problem (1), (2) on $[0, T]$.

This problem was researched in the paper [3] provided that the operator $A(t, \upsilon)$ generates analytic semi-group of the class (C_0), the domain of definition $D(t)$ of the operator $A(t,\upsilon)$ is densely set in E, the right hand side in (1) is Holderian continuous. In the paper [4] problem (1), (2) was studied in the case when the operator $A(t,\upsilon)$ is a generating operator of the class $(0,A)$. The right hand side $f(t,A_0^{-1}u)$ on $[0,T]\times S_0$ ($S_0\subset E$ is some ball) has partial derivatives continuous in totality of variables, satisfying the Lipschitz condition.

In this paper the problem (1)-(2) is considered in the case of variable domain of definition $D(t)$ of the operator function $A(t,\upsilon)$. Here we consider a more general case, when the set $D(t)$ may depend on t; $D(t)$ is non densely set in E, and the operator $A(t_0,\upsilon)$ for each $t_0\in[0,T]$ and $\upsilon\in E$ generates a strongly continuous semi-group having power singularity in zero.

The following conditions are imposed everywhere on the operator function $A(t,\upsilon)$:

(I). For each $t\in[0,T]$, $\upsilon\in E$ the operator $A(t,A_0^{-\alpha}\upsilon)$ has a bounded inverse operator $A(t,A_0^{-\alpha}\upsilon)$ and for some $\eta>0$, $\beta\in\left(\frac{1}{2},1\right]$ satisfies the condition

$$\left\|R(\lambda;A(t,A_0^{-\alpha}\upsilon))\right\|\le C\left|\lambda\right|^{-\beta},\ \ |\arg\lambda|<\frac{\pi}{2}+\eta,\ \ |\lambda|\to\infty.$$

(II). For all $0\le\tau\le t\le T$ and for some $\alpha\in[1-\beta,1)$ the inclusions $D(A(\tau,A_0^{-\alpha}u))\subset D(A(t,A_0^{-\alpha}u))$ hold; for any $0\le s\le\tau,\ t\le T$ it is fulfilled the inequality

$$\left\|\left[A(t)-A(\tau)\right]A^{-1}(s)\right\|\le C\omega\left(|t-\tau|\right),$$

where $\omega(t)$ is a positive, monotonically increasing, determined on the interval $(0,\infty)$ function that for some $p\in\left[\frac{1}{2\beta-1},\infty\right)$ satisfies the following condition (for $\beta=1$, $p=1$ see [9])

$$\int\limits_0^{T_0}\frac{\omega^p(r)}{r^p}\left|\log r\right|^p dr<\infty. \tag{4}$$

For example, the functions:

$$\omega(r)=c\left|1ogr\right|^{\alpha},\quad for\quad \alpha<-2,\ p=1;$$
$$\omega(r)=c\frac{\log(1+r)}{r^{\frac{1}{p}}\left|\log r\right|^{\alpha}}\quad for\quad \alpha>\frac{p+1}{p},\ p\ge1.$$

satisfy the condition (4).

Notice that these functions are not uniformly continuous by Holder in the vicinity $r=0$.

Fractional powers may be determined for the operators satisfying condition (I) (see [1]). Negative fractional powers $A^{-\delta(t)}A(t,A_0^{-\alpha}\upsilon)\equiv A(t)$ as in the case of strongly continuous exponentially decreasing semi-groups are derived by the formula

$$A^{-\delta}(t)=\frac{1}{\Gamma(\delta)}\int\limits_0^{\infty}s^{\delta-1}\exp\left\{-s\ A(t)ds\right\}.$$

However, unlike the strongly continuous semi-groups, only for $\beta+\delta>1$ this formula determines bounded operators. Positive fractional powers $A^{\delta}(t)$ are determined as the inverse operators to negative powers $A^{-\delta}(t)$. Another approach to definition of fractional powers of "badly positive" operators is given in [18]. By $D(A^{-\delta}(t))$ we denote a set of elements $\upsilon\in E$, for which the integral $\int\limits_0^\infty s^{\delta-1}\exp\{-s\ A(t)\}\,\upsilon ds \quad (\delta>0)$ converges improperly (in zero). For such ν we assume

$$A^{-\delta}(t)\upsilon=\frac{1}{\Gamma(\delta)}\int\limits_0^\infty s^{\delta-1}e^{-sA(t)}\upsilon ds \qquad (\delta>0).$$

Continuous imbedding $D(A(t))\subset D(A^{-\delta}(t))$ for each $t\in[0,T]$ easily follows from this definition. If $\delta>1-\beta$, then $D(A^{-\delta}(t))=E$ and the operators $A^{-\delta}(t)$ are bounded. For $0<\delta<1-\beta$ as the example from [1] shows, these operators may be unbounded.

On the elements $\nu\in D\left(A^{-\delta}(t)\right)$ $(0<\delta<1)$ the equality $A^{-(2-\delta)}(t)\ A^{-\delta}(t)\nu=A^{-2}(t)\nu$ is valid. It follows of this that $\nu=0$, if $\nu\in D\left(A^{-\delta}(t)\right)$ and $A^{-\delta}(t)\upsilon=0$, therefore there exist inverse operators $[A^{-\delta}(t)]^{-1}$. Now, let's define positive fractional operators $A(t)$ by means of the equality $A^{\delta}(t)=[A^{-\delta}(t)]^{-1}$ $(0<\delta\le 1)$. For investigation of problem (1)-(2) we apply the methods of semi-groups theory.

1. Solvability of problem (1)-(2). For the problem (1)-(2) we prove the following solvability theorem.

Theorem 1. *Let conditions (I) and (II) be fulfilled.*

Let, further:

(III). *For any* $t,s\in[0,T];\nu,w\in E$ with $\|\nu\|,\|w\|\le R$ *it hold the inequality*

$$\left\|\left[A(t,A^{-\alpha}\nu)-A(s,A_0^{-\alpha}w\right]A_0^{-1}\right\|\le C(R)\left[\omega(|t-s)|+\|\nu-w\|\right];$$

(IV). *For any* $t,s\in\{0,T];\ \upsilon,w\in E$ *it hold the inequality*

$$\left\|f(t,A_0^{-\alpha}\nu)-f(s,A_0^{-\alpha}w\right\|\le C(R)\left[\omega(|t-s)|+\|\nu-w\|\right];$$

(V). *For any* $t,s\in[0,T];\upsilon\in E$ *it hold the estimation*

$$\left\|f(t,A_0^{-\alpha}\nu)\right\|\le C(1+\|\nu\|);$$

(VI). $u_0\in D(A_0^{\delta})$ *for some* $\delta>\alpha$.

Then problem (1)-(2) has a unique solution on $[0,T]$.

Proof. Under conditions (I), (II), (III) and (VI) problem (1)-(2) is equivalent to the integral equation (see, e.g. [3]).

$$u(t)=U_u(t,0)u_0+\int\limits_0^t U_u(t,s)f_u(s)ds, \tag{5}$$

where $U_u(t,s)$ is an evolution operator generated by the operator

$$A(t, A_0^{-\alpha}u(t)),\ \ f_u(t) = f(t, A_0^{-\alpha}u(t)).$$

It is known the following estimation [see (10)]

$$\left\| A^{\delta}(t) U(t,s) A^{-\alpha}(t) \right\| \le C(\alpha,\beta,\delta) \left| t-s \right|^{\alpha+\beta-\delta-1}. \qquad (\alpha)$$

In this equation the integrated summand admits the estimation

$$\|U_u(t,0)u_0\| = \left\| U_u(t,0)A_0^{-\delta} \right\| \cdot \left\| A_0^{\delta}u_0 \right\| \le Ct^{\beta+\delta-1} \left\| A_0^{\delta}u_0 \right\| \le C(\beta,\delta) \left\| A_0^{\delta}u_0 \right\|,$$

since here $\delta > 1-\beta$, and for $\delta > 1-\beta$ the operator function $U_u(t,0)A_0^{-\delta}$ is continuous on $t \in [0,T]$ (see [1]). Therefore the principle of contracted mappings is applied to equation (5). Consequently, equation (5) on some interval $[0,t_0]$, where $t_0 \in (0,T]$, has a unique solution, that may be found by successive approximations method. Let $u(t)$ be any solution of equation (5). Then

$$\|u(t)\| \le \left\| U_u(t,0)A_0^{-\delta} \right\| \cdot \left\| A_0^{\delta}u_0 \right\| + \int\limits_0^t \|U_u(t,s)\| \cdot \|f_u(s)\|\, ds, \qquad (6)$$

since $\left\| U_u(t,0)A_0^{-\delta} \right\| \le C(\beta,\delta)$ and

$$\|f_u(s)\| = \left\| f(s, A_0^{-\alpha}u(s)) \right\| \le C(1+\|u(s)\|), \qquad (7)$$

we get from (IV) and (V)

$$\|u(t)\| \le C \left\| A_0^{\delta}u_0 \right\| + \int\limits_0^t c\left| t-s \right|^{\beta-1} (1+\|u(s)\|) ds \le C + \int\limits_0^t C\left| t-s \right|^{\beta-1} ds +$$

$$+C\int\limits_0^t C\left| t-s \right|^{\beta-1} \|u(s)\|) ds \le C + C\int\limits_0^t C\left| t-s \right|^{\beta-1} \|u(s)\|) ds.$$

Hence, by the theorem on integral inequalities [[6] p. 206] we get estimation $\|u(t)\| \le CE(\theta,t),\ t \in [0,T]$,

where $\theta = (C\Gamma(\beta))^{\frac{1}{\beta}},\ E_\beta(t) = \sum\limits_{n=0}^{\infty} \dfrac{z^{n^\beta}}{\Gamma(n\beta+1)}$; $\left(E_\beta(z) \approx \dfrac{1}{\beta} e^z \quad for \quad z \to +\infty \right)$

Consequently for any solution $u(t)$ we have a priori estimation $\|u(t)\| \le C$. Now, acting by the operator $A^{\delta}(t, A_0^{-\alpha}u(t))$ in (5) we get

$$A^{\delta}(t, A_0^{-\alpha}u(t))u(t) = A^{\delta}(t, A_0^{-\alpha}u(t))U_u(t,0)u_0 + \int\limits_0^t A^{\delta}(t, A_0^{-\alpha}u(t))U_u(t,s)f_u(s)ds.$$

From estimation (α) we have:

$$\left\| A^{\delta}(t, A_0^{-\alpha}u(t)) U_u(t,s) \right\| \le C(\beta,\delta) \left| t-s \right|^{\beta-\delta-1}$$

Besides, by condition (I) and inequality of moments for fractional powers of operators [7] for any $\nu \in E$ c $\|\nu\| \leq R$ it holds the inequality $\left\|A^{-\delta}(t, A^{-\alpha}u(t))\right\| \leq C(R)$. Then, by (IV) and ($\alpha$), taking into account a priori estimation $\|u(t)\| \leq C$ we get

$$\left\|A^{\delta}(t, A_0^{-\alpha}u(t))u(t)\right\| \leq \left\|A^{\delta}(t, A_0^{-\alpha}u(t))U_u(t,0)A_0^{-\rho}\right\| \cdot \left\|A_0^{-\rho}u_0\right\| +$$
$$+\int_0^t C(\beta,\delta)\left|t-s\right|^{\beta-\delta-1}(1+\|u(s)\|)ds \leq C(\alpha,\delta,\beta,\rho)+$$
$$+\int_0^t C(\beta,\delta)\left|t-s\right|^{\beta-\delta-1}\left\|A^{\delta}(s, A_0^{-\alpha}u(s))\right\| ds$$

Here $\rho > 1+\delta-\beta$. By the theorem on integral inequalities we again have:

$$\left\|A^{\delta}(t, A^{-\alpha}u(t))U(t)\right\| \leq C(\alpha,\delta,\beta,\rho)E_d(\theta t) \leq C,$$

where $d = \beta - \delta$. So, by [3] the theorem is completely proved.

Remark. If $u(t)$ is the solution of problem (1), then under the conditions of the theorem $A(t,u(t))u(t)$ will be continuous on $(0,T]$. Really, by condition (III) and by inequality (3) we get

$$\|f(t,u(t)) - f(s,u(s))\| \leq C\left[\omega(|t-s|) + \|A_0^{\alpha}(u(t)-u(s))\|\right] \leq C\omega(|t-s|).$$

This shows that the vector-function $f(t,u(t))$ is continuous on $[0,T]$. But since by definition $u'(t)$ is continuous on $(0,T]$, continuity of $A(t,u(t))u(t)$ on $(0,T]$ is obtained from equation (1).

2. Now go over the problem for a second order equation

$$u''(t) + A(t,u(t),u'(t))u'(t) = f(t,u(t),u'(t)); \tag{8}$$

$$u(0) = u_0, \quad u'(0) = u_1 \tag{9}$$

Definition 2. *The function $u(t)$ continuously differentiable on $[0,T]$, twice continuously differentiable on $(0,T]$ with values in E, satisfying for each $t \in (0,T]$ equation (8) and initial conditions of (9), for which it holds the estimation*

$$\left\|A_0^{\alpha}u'(t) - A_0^{\alpha}u'(s)\right\| \leq C\omega\left(|t-s|\right),$$

where $A_0 = A(0,u_0,u_1)$; the function $\omega(r)$ satisfies the condition (4), will be said to be a solution of problem (8) (9) on $[0,T]$.

Theorem 2. *Let the following conditions be fulfilled:*

(I) *For each $t \in [0,T], u,\nu \in E, A(t) = A(t, A_0^{\alpha}u, A_{\upsilon}^{-\alpha}\nu)$ is a closed linear operator with domain of definition $D(A(t))$, has a bounded inverse operator $A^{-1}(t, A_0^{\alpha}u, A_0^{-\alpha}\upsilon)$ and for some $\eta > 0, \beta \in \left(\frac{1}{2}, 1\right]$*

$$\|R(\lambda; -A(t))\| \leq C\left|\lambda\right|^{-\beta}, |\arg\lambda| < \frac{\pi}{2}+\eta, \ \ |\lambda| \to \infty.$$

(II) *For some* $\alpha \in (1-\beta, 1]$ *and any* $u, \upsilon, u', \upsilon' \in E$, with $\|u\| + \|\upsilon\| < R$, $\|u'\| + \|\upsilon'\| \leq R$ *where* R *is some positive number and for* $0 \leq \tau \leq t \leq T$; $D(A(\tau, A_0^{-\alpha}u, A_0^{-\alpha}\upsilon)) \subset D(A(t, A_0^{-\alpha}u, A_0^{-\alpha}\upsilon))$; *for any* $0 \leq s \leq \tau, t \leq T$,

$$\left\| \left[A(t, A_0^{-\alpha}u, A_0^{-\alpha}\upsilon) - A(\tau, A_0^{-\alpha}u', A_0^{-\alpha}\upsilon') \right] A_0^{-1}(s) \right\| \leq$$
$$\leq C \left[\omega\left(|t-\tau|\right) + \|u-u'\| + \|\upsilon - \upsilon'\| \right];$$

where $A_0 = A(0, u_0, u_1)$; the function $\omega(t)$ satisfies the condition (4).

(III) For any $t, \tau \in [0, T]; u, \upsilon, u', \upsilon' \in E$ with $\|u\| + \|\upsilon\| < R$, $\|u'\| + \|\upsilon'\| \leq R$ it holds

$$\left\| f(t, A_0^{\alpha}u, A^{-\alpha}\upsilon) - f(\tau, A_0^{-\alpha}u', A_0^{-\alpha}\upsilon') \right\| \leq C \left[\omega\left(|t-\tau|\right) + \left\|u-u'\right\| + \left\|\upsilon - \upsilon'\right\| \right].$$

(IV) $u_0 \in D(A_0^{\alpha}), u_1 \in D(A_0^{\delta})$ *for some* $\delta > \alpha$.

Then on some segment $[0, t_0]$ where $t_0 \in [0, T]$, there exists a unique solution of the problem (8)-(9).

Let $u(t)$ be a solution of the problem (8)-(9). Then the functions

$$V_1(t) = A_0^{\alpha}u(t),\ V_2(t) = u'(t) \tag{10}$$

are continuous on $[0, T]$ and continuously differentiable on $(0, T]$, since the functions $A_0^{\alpha}u(t)$ and $u'(t)$ possess these properties. Then differentiating on $(0, T]$ each relation of (10) and taking into account that $u(t)$ satisfies equation (8) on $(0, T]$ we get that the function $\upsilon(t) = (\upsilon_1(t), \upsilon_2(t))$ in the Banach space $E \times E$ satisfies the problem

$$\nu'(t) + \mathfrak{U}(t, \nu(t))\nu(t) = F(t, \nu(t)); \tag{11}$$

$$\nu(0) = \nu_0 \tag{12}$$

where

$$\mathfrak{U}(t, \upsilon) = \begin{pmatrix} I & 0 \\ 0 & A(t, A_0^{-\alpha}\nu_1, \nu_2) \end{pmatrix}, \tag{13}$$

$$F(t, \upsilon) = \begin{pmatrix} \upsilon_1 + A_0^{\alpha}\nu_2 \\ f(t, A^{-\alpha}\nu_1, \nu_2) \end{pmatrix}, \upsilon_0 = \begin{pmatrix} A_1^{\alpha}u_0 \\ u_1 \end{pmatrix}. \tag{14}$$

Directly calculating, we find

$$(\mathfrak{U}_0 + \lambda I)^{-1} = \begin{pmatrix} (\lambda+1)^{-1} & 0 \\ 0 & (A_0 + \lambda I)^{-1} \end{pmatrix}, \tag{15}$$

where $\mathfrak{U}_0 = \mathfrak{U}(0, \nu_0)$.

Hence, by the formula $\mathfrak{U}_0^{-\alpha} = \dfrac{\sin \pi\alpha}{\pi} \displaystyle\int_0^{\infty} t^{-\alpha}(tI + \mathfrak{U}_0)^{-1}dt$ we get $\mathfrak{U}_0^{-\alpha} =$

$$= \begin{pmatrix} I & 0 \\ 0 & A_0^{-\alpha} \end{pmatrix}.$$

Now, let's prove that the function $\upsilon(t)$ satisfies the inequality

$$\|\mathfrak{U}_0^{\alpha}\nu(t) - \mathfrak{U}_0^{\alpha}\nu(s)\| \leq C\omega\left(|t-s|\right).$$

Since

$$\mathfrak{U}_0^\alpha \nu(t) - \mathfrak{U}_0^\alpha \nu(s) = \left(\mathcal{A}_0^\alpha u(t) - \mathcal{A}_0^\alpha u(s),\ \mathcal{A}_0^\alpha u'(t) - \mathcal{A}_0^\alpha u'(s)\right),$$

hence we get the estimation

$$\|\mathfrak{U}_0^\alpha \nu(t) - \mathfrak{U}_0^\alpha \nu(s)\| \le \|\mathcal{A}_0^\alpha u(t) - \mathcal{A}_0^\alpha u(s)\| + \left\|\mathcal{A}_0^\alpha u'(t) - \mathcal{A}_0^\alpha u'(s)\right\| \le C\omega\left(|t-s|\right).$$

We showed that if $u(t)$ is a solution of the problem (8)-(9), the function $\nu(t) = (\mathcal{A}_0^\alpha u(t), u'(t))$ will be a solution of the problem (11)-(12). The inverse statement is easily proved. Thus, the problems (9)-(10) and (11)-(12) are equivalent.

Now, prove that under the conditions of theorem 2 the conditions (I), (II), (III) and (VI) of theorem 1 are satisfied. The operator $\mathfrak{U}_0$ has a domain of definition $D(\mathfrak{U}_0) = E \times D(A_0) \subset E \times E$. Besides, it is clear from (15) that for any λ that $|\arg\lambda| \le \dfrac{\pi}{2} + \eta$, it holds

$$\left\|(\mathfrak{U}_0 + \lambda)^{-1}\right\| \le C\,|\lambda|^{-\beta}.$$

Now, let $\nu = (\nu_1, \nu_2)$ where $\nu_1, \nu_2 \in E$. Then we can easily get

$$\mathfrak{U}(t, \mathfrak{U}_0^{-\alpha}\nu) = \begin{pmatrix} I & 0 \\ 0 & A(t, A_0^{-\alpha}\nu_1, \nu_2) \end{pmatrix}.$$

By condition (II) the operator $\mathfrak{U}(t, \mathfrak{U}_0^{-\alpha}\nu)$ is determined on $D(\mathfrak{U}_0)$. Besides, for $0 \le s \le t \le T$ it holds the inclusion $D(\mathfrak{U}(\tau, \mathfrak{U}_0^{-\alpha}\nu) \subset D(\mathfrak{U}(t, \mathfrak{U}_0^{-\alpha}\nu))$ and for any $\nu = (\nu_1, \nu_2), \nu' = (\nu_1', \nu_2')c\,\|\nu\|, \|\nu'\| \le R$ we have

$$\left[\mathfrak{U}(t, \mathfrak{U}_0^{-\alpha}\nu) - \mathfrak{U}(s, \mathfrak{U}_0^{-\alpha}\nu')\right]\mathfrak{U}_0^{-1} = .$$
$$= \begin{pmatrix} I & 0 \\ 0 & \left[A(t, A_0^{-\alpha}\nu_1, \nu_2) - A(s, A^{-\alpha}\nu_1', \nu_2')\right]A_0^{-1} \end{pmatrix}$$

Hence, by condition (II) we get

$$\left\|\left[\mathfrak{U}(t, \mathfrak{U}_0^{-\alpha}\nu) - \mathfrak{U}(s, \mathfrak{U}_0^{-\alpha}\nu')\right]\mathfrak{U}_0^{-1}\right\| \le C\left[\omega\left(|t-s|\right) + \left\|\nu - \nu'\right\|\right].$$

Besides, since $F(t, \mathfrak{U}_0^{-\alpha}\nu) = (\nu_1 + A_0^\alpha\nu_2, , f(t, A_0^\alpha\nu_1, \nu_2))$, then for any $t, s \in$
$\in [0, T], \nu = (\nu_1, \nu_2), \nu' = (\nu_1', \nu_2')c\,\|\nu\|, \|\nu'\| \le R$ we get

$$F(t, \mathfrak{U}_0^{-\alpha}\nu) - \mathcal{F}(s, \mathfrak{U}_0^{-\alpha}\nu') = (\nu_1 - \nu_1' + A_0^\alpha\nu_2 - A_0\nu_2',$$
$$f(t, A_0^{-\alpha}\nu_1, \nu_2) - f(s, A^{-\alpha}\nu_1', \nu_2'),$$

Consequently, by condition (III) we have:

$$\left\|\mathcal{F}(t, \mathfrak{U}_0^{-\alpha}\nu) - \mathcal{F}(s, \mathfrak{U}_0^{-\alpha}\nu')\right\| \le C\left[\omega\left(|t-s|\right) + \left\|\nu - \nu'\right\|\right].$$

Finally, since $\mathfrak{U}_0^\delta\nu_0 = \begin{pmatrix} \mathcal{A}_0^\delta u_0 \\ \mathcal{A}_0^\delta u_1 \end{pmatrix}$ we get from condition (IV) that $\nu_0 \in D(\mathcal{U}_0^\delta)$.

Thus, the conditions (I-III) and (VI) of theorem 1 hold. Then by [10] there exists a unique solution of the problem (11)-(12) on $[0, t_0]$. Therewith we proved the existence of a unique solution of the problem (8)-(9) on $[0, t_0]$ where $t_0 \in$
$\in (0, T]$.

Theorem 3. *Let the following conditions be fulfilled:*

I. *For any* $t_0 \in (0,T], \nu_1, \nu_2 \in E, A(t, A_0^{-\alpha}\nu_1, A_0^{-\alpha}\nu_2) = A(t)$ *a closed linear operator with domain of definition* $D(A(t))$ *has a bounded inverse operator* $A^{-1}(t, A_0^{-\alpha}\nu_1, A_0^{-\alpha}\nu_2)$, *moreover for some* $\eta > 0, \beta \in \left(\frac{1}{2}, 1\right]$

$$\|R(\lambda; -A(t))\| \leq C\,|\lambda|^{-\beta}, |\arg\lambda| \leq \frac{\pi}{2} + \eta, \ \ |\lambda| \to \infty.$$

II. *For some* $\alpha \in (1-\beta, 1]$ *for any* $\nu_1, \nu_2, w_1, w_2 \in E$ *with* $\|\nu_1\| + \|\nu_2\| \leq$ $\leq R$, $\|w_1\| + \|w_2\| \leq R$, *and for* $0 \leq \tau \leq t \leq T$; $D(A(\tau, A_0^{-\alpha}\nu_1, A_0^{-\alpha}\nu_2)) \subset$ $\subset D(A(t, A^{-\alpha}\nu_1, A^{-\alpha}\nu_2))$; $0 \leq s \leq \tau, t \leq T$ *for any*

$$\begin{gathered}\left\|\left[A(t, A_0^{-\alpha}\nu_1, A_0^{-\alpha}\nu_2) - A(\tau, A_0^{-\alpha}w_1, A_0^{-\alpha}w_2)\right] A_0^{-1}(s)\right\| \leq \\ \leq C\left[\omega\left(|t-\tau|\right) + \|\nu_1 - w_1\| + \|\nu_2 - w_2\|\right];\end{gathered}$$

III. *For any* $t, \tau \in [0,T]$; $\|\nu_i\| \leq R, \|w_i\| \leq R, (i=1,2)$ *it holds*

$$\begin{gathered}\left\|f(t, A_0^{-\alpha}\nu_1, A_0^{-\alpha}\nu_2) - f(\tau, A_0^{-\alpha}w_1, A_0^{-\alpha}w_2)\right\| \leq \\ \leq C\left[\omega\left(|t-\tau|\right) + \|\nu_1 - w_1\| + \|\nu_2 - w_2\|\right].\end{gathered}$$

IV. *For any* $t_0 \in [0,T], \nu_1, \nu_2 \in E$ *the estimation*

$$\left\|f(t, A_0^{-\alpha}\nu_1, A_0^{-\alpha}\nu_2)\right\| \leq C\left(1 + \|\nu_1\| + \|\nu_2\|\right)$$

is satisfied.

V. $u_0 \in D(A_0^{\alpha}), u_1 \in D(A_0^{\delta})$ *for some* $\delta > \alpha$. *Then, problem (8)-(9)* has a unique solution on $[0,T]$.

Proof. We prove that in the given case all the conditions of theorem 1 are satisfied for the problem (11)-(12). The conditions (II),(III) and (V) are verified by the theorem.

2. In order to verify condition (I) of theorem 1, it suffices to remark that for any $t \in [0,T]$

$$\left[\mathfrak{U}(t, \mathfrak{U}_0^{-\alpha}v) + \lambda I\right]^{-1} = \begin{pmatrix} (\lambda+1)^{-1}I & 0 \\ 0 & \left[A(t, A_0^{-\alpha}\nu_1, A_0^{-\alpha}\nu_2) + \lambda I\right]^{-1}\end{pmatrix}.$$

Hence, it follows that for any $\nu \in E \times E$ it holds the estimation

$$\left\|\left[\mathfrak{U}(t, \mathfrak{U}_0^{-\alpha}\nu) + \lambda I\right]^{-1}\right\| \leq C\,|\lambda|^{-\beta}, \quad |\arg\lambda| \leq \frac{\pi}{2} + \eta, \ \ |\lambda| \to \infty.$$

Now, let's verify condition (IV). Let $\nu = (\nu_1, \nu_2)$ where $\nu_1, \nu_2 \in E$. Then by (14) and (IV) we have

$$\left\|\mathcal{F}(t, \mathfrak{U}_0^{-\alpha}\nu)\right\| = \|\nu_1 + \nu_2\| + f\left\|(t, A_0^{-\alpha}\nu_1, A_0^{-\alpha}\nu_2)\right\| \leq C(1 + \|\nu\|).$$

So, on the basis of theorem 1 problem (11)-(12) has a unique solution on $[0,T]$. Thereby we proved (8)-(9) has a unique solution on $[0,T]$.

3. Let's consider an initial boundary value problem with non-local boundary conditions:

$$\begin{gathered}\frac{\partial u(t,x)}{\partial t}-a_0\left((t,x,u(t,x),\frac{\partial u(t,x)}{\partial x}\right)\frac{\partial^2 u(t,x)}{\partial x^2}+\\ +a_1\left((t,x,u(t,x),\frac{\partial u(t,x)}{\partial x}\right)u(t,x)=\\ =f(t,x,u(t,x)),\quad (t,x)\in[0,T]\times[0,1],\end{gathered}\tag{16}$$

$$\begin{cases} L_1u=\int\limits_0^1\varphi_1(x)u(t,x)dx=0,\\ L_2u=\int\limits_0^1\varphi_2(x)u(t,x)dx=0,\end{cases}\tag{17}$$

$$u(0,x)=\varphi_0(x);\tag{18}$$

We'll consider this problem in the space $L_2(0,1)$ and look for its classic solution .

Let the following conditions be fulfilled:

$$\begin{gathered}\left\|a_k\left(t,x,u_0(t,x),\frac{\partial u_0}{\partial x}\right)-a_k\left(\tau,x,\nu_0(t,x),\frac{\partial \nu_0}{\partial x}\right)\right\|_{L_2(0,1)}\le\\ \le C\left[|\log|\, t-\tau\,\|^{\alpha}+\|\,u_0-\nu_0\|_{W_2(0,1)}\right],\end{gathered}$$

$$\|f(t,x,u_0(t,x))-f(\tau,x,\nu_0(t,x))\|\le C\left[|\log|\, t-\tau\,\|^{\alpha}+\|\,u_0-\nu_0\|_{L_2(0,1)}\right]$$

where $k=0,1;\alpha<-2$.

It we introduce the operator $A(t,u(t))$:$L_2(0,1)$ determined by the formula

$$D(A(t,u))=\left\{u\in W_2^2(0,1);L_ju=0,\ j=1,2\right\},$$

$$A(t,u)u=a_{0k}\left(t,x,u,\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2}+a_1\left(t,x,u,\frac{\partial u}{\partial x}\right)u,$$

and additionally assume $a_0(t,x,...)>M>0$, $t\in[0,T]$, $x\in[0,1]$, $\frac{\partial a_0(t,0,...)}{\partial x}=$ $=\frac{\partial a_0(t,1,...)}{\partial x}=0$, $a_1(t,x,...)\in C[0,1]$, $\triangle_\varphi=\varphi_1(0)\ \varphi_2(1)-\varphi_2(0)\ \varphi_1(1)\neq$ $\neq 0$, $\varphi_j\in C[0,1]$, $j=1,2$ are linearly independent functions. Then for the operator $A(t,u)$ we have: $D(A(t,u))$ is not a compact set in $L_2(0,1)$ and $\|R(\lambda,A(t,u))\|\le C\,|\lambda|^{-3/4}$; (see [11]).

Then problem (16)-(17)-(18) is reduced to the Cauchy abstract problem

$$\begin{cases}u'(t)+A(t,u)u=f(t,u),\\ u(0)=u_0.\end{cases}$$

in functional space $L_2(0,1)$.

Now, applying theorem 1 to this problem we obtain its solvability in the class of classic solutions.

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