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AN OPTIMAL CONTROL PROBLEM WITH NONLOCAL CONDITIONS FOR WAVE EQUATION

Abstract

Some problems of modern physics and technology can be effectively described in terms of nonlocal problems for differential equations. These nonlocal conditions show that some data on domain are inaccessible to the measurement. Therefore some averaged data of problem are given [see 1,2]. In the present paper we derive the necessary condition of optimality in the optimal control problem for wave equation with nonlocal boundary conditions.

1. Statement of the problem. Let the controlled process be described by the following equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f\left(x,t,u\left(x,t\right),\vartheta\left(x,t\right)\right), \tag{1.1}$$

with the initial conditions

$$u\left(x,0\right) = \varphi\left(x\right), \frac{\partial u\left(x,0\right)}{\partial t} = \psi\left(x\right), \tag{1.2}$$

and nonlocal condition

$$\left.\frac{\partial u}{\partial \nu}\right|_S = \int\limits_{\Omega} k\left(x,y\right) u\left(y,t\right) dy, \tag{1.3}$$

here $u\left(x,t\right)$ characterizes the state of the process, $\vartheta\left(x,t\right)$ is a control function, $Q = \{(x,t) : x \in \Omega,\ 0 < t < T\}$, where Ω is a bounded domain in R^n with smooth boundary $(n \le 3)$, $S\{(x,t) : x \in \partial\Omega,\ 0 < t < T\}$ is lateral surface of cylinder Q, ν is an outer normal to S, $\varphi \in W_2^1(\Omega)$, $\psi \in L_2(\Omega)$, $k(x,y) \in L_2(\Omega \times \Omega)$, $k(x,y) = k(y,x)$, $\iint\limits_{\Omega\,\Omega} \left(k^2(x,y) + \frac{1}{2}\left|\nabla_x k(x,y)\right|^2\right) dxdy = L < \infty$. As a class of admissible controls we take the set of the functions $V = \{\vartheta(x,t) : \vartheta(x,t) \in L_\infty(Q), \vartheta \in [\alpha,\beta]\}$, α,β are given numbers.

It is required to find such a control from V that together with solution of problem (1.1)-(1.3) it delivers minimum to the functional

$$J\left(\vartheta\right) = \int\limits_{Q} f_0\left(x,t,u\left(x,t\right),\vartheta\left(x,t\right)\right) dxdt. \tag{1.4}$$

If some control $\vartheta_0(x,t)$ from V delivers minimum value to functional (1.4) then this control is said to be optimal control. Solution of problem (1.1)-(1.3), which corresponds to the optimal control. $\vartheta_0(x,t)$ we'll denote by $u_0(x,t)$. Then the pair $(\vartheta_0(x,t), u_0(x,t))$ is said to be an optimal pair.

At is assumed that the functions $f(x,t,u,\vartheta)$ and $f_0(x,t,u,\vartheta)$ are continuous on $\overline{Q}\times R\times[\alpha,\beta]$ and have continuous derivatives $\dfrac{\partial f}{\partial u}$, $\dfrac{\partial f_0}{\partial u}$, moreover $\dfrac{\partial f}{\partial u}$ is bounded and by u it satisfies the Hölder condition with exponent λ $\frac{2}{3}\leq\lambda\leq1$, $\dfrac{\partial f_0}{\partial u}$, by u satisfies Lipcshitz condition. At these conditions we can prove that for each admissible control $\vartheta(x,t)\in V$ problem (1.1)-(1.3) has a unique generalized solution $u(x,t)$ from $W_2^1(Q)$.

Under the generalized solution of problem (1.1)-(1.3) we understand such a function $u(x,t)$ from $W_2^1(Q)$ that for any function $\phi(x,t)\in W_2^1(Q)$, $\phi(x,T)=0$ the following integral identity

$$\int_0^T\int_\Omega\left(-\frac{\partial u}{\partial t}\frac{\partial\phi}{\partial t}+\nabla u\nabla\phi\right)dxdt-\int_0^T\int_{\partial\Omega}\phi(x,t)\int_\Omega k(x,y)\,u(y,t)\,dydsdt=$$

$$=\int_Q f(x,t,u(x,t),\vartheta(x,t))\,\phi(x,t)\,dxdt+\int_\Omega\psi(x)\,\phi(x,0)\,dx, \tag{1.5}$$

is fulfilled, moreover the fulfilment of condition $u(x,0)=\varphi(x)$ is understandood in the sense of $\lim\limits_{t\to+0}\int\limits_\Omega(u(x,t)-\varphi(x))^2\,dx=0$.

For admissible control $\vartheta_0(x,t)$ we introduce the following conjugate problem

$$\frac{\partial^2\psi}{\partial t^2}-\triangle\psi=\int_{\partial\Omega}k(\xi,x)\,\psi(\xi,t)\,ds+\frac{\partial H(x,t,u_0(x,t),\vartheta_0(x,t),\psi(x,t))}{\partial u} \tag{1.6}$$

$$\psi(x,T)=0,\ \frac{\partial\psi(x,T)}{\partial t}=0, \tag{1.7}$$

$$\left.\frac{\partial\psi}{\partial\nu}\right|_S=0, \tag{1.8}$$

here $H(x,t,u,\vartheta,\psi)=\psi f(x,t,u,\vartheta)-f_0(x,t,u,\vartheta)$ is a Pontryagin function, and $u_0(x,t)$ is a solution of boundary-value problem (1.1)-(1.3), corresponding to $\vartheta_0(x,t)$. Equation (1.6) is a linear equation and we can show that problem (1.6)-(1.8) in $W_2^1(Q)$ has a unique solution [see 1].

2. Some constructions and lemma. Let's introduce some constructions for deriving the necessary optimality conditions.

Let $(\sigma,\tau)\in Q$ be Lebesgue point for all functions participating in the problem and $\varepsilon>0$ be a sufficiently small number.

Let's assume

$$\vartheta_\varepsilon(x,t)=\begin{cases}\vartheta,\ (x,t)\in\Pi_\varepsilon,\\ \vartheta_0(x,t),(x,t)\in Q\backslash\Pi_\varepsilon,\end{cases}$$

where $\vartheta\in[\alpha,\beta]$ is an arbitrary value and

$$\Pi_\varepsilon=\left\{(x,t):\sigma_i<x_\iota<\sigma_i+\varepsilon,\ i=\overline{1,n},\ \tau<t<\tau+\varepsilon\right\}\subset Q.$$

We'll denote by $u_\varepsilon(x,t)$ the generalized solution of problem (1.1)-(1.3) corresponding to $\vartheta_\varepsilon(x,t)$.

Then $\delta u_\varepsilon = u_\varepsilon - u_0$ is generalized solution of the following problem:

$$\frac{\partial^2 \delta u_\varepsilon}{\partial t^2} - \Delta \delta u_\varepsilon = f(x,t,u_0+\delta u_\varepsilon,\vartheta_\varepsilon) - f(x,t,u_0,\vartheta_0) \tag{2.1}$$

$$\delta u_\varepsilon(x,0)=0,\ \frac{\partial \delta u_\varepsilon(x,0)}{\partial t}=0, \tag{2.2}$$

$$\left.\frac{\partial \delta u_\varepsilon}{\partial \nu}\right|_S = \int\limits_\Omega k(x,y)\,\delta u_\varepsilon(y,t)\,dy. \tag{2.3}$$

Lemma. *At above imposed conditions on data of problem for solution of problem (2.1)-(2.3) it holds the estimation*

$$\|\delta u_\varepsilon\|^2_{L_2(\Omega)} + \left\|\frac{\partial \delta u_\varepsilon}{\partial t}\right\|^2_{L_2(\Omega)} + \|\nabla \delta u_\varepsilon\|^2_{L_2(\Omega)} \le c\varepsilon^{n+2}\quad \forall t\in[0,T]. \tag{2.4}$$

Proof. Let $(x,t)\in\Omega\times(0,\tau)$. In this case $\vartheta_\varepsilon(x,t)=\vartheta_0(x,t)$ and problem (2.1)-(2.3) takes the form

$$\frac{\partial^2 \delta u_\varepsilon}{\partial t^2} - \triangle \delta u_\varepsilon = f(x,t,u_0+\delta u_\varepsilon,\vartheta_0) - f(x,t,u_0,\vartheta_0) \tag{2.5}$$

$$\delta u_\varepsilon(x,0)=0,\ \frac{\partial \delta u_\varepsilon(x,0)}{\partial t}=0, \tag{2.6}$$

$$\left.\frac{\partial \delta u_\varepsilon}{\partial \nu}\right|_S = \int\limits_\Omega k(x,y)\,\delta u_\varepsilon(y,t)\,dy. \tag{2.7}$$

It is clear that $\delta u_\varepsilon(x,t)\equiv 0$ is a solution of this problem. Then by virtue of uniqueness of generalized solution of problem (2.5)-(2.7) at $(x,t)\in\Omega\times(0,\tau)$ estimation (2.4) is automatically fulfilled.

Now let $(x,t)\in\Omega\times(\tau,\tau+\varepsilon)$. Here $\vartheta_\varepsilon(x,t)=\vartheta_0(x,t),(x,t)\bar{\in}\Pi_\varepsilon,\ \vartheta_\varepsilon(x,t)=\vartheta,\ (x,t)\in\Pi_\varepsilon$.

Let's apply Galerkin method for getting estimation (2.4) in $\Omega\times(\tau,\tau+\varepsilon)$. Let $\{\varphi_k(x)\}$ be a fundamental system in $W_2^1(\Omega)$ and the orthonormal property be fulfilled

$$(\varphi_k,\varphi_l)=\int\limits_\Omega \varphi_k\varphi_l dx=\delta_k^l.$$

We seek the approximate solution of problem (2.1)-(2.3) in the form

$$\delta u_\varepsilon^N(x,t)=\sum_{k=1}^N c_k^N(t)\,\varphi_k(x) \tag{2.8}$$

from the relations

$$\int\limits_\Omega \frac{\partial^2 \delta u_\varepsilon^N}{\partial t^2}\varphi_l dx + \int\limits_\Omega \sum_{k=1}^N \frac{\partial^2 \delta u_\varepsilon^N}{\partial x_i}\frac{\partial \varphi_l}{\partial x_i}dx - \int\limits_{\partial\Omega}\varphi_l(x)\int\limits_\Omega k(x,y)\,\delta u_\varepsilon^N(y,t)\,dyds =$$

$$=\int\limits_{\Omega}\left(f\left(x,t,u_0+\delta u_{\varepsilon}^{N},\vartheta_{\varepsilon}\right)-f\left(x,t,u_0,\vartheta_0\right)\right)\varphi_l\left(x\right)dx,\quad l=1,2,...,N, \tag{2.9}$$

$$c_k^N\left(\tau\right)=0, \tag{2.10}$$

$$\left.\frac{d}{dt}c_k^N\left(t\right)\right|_{t=\tau}=0. \tag{2.11}$$

Substituting (2.8) into (2.9) we'll obtain:

$$c_l^N\left(t\right)+\sum_{k=1}^{N}c_k^N\left(t\right)\int\limits_{\Omega}\sum_{i=1}^{n}\varphi_{k_{x_i}}\left(x\right)\varphi_{l_{x_i}}\left(x\right)dx-$$

$$-\sum_{k=1}^{N}c_k^N\left(t\right)\int\limits_{\partial\Omega}\varphi_l\left(x\right)\int\limits_{\Omega}k\left(x,y\right)\varphi_k\left(y\right)dyds=$$

$$=\int\limits_{\Omega}\left(f\left(x,t,u_0+\delta u_{\varepsilon}^{N},\vartheta_{\varepsilon}\right)-f\left(x,t,u_0,\vartheta_0\right)\right)\varphi_l\left(x\right)dx,\quad l=1,2,...,N, \tag{2.12}$$

$$c_k^N\left(\tau\right)=0, \tag{2.13}$$

$$\left.\frac{d}{dt}c_k^N\left(t\right)\right|_{t=\tau}=0. \tag{2.14}$$

Applying the inequality

$$\int\limits_{\partial\Omega}\left|w\right|ds\leq\alpha\int\limits_{\Omega}\left(\left|\nabla w\right|+\left|w\right|\right)dx \tag{2.15}$$

valid for $\forall w\in W_1^1\left(\Omega\right)$ and for domain Ω with smooth boundary [see 3], and then Cauchy-Bunyakovskiy inequality, we obtain

$$\left|\int\limits_{\partial\Omega}\varphi_l\left(x\right)\int\limits_{\Omega}k\left(x,y\right)\varphi_k\left(y\right)dyds\right|\leq$$

$$\leq c+\frac{c}{2}\int\limits_{\Omega}\left|\nabla\varphi\right|^2dx+\frac{c}{2}\int\limits_{\Omega}\int\limits_{\Omega}\left(2k^2+\left|\nabla_x k\right|^2\right)dxdy.$$

Here and in future by c we'll denote the different constants that don't depend on admissible controls and estimated values.

It is clear that system (2.12) is a system of ordinary differential equations of the second order by t for unknowns $c_k^N\left(t\right),\ k=1,2,...,N$, solvable relative to $\frac{d^2c_k^N}{dt^2}$. Thus $\forall N$ system (2.12) is uniquely solvable at initial conditions (2.13), (2.14) (see [4]), moreover $\frac{d^2c_k^N}{dt^2}\in L^2\left(0,T\right)$.

Let's show that for $\delta u_\varepsilon^N(x,t)$ estimation (2.4) is true. Really, multiplying each of equalities of (2.9) by its $\frac{dc_l^N}{dt}$, we'll come to equality

$$\int\limits_\Omega \frac{\partial^2 \delta u_\varepsilon^N(x,t)}{\partial t^2}\frac{\partial \delta u_\varepsilon^N(x,t)}{\partial t}dx + \int\limits_\Omega \sum_{i=1}^n \frac{\partial \delta u_\varepsilon^N(x,t)}{\partial x_i}\frac{\partial^2 u_\varepsilon^N(x,t)}{\partial x_i \partial t}dx -$$

$$-\int\limits_\Omega \frac{\partial \delta u_\varepsilon^N(x,t)}{\partial t}\int\limits_{\partial\Omega} k(x,y)\,\delta u_\varepsilon^N(y,t)\,dyds =$$

$$= \int\limits_\Omega \left(f\left(x,t,u_0+\delta u_\varepsilon^N,\vartheta_\varepsilon\right) - f\left(x,t,u_0,\vartheta_0\right)\right)\frac{\partial \delta u_\varepsilon^N(x,t)}{\partial t}dx.$$

Integrating it with respect to t from τ to t, $t\in(\tau,\tau+\varepsilon)$ and multiplying the both parts by 2, we get

$$\int\limits_\Omega \left(\left(\frac{\partial^2 \delta u_\varepsilon^N}{\partial t}\right)^2 + \left|\nabla \delta u_\varepsilon^N\right|^2\right)dx -$$

$$-2\int\limits_\tau^t\int\limits_{\partial\Omega} \frac{\partial \delta u_\varepsilon^N(x,t)}{\partial t}\int\limits_\Omega k(x,y)\,\delta u_\varepsilon^N(y,t)\,dydsdt =$$

$$= 2\int\limits_\tau^t\int\limits_\Omega \left(f\left(x,t,u_0+\delta u_\varepsilon^N,\vartheta_\varepsilon\right) - f\left(x,t,u_0,\vartheta_0\right)\right)\frac{\partial \delta u_\varepsilon^N(x,t)}{\partial t}dxdt. \qquad (2.16)$$

In the right part subtracting and adding the expression

$$2\int\limits_\tau^t\int\limits_{\partial\Omega} f\left(x,t,u_0,\vartheta_\varepsilon\right)\frac{\partial \delta u_\varepsilon^N(x,t)}{\partial t}dxdt$$

we get

$$\int\limits_\Omega \left(\left(\frac{\partial^2 \delta u_\varepsilon^N}{\partial t}\right)^2 + \left|\nabla \delta u_\varepsilon^N\right|^2\right)dx -$$

$$-2\int\limits_\tau^t\int\limits_{\partial\Omega} \frac{\partial \delta u_\varepsilon^N(x,t)}{\partial t}\int\limits_\Omega k(x,y)\,\delta u_\varepsilon^N(y,t)\,dydsdt = \qquad (2.17)$$

$$= 2\int\limits_\tau^t\int\limits_\Omega \left[\left(f\left(x,t,u_0+\delta u_\varepsilon^N,\vartheta_\varepsilon\right) - f\left(x,t,u_0,\vartheta_\varepsilon\right)\right)\frac{\partial \delta u_\varepsilon^N}{\partial t}\right.$$

$$\left. + \left(f\left(x,t,u_0,\vartheta_\varepsilon\right) - f\left(x,t,u_0,\vartheta_0\right)\right)\frac{\partial \delta u_\varepsilon^N}{\partial t}\right]dxdt.$$

We transform the integral by lateral surface in the following way

$$\int\limits_\tau^t\int\limits_{\partial\Omega} \frac{\partial \delta u_\varepsilon^N(x,t)}{\partial t}\int\limits_\Omega k(x,y)\,\delta u_\varepsilon^N(y,t)\,dydsdt = i_1+i_2+i_3,$$

where

$$i_1 = -\int\limits_{\partial\Omega}\int\limits_{\tau}^{t} \delta u_\varepsilon^N(x,t) \int\limits_{\Omega} k(x,y) \frac{\partial \delta u_\varepsilon^N(y,t)}{\partial t} dydtds,$$

$$i_2 = \int\limits_{\partial\Omega} \delta u_\varepsilon^N(x,t) \int\limits_{\Omega} k(x,y)\, \delta u_\varepsilon^N(y,t)\, dyds,$$

$$i_3 = -\int\limits_{\partial\Omega} \delta u_\varepsilon^N(x,\tau) \int\limits_{\Omega} k(x,y)\, \delta u_\varepsilon^N(y,\tau)\, dyds.$$

By virtue of $\delta u_\varepsilon^N(x,\tau) = 0$, hence it follows that $i_3 = 0$. Using inequality (2.15) and then Cauchy-Bunyakovskiy inequality we obtain

$$|i_1| = \left| \int\limits_{\tau}^{t}\int\limits_{\partial\Omega} \delta u_\varepsilon^N(x,t) \int\limits_{\Omega} k(x,y) \frac{\partial \delta u_\varepsilon^N(y,t)}{\partial t} dydsdt \right| \le$$

$$\le c\int\limits_{\tau}^{t}\int\limits_{\partial\Omega} \left(\left| \delta u_\varepsilon^N(x,t) \int\limits_{\Omega} k(x,y) \frac{\partial \delta u_\varepsilon^N(y,t)}{\partial t} dy \right| + \right.$$

$$+ \left| \nabla \delta u_\varepsilon^N(x,t) \int\limits_{\Omega} k(x,y) \frac{\partial \delta u_\varepsilon^N(y,t)}{\partial t} dy + \right.$$

$$\left. \left. + \delta u_\varepsilon^N(x,t) \int\limits_{\Omega} \nabla_x k(x,y) \frac{\partial \delta u_\varepsilon^N(y,t)}{\partial t} dy \right| \right) dxdt \le$$

$$\le \frac{c}{2}\int\limits_{\tau}^{t}\int\limits_{\Omega} \left(\left(\delta u_\varepsilon^N(x,t)\right)^2 + \left(\int\limits_{\Omega} k(x,y) \frac{\partial \delta u_\varepsilon^N(y,t)}{\partial t} dy \right)^2 + \right. \tag{2.18}$$

$$+ \left|\nabla \delta u_\varepsilon^N(x,t)\right|^2 + \left| \int\limits_{\Omega} k(x,y) \frac{\partial \delta u_\varepsilon^N(y,t)}{\partial t} dy \right|^2 +$$

$$\left. + \left(\delta u_\varepsilon^N(x,t)\right)^2 + \left| \int\limits_{\Omega} \nabla_x k(x,y) \frac{\partial \delta u_\varepsilon^N(y,t)}{\partial t} dy \right|^2 \right) dxdt \le$$

$$\le c\int\limits_{\tau}^{t}\int\limits_{\Omega} \left(\left(\delta u_\varepsilon^N(x,t)\right)^2 + \left|\nabla \delta u_\varepsilon^N(x,t)\right|^2 + \right.$$

$$\left. + \int\limits_{\Omega} \left(k^2(x,y) + \frac{1}{2}\left|\nabla_x k(x,y)\right|^2 \right) dy \int\limits_{\Omega} \left(\frac{\partial \delta u_\varepsilon^N(y,t)}{\partial t} \right)^2 dy \right) dxdt =$$

$$= c\int\limits_{\tau}^{t}\int\limits_{\Omega} \left(\left(\delta u_\varepsilon^N(x,t)\right)^2 + \left|\nabla \delta u_\varepsilon^N(x,t)\right|^2 \right) dxdt + cL\int\limits_{\tau}^{t}\int\limits_{\Omega} \left(\frac{\partial \delta u_\varepsilon^N(y,t)}{\partial t} \right)^2 dydt.$$

Further we have the chain of inequalities

$$|i_2| = \left| \int\limits_{\partial\Omega} \delta u_\varepsilon^N(x,t) \int\limits_{\Omega} k(x,y)\, \delta u_\varepsilon^N(y,t)\, dyds \right| \leq$$

$$\leq \left(\int\limits_{\partial\Omega} \left(\delta u_\varepsilon^N(x,t)\right)^2 ds \right)^{\frac{1}{2}} \left(\int\limits_{\partial\Omega}\int\limits_{\Omega} k^2(x,y)\, dy \int\limits_{\Omega} \left(\delta u_\varepsilon^N(y,t)\right)^2 dyds \right)^{\frac{1}{2}} \leq$$

$$\leq c^{\frac{1}{2}} \left(\int\limits_{\Omega} \left(\delta u_\varepsilon^N(x,t) \left|\nabla \delta u_\varepsilon^N(x,t)\right| + \left(\delta u_\varepsilon^N(x,t)\right)^2 \right) dx \right)^{\frac{1}{2}} \times$$

$$\times c^{\frac{1}{2}} \left(\int\limits_{\Omega} \left(\delta u_\varepsilon^N(x,t)\right)^2 dx \right)^{\frac{1}{2}} \leq$$

$$\leq c \left(\int\limits_{\Omega} \left(\left|\nabla \delta u_\varepsilon^N(x,t)\right|^2 + \left(\delta u_\varepsilon^N(x,t)\right)^2 \right) dx \right)^{\frac{1}{2}} \left(\int\limits_{\Omega} \left(\delta u_\varepsilon^N(x,t)\right)^2 dx \right)^{\frac{1}{2}}. \quad (2.19)$$

Introduce the denotation

$$z^N(t) = \int\limits_{\Omega} \left(\left(\delta u_\varepsilon^N\right)^2 + \left(\frac{\partial \delta u_\varepsilon^N}{\partial t}\right)^2 + \left|\nabla \delta u_\varepsilon^N\right|^2 \right) dx.$$

Use the inequality [see 3]

$$\int\limits_{\Omega} \left(\delta u_\varepsilon^N(x,t)\right)^2 dx \leq 2 \int\limits_{\Omega} \left(\delta u_\varepsilon^N(x,\tau)\right)^2 dx + 2t \int\limits_{\tau}^{t} y^N(t)\, dt, \quad (2.20)$$

where

$$y^N(t) = \int\limits_{\Omega} \left(\left(\frac{\partial \delta u_\varepsilon^N(x,t)}{\partial t}\right)^2 + \left|\nabla \delta u_\varepsilon^N(x,t)\right|^2 \right) dx. \quad (2.21)$$

Since $\delta u_\varepsilon^N(x,\tau) = 0$ then from (2.19) it follows

$$|i_2| \leq c \left(z^N(t)\right)^{\frac{1}{2}} \left(2t \int\limits_{\tau}^{t} z^N(t)\, dt \right)^{\frac{1}{2}}. \quad (2.22)$$

Under the conditions on data of problem from (2.17), (2.18) and (2.22) we have

$$z^N(t) \leq c \int\limits_{\tau}^{t} z^N(t)\, dt + 2t \int\limits_{\tau}^{t} z^N(t)\, dt + c \left(z^N(t)\right)^{\frac{1}{2}} \left(2t \int\limits_{\tau}^{t} z^N(t)\, dt \right)^{\frac{1}{2}} +$$

$$+2\int_{\tau}^{t}\left(z^N(t)\right)^{\frac{1}{2}}\left(\int_{\Omega}\left|f(x,t,u_0,\vartheta_\varepsilon)-f(x,t,u_0,\vartheta_0)\right|^2dx\right)^{\frac{1}{2}}dt. \qquad (2.23)$$

Denote by $\max\limits_{\tau\le\xi\le\tau+\xi} z^N(\xi)=\bar{z}^N(t)$ and

$$g(t)=c\left(\int_{\Omega}\left|f(x,t,u_0,\vartheta_\varepsilon)-f(x,t,u_0,\vartheta_0)\right|^2dx\right)^{\frac{1}{2}},$$

then from (2.23) it follows that

$$\bar{z}^N(t)\le(c+2t)(t-\tau)\bar{z}^N(t)+c\left(2t(t-\tau)\right)^{\frac{1}{2}}\bar{z}^N(t)+$$

$$+2\int_{\tau}^{t}\|g(t)\|_{L_2(\Omega)}dt\left(\bar{z}^N(t)\right)^{\frac{1}{2}}.$$

Determining t_l from the condition $(c+2t_1)(t_1-\tau)=\frac{1}{2}$ we obtain

$$\bar{z}^N(t)\le 2c\sqrt{2t(t-\tau)}\bar{z}^N(t)+4\int_{\tau}^{t}\|g(t)\|_{L_2(\Omega)}dt\left(\bar{z}^N(t)\right)^{\frac{1}{2}}.$$

Now, selecting t_2 from the condition $2c\sqrt{2t_2(t_2-\tau)}=\frac{1}{2}$ at $t\le\min(t_1,t_2,\tau+\varepsilon)$ we obtain

$$\left(\bar{z}^N(t)\right)^{\frac{1}{2}}\le 8\int_{\tau}^{t}\|g(t)\|_{L_2(\Omega)}dt.$$

Allowing for the form of the function $g(t)$ and definition of impulse variation $\vartheta_\varepsilon(x,t)$, hence we have the estimation

$$\left(\bar{z}^N(t)\right)^{\frac{1}{2}}\le c\varepsilon^{\frac{n}{2}+1}$$

or

$$z^N(t)\le c\varepsilon^{n+2},\ \tau\le t\le\min(t_1,t_2,\tau+\varepsilon).$$

Continuing this process, for a finite number of steps we obtain the validity of the estimation

$$z^N(t)\le c\varepsilon^{n+2},\ \tau\le t\le\tau+\varepsilon. \qquad (2.24)$$

Now, we'll obtain the estimation for $\delta u_\varepsilon^N(x,t)$ on the segment $[\tau+\varepsilon,T]$. On this segment $\vartheta_\varepsilon(x,t)=\vartheta_0(x,t)$. Then again by means of Galerkin method from (2.1)-(2.3) we obtain

$$\int_{\Omega}\frac{\partial^2\delta u_\varepsilon^N(x,t)}{\partial t^2}\frac{\partial\delta u_\varepsilon^N(x,t)}{\partial t}dx+\int_{\Omega}\sum_{i=1}^{n}\frac{\partial\delta u_\varepsilon^N(x,t)}{\partial x_i}\frac{\partial^2u_\varepsilon^N(x,t)}{\partial x_i\partial t}dt-$$

$$-\int_{\partial\Omega}\frac{\partial\delta u_\varepsilon^N(x,t)}{\partial t}\int_{\Omega}k(x,y)\,\delta u_\varepsilon^N(y,t)\,dyds=$$

$$=\int_{\Omega}\left(f\left(x,t,u_0+\delta u_\varepsilon^N,\vartheta_0\right)-f\left(x,t,u_0,\vartheta_0\right)\right)\frac{\partial\delta u_\varepsilon^N(x,t)}{\partial t}dx.$$

Integrating it with respect to t from t to $\tau+\varepsilon$ and multiplying the both sides by 2, we obtain the equality

$$\int_{\Omega}\left(\left(\frac{\partial\delta u_\varepsilon^N(x,t)}{\partial t}\right)^2+\left|\nabla\delta u_\varepsilon^N(x,t)\right|^2\right)dx=$$

$$=\int_{\Omega}\left(\left(\frac{\partial\delta u_\varepsilon^N(x,\tau+\varepsilon)}{\partial t}\right)+\left|\nabla\delta u_\varepsilon^N(x,\tau+\varepsilon)\right|^2\right)dx+$$

$$+2\int_{\tau+\varepsilon}^{t}\int_{\partial\Omega}\frac{\partial\delta u_\varepsilon^N(x,t)}{\partial t}\int_{\Omega}k(x,y)\,\delta u_\varepsilon^N(y,t)\,dydsdt+$$

$$+2\int_{\tau+\varepsilon}^{t}\int_{\partial\Omega}\left(f\left(x,t,u_0+\delta u_\varepsilon^N,\vartheta_0\right)-f\left(x,t,u_0,\vartheta_0\right)\right)\frac{\partial\delta u_\varepsilon^N(x,t)}{\partial t}dxdt. \qquad (2.25)$$

Let's transform the integral by the lateral surface in the following form:

$$2\int_{\tau+\varepsilon}^{t}\int_{\partial\Omega}\frac{\partial\delta u_\varepsilon^N(x,t)}{\partial t}\int_{\Omega}k(x,y)\,\delta u_\varepsilon^N(y,t)\,dydsdt=i_1+i_2+i_3,$$

here

$$i_1=-\int_{\tau+\varepsilon}^{t}\int_{\partial\Omega}\delta u_\varepsilon^N(x,t)\int_{\Omega}k(x,y)\,\frac{\partial\delta u_\varepsilon^N(y,t)}{\partial t}dydtds,$$

$$i_2=\int_{\partial\Omega}\delta u_\varepsilon^N(x,t)\int_{\Omega}k(x,y)\,\delta u_\varepsilon^N(y,t)\,dyds,$$

$$i_3=-\int_{\partial\Omega}\partial\delta u_\varepsilon^N(x,\tau+\varepsilon)\int_{\Omega}k(x,y)\,\delta u_\varepsilon^N(y,\tau+\varepsilon)\,dyds.$$

Using inequality (2.15) and then Cauchy-Bunyakovskiy inequality we obtain

$$|i_1|\le c\int_{\tau+\varepsilon}^{t}\int_{\partial\Omega}\left(\left(\delta u_\varepsilon^N(x,t)\right)^2+\left|\nabla\delta u_\varepsilon^N(x,t)\right|^2\right)dxdt+$$

$$+cL\int_{\tau+\varepsilon}^{t}\int_{\partial\Omega}\left(\frac{\partial\delta u_\varepsilon^N(y,t)}{\partial t}\right)^2dydt, \qquad (2.26)$$

$$|i_2| = \left| \int\limits_{\partial\Omega} \left(\delta u_\varepsilon^N(x,t) \int\limits_\Omega k(x,y)\delta u_\varepsilon^N(y,t)dy \right) ds \right| \leq$$

$$\leq \alpha \int\limits_\Omega \left(\left| \nabla \delta u_\varepsilon^N(x,t) \int\limits_\Omega k(x,y)\delta u_\varepsilon^N(y,t)dy \right| + \right.$$

$$+ \left| \delta u_\varepsilon^N(x,t) \int\limits_\Omega \nabla_x k(x,y)\delta u_\varepsilon^N(y,t)dy \right| +$$

$$\left. + \, \delta u_\varepsilon^N(x,t) \int\limits_\Omega k(x,y)\delta u_\varepsilon^N(y,t)dy \right) dx \leq$$

$$\leq \alpha \int\limits_\Omega \left(\frac{1}{2} \left| \nabla \delta u_\varepsilon^N(x,t) \right|^2 + \frac{1}{2} \left(\int\limits_\Omega k(x,y)\delta u_\varepsilon^N(y,t)dy \right)^2 + \right.$$

$$+ \frac{1}{2} \left| \delta u_\varepsilon^N(x,t) \right|^2 + \frac{1}{2} \left(\int\limits_\Omega \nabla_x k(x,y)\delta u_\varepsilon^N(y,t)dy \right)^2 +$$

$$\left. + \frac{1}{2} \left| \delta u_\varepsilon^N(x,t) \right|^2 + \frac{1}{2} \left(\int\limits_\Omega k(x,y)\delta u_\varepsilon^N(y,t)dy \right)^2 \right) dx \leq$$

$$\leq \alpha \int\limits_\Omega \left(\left(\delta u_\varepsilon^N(x,t) \right)^2 + \left| \nabla \delta u_\varepsilon^N(x,t) \right|^2 + \right.$$

$$+ L \int\limits_\Omega \left(\delta u_\varepsilon^N(y,t) \right)^2 dy)dx = \alpha \int\limits_\Omega \left(\left(\delta u_\varepsilon^N(x,t) \right)^2 + \right.$$

$$\left. + \left| \nabla \delta u_\varepsilon^N(x,t) \right|^2 \right) dx + \alpha L mes\Omega \int\limits_\Omega \left(\delta u_\varepsilon^N(x,t) \right)^2 dx \leq \alpha \left(1 + L mes\Omega \right) z^N(t), \quad (2.27)$$

$$|i_3| \leq c \left(z^N(\tau+\varepsilon) \right)^{\frac{1}{2}} \left(z^N(\tau+\varepsilon) \right)^{\frac{1}{2}} = cz^N(\tau+\varepsilon). \quad (2.28)$$

Let's use the inequality

$$\int\limits_\Omega \left(\delta u_\varepsilon^N(x,t) \right)^2 dx \leq 2 \int\limits_\Omega \left(\delta u_\varepsilon^N(x,\tau+\varepsilon) \right)^2 dx + 2t \int\limits_{\tau+\varepsilon}^t z^N(t)\, dt. \quad (2.29)$$

Now adding (2.29) and (2.25), allowing for the estimations $|i_1|, |i_2|, |i_3|$ we obtain:

$$z^N(t) \leq 2z^N(\tau+\varepsilon) + 2t \int\limits_{\tau+\varepsilon}^t z^N(t)\, dt +$$

$$+c\int_{\tau+\varepsilon}^{t} z^N(t)\,dt + Az^N(t) + cz^N(\tau+\varepsilon),$$

where $A=\alpha(1+Lmes\Omega)$.

Let $A<1$. Then from previous inequality and from inequality (2.24) at $t=\tau+\varepsilon$ it follows that

$$z^N(t)\le c\varepsilon^{n+2}+c\int_{\tau+\varepsilon}^{t} z^N(t)\,dt.$$

Applying here the Gronwall lemma we have the estimation

$$z^N(t)\le c\varepsilon^{n+2}\cdot\varepsilon^{c(T-\tau-\varepsilon)}\le c\varepsilon^{n+2},\quad \tau+\varepsilon\le t\le T. \tag{2.30}$$

By virtue of $\delta u_\varepsilon(x,t)\equiv 0,\ (x,t)\in\Omega\times(0,\tau)$ and (2.24), (2.30) we obtain that

$$z^N(t)\le c\varepsilon^{n+2}, 0\le t\le T$$

or

$$\int_\Omega\left(\left(\delta u_\varepsilon^N\right)^2+\left(\frac{\partial\delta u_\varepsilon^N}{\partial t}\right)^2+\left|\nabla\delta u_\varepsilon^N\right|^2\right)dx\le c\varepsilon^{n+2},\ 0\le t\le T. \tag{2.31}$$

Then we can assume that as $N\to\infty$ $\delta u_\varepsilon(x,t)$ is weak limit of sequence $\left\{\delta u_\varepsilon^N\right\}$ in $W_2^1(Q)$.

Since the norm is weakly lower semi continuous in Hilbert space, hence it follows that for $\delta u_\varepsilon(x,t)$ estimation (2.4) is true. The lemma is proved.

3. Increment of functional and estimation of remainder term. Using expansion

$$f_0(x,t,u_0+\delta u_\varepsilon,\vartheta_0+\delta\vartheta_\varepsilon)-f(x,t,u_0,\vartheta_0+\delta\vartheta_\varepsilon)=$$
$$=\frac{\partial f(x,t,u_0,\vartheta_0+\delta\vartheta_\varepsilon)}{\partial u}\delta u_\varepsilon+\omega(u_0,\delta u_\varepsilon),$$
$$f_0(x,t,u_0+\delta u_\varepsilon,\vartheta_0+\delta\vartheta_\varepsilon)-f_0(x,t,u_0,\vartheta_0+\delta\vartheta_\varepsilon)=$$
$$=\frac{\partial f_0(x,t,u_0,\vartheta_0+\delta\vartheta_\varepsilon)}{\partial u}\delta u_\varepsilon+\omega_0(u_0,\delta u_\varepsilon)$$

and considering that functions $\delta u_\varepsilon(x,t)$ and $\psi(x,t)$ are generalized solutions of problem (2.1)-(2.3) and (1.6)-(1.8) for increment of functional we obtain the following expression:

$$\delta J(\vartheta)=-\int_Q\left(H(x,t,u_0,\vartheta_\varepsilon,\psi)-H(x,t,u_0,\vartheta_0,\psi)\right)dxdt+\eta(\varepsilon) \tag{3.1}$$

where

$$\eta(\varepsilon)=\int_Q\left[\psi(x,t)\,\omega(u_0,\delta u_\varepsilon)+\omega_0(u_0,\delta u_\varepsilon)+\frac{\partial\delta_{\vartheta_\varepsilon}H(x,t,u_0,\vartheta_0,\psi)}{\partial u}\delta u_\varepsilon\right]dxdt,$$

$$\delta_{\vartheta_\varepsilon} H(x,t,u_0,\vartheta_0,\psi) = H(x,t,u_0,\vartheta_\varepsilon,\psi) - H(x,t,u_0,\vartheta_0,\psi).$$

By virtue of conditions on the function $f(x,t,u,\vartheta)$ the theorem on mean value [see 5] and embedding theorem $W_2^1(\Omega) \subset L_6(\Omega)$ (at $n \leq 3$) we have

$$\left| \int_Q \psi(x,t)\,\omega(u_0,\delta u_\varepsilon)\,dxdt \right| \leq \int_Q |\psi(x,t)|\,|f(x,t,u_0+\delta u_\varepsilon,\vartheta_\varepsilon) -$$

$$-f(x,t,u_0,\vartheta_\varepsilon) - \frac{\partial f(x,t,u_0,\vartheta_\varepsilon)}{\partial u}\delta u_\varepsilon \Big| \,dxdt \leq$$

$$\leq \int_Q |\psi(x,t)| \sup_{0\leq\theta\leq 1} \left| \frac{\partial f(x,t,u_0+\theta\delta u_\varepsilon,\vartheta_\varepsilon)}{\partial u} - \frac{\partial f(x,t,u_0,\vartheta_\varepsilon)}{\partial u} \right| \times$$

$$\times |\delta u_\varepsilon|\,dxdt \leq c\int_0^T \left(\int_\Omega |\psi(x,t)|^p\,dx \right)^{1/p} \times$$

$$\times \left(\int_\Omega |\delta u_\varepsilon|^{(1+\lambda)q}\,dx \right)^{1/q} dt = 0\left(\varepsilon^{n+1}\right),$$

here we take into account that at $p \geq 6$, $\frac{6}{5} \geq q \geq 1$, $\frac{2}{3} \leq \lambda \leq 1$ we can assume $(1+\lambda)\,q = 2$.

Analogously we have

$$\left| \int_Q \omega_0(u_0,\delta u_\varepsilon)\,dxdt \right| \leq$$

$$\leq c\int_Q \sup_{0\leq\theta\leq 1} \left| \frac{\partial f_0(x,t,u_0+\theta\delta u_\varepsilon,\vartheta_\varepsilon)}{\partial u} - \frac{\partial f_0(x,t,u_0,\vartheta_\varepsilon)}{\partial u} \right| \times$$

$$\times |\delta u_\varepsilon|\,dxdt \leq c\int_Q |\delta u_\varepsilon|^2\,dxdt = 0\left(\varepsilon^{n+1}\right).$$

Thus

$$\int_Q \left(\psi(x,t)\,\omega(u_0,\delta u_\varepsilon) + \omega_0(u_0,\delta u_\varepsilon)\right)dxdt = 0\left(\varepsilon^{n+1}\right). \tag{3.2}$$

By virtue of estimation (2.4) and determination of variation of control we have

$$\left| \int_Q \frac{\partial \delta_{\vartheta_\varepsilon} H(x,t,u_0,\vartheta_0,\psi)}{\partial u}\delta u_\varepsilon dxdt \right| \leq$$

$$\leq \left(\int_{\Pi_\varepsilon} \left| \frac{\partial\left(H(x,t,u_0,\vartheta,\psi) - H(x,t,u_0,\vartheta_0,\psi)\right)}{\partial u} \right|^2 dxdt \right)^{\frac{1}{2}} \times$$

$$\times\left(\int\limits_Q (\delta u_\varepsilon)^2\,dxdt\right)^{\frac{1}{2}} = 0\left(\varepsilon^{n+1}\right). \tag{3.3}$$

Then from (3.2) and (3.3) we obtain

$$\lim_{\varepsilon\to 0}\frac{\eta(\varepsilon)}{\varepsilon^{n+1}} = 0.$$

Therefore from (3.1) for the first variation of the functional $J(\vartheta)$ we obtain the following expression

$$\delta_1 J(\vartheta_0) = \lim_{\varepsilon\to 0}\frac{\delta J(\vartheta)}{\varepsilon^{n+1}} =$$
$$= -\left[H(\sigma,\tau,u_0(\sigma,\tau),\vartheta,\psi(\sigma,\tau)) - H(\sigma,\tau,u_0(\sigma,\tau),\vartheta_0(\sigma,\tau),\psi(\sigma,\tau))\right].$$

If $(\vartheta_0(x,t), u_0(x,t))$ is an optimal pair, then $\delta_1 J(\vartheta_0) \geq 0$.

Thus the following theorem is proved.

Theorem. *Let the above imposed conditions on data of problem (1.1)- (1.4) be fulfilled. Besides,* $A = \alpha(1 + L mes\Omega) < 1$. *If* (ϑ_0, u_0) *is an optimal pair, and* $\psi(x,t)$ *is a corresponding generalized solution of conjugate problem (1.6)-(1.8), then almost for all* $(x,t) \in Q$ *and for all* $\vartheta \in [\alpha,\beta]$ *the inequality*

$$H(x,t,u_0(x,t),\vartheta,\psi(x,t)) \leq H(x,t,u_0(x,t),\vartheta_0(x,t),\psi(x,t)).$$

is true.

The proved theorem is a necessary condition of optimality of Pontraygin maximum principle type for the considered problem.

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