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BOUNDARY CONTROL PROBLEM FOR STRING VIBRATIONS EQUATION. II

Abstract

In the paper we consider string vibrations equation. Assuming that the state of a string is given at initial time, we study a problem on finding such boundary controls at the ends of a string that reduce the state of a string the to given state at finite time. In the paper we establish necessary and sufficient conditions on initial and finite functions and in fulfilling these conditions we find obvious form of desired controls.

The present paper is continuation of the paper [1]. Therefore statement of the text and enumeration of formulae of the present paper is direct continuation of the paper [1]. Notice that by accomplishing this paper we have used some facts from the paper [2].

§1. Necessary conditions of existence of solutions in $\widehat{W}_2^2[Q_l]$ of problem III from [1]

In this section we'll establish necessary conditions of existence of the solutions in $\widehat{W}_2^2[Q_l]$ of problem III from the paper [1] provided $T = l$.

Theorem 3. *It $T = l$ and for arbitrary four functions $\varphi(x) \in W_2^2[0,l]$, $\psi(x) \in W_2^1[0,l]$, $\varphi_1(x) \in W_2^2[0,l]$ and $\psi_1(x) \in W_2^1[0,l]$ there exists the solution $u(x,t)$ from $\widehat{W}_2^2[Q_l]$ of problem III, this solution satisfies the following three requirements:*

$$u_t(0,0) - u_x(0,0) - u_t(l,l) + u_x(l,l) = 0, \tag{30}$$

$$u_t(l,0) + u_x(l,0) - u_t(0,l) - u_x(0,l) = 0, \tag{31}$$

$$\int_0^l u_t(x,0)dx + u(0,0) + u(l,0) + \int_0^l u_t(x,l)dt + u(0,l) - u(l,l) = 0. \tag{32}$$

Proof. At first we prove this theorem for the special case $u(x,0) = \varphi(x) \equiv 0$ and $u_t(x,0) = \psi(x) \equiv 0$ for $0 \le x \le l$, i.e. we prove that for the special case the solution from $\widehat{W}_2^2[Q_l]$ of problem III satisfies the three requirements:

$$-u_t(l,l) + u_x(l,l) = 0, \tag{30*}$$

$$u_t(0,l) + u_x(0,l) = 0, \tag{31*}$$

$$\int_0^l u_t(x,l)dx - u(0,l) - u(l,l) = 0. \tag{32*}$$

Since the boundary values $u(0,t) = \mu(t)$ and $u_x(l,t) = \nu(t)$ of the solution $u(x,t)$ from $\widehat{W}_2^2[Q_l]$ of problem III belong with respect to t to the classes $W_2^2[0,l]$ and

$W_2^1[0,l]$, respectively and for $t=0$ satisfy the agreement conditions with $\varphi(x)\equiv 0$ and $\psi(x)\equiv 0$, then

$$\mu(0)=0,\ \nu(0)=0,\ \mu'(0)=0. \tag{33}$$

Having continued the boundary values $\mu(t)$ and $\nu(t)$ by an identity zero on the values of $t<0$, by the conditions (33) we'll get that so continued functions (we denote them by the symbols $\underline{\mu}(t)$ and $\underline{\nu}(t)$) will belong to the classes $W_2^2[0,l]$ and $W_2^1[0,l]$, respectively. We can verify that a unique (by theorem 1) solution from $\widehat{W}_2^2[Q_l]$ of the mixed problem (1)-(3) for $\varphi(x)\equiv 0$ and $\psi(x)\equiv 0$ has the form:

$$u(x,t)=\underline{\mu}(t-x)+\underline{\nu}_1(t+x-l), \tag{34}$$

where $\underline{\nu}_1(t)=\int_0^l \underline{\nu}(s)ds$. Differentiating (34) with respect to t and x and then assuming $t=l$, for all x from $[0,l]$ we get

$$u_t(x,l)=\underline{\mu}'(l-x)+\underline{\nu}_1'(x), \tag{35}$$

$$u_x(x,l)=-\underline{\mu}'(l-x)+\underline{\nu}_1'(x). \tag{36}$$

Assuming in (35) and (36) at first $x=0$ and then $x=l$, and using equality (33) $\nu_1'(0)=\nu(0)=0$ we find

$$u_t(0,l)=\underline{\mu}'(l)+\underline{\nu}_1'(0)=\underline{\mu}'(l), \tag{37}$$

$$u_x(0,l)=-\underline{\mu}'(l)+\underline{\nu}_1'(0)=-\underline{\mu}'(l), \tag{38}$$

$$u_t(l,l)=\underline{\mu}'(0)+\underline{\nu}_1'(l)=\underline{\nu}_1'(l), \tag{39}$$

$$u_x(l,l)=-\underline{\mu}'(0)+\underline{\nu}_1'(l)=\underline{\nu}_1'(l). \tag{40}$$

Equality (31*) follows from comparison of (37) and (38), equality (30*) from comparison of (39) and (40). To prove equality (32*) we integrate relation (35) with respect to x from 0 to l and again use the equalities (33).

We get

$$\int_0^l u_t(x,l)dx=\int_0^l\left[\underline{\mu}'(l-x)+\underline{\nu}_1'(x)dx\right]dx=\underline{\mu}(l)-\underline{\mu}(0)+\underline{\nu}_1(l)-\underline{\nu}_1(0)=$$
$$=\underline{\mu}(l)+\underline{\nu}_1(l)=u(0,l)+u(l,l).$$

Thereby, for the special case $\varphi(x)\equiv 0$ and $\psi(x)\equiv 0$ theorem 3 is proved.

Now, let $u(x,t)$ be a solution from $\widehat{W}_2^2[Q_l]$ of general problem III with arbitrary $\varphi(x)\in W_2^2[0,l]$ and $\psi(x)\in W_2^1[0,l]$. Continue the functions $\varphi(x)$ and $\psi(x)$ on the segment $[-l,0]$ and $[l,2l]$ so that the continued functions $\varphi(x)$ and $\psi(x)$ belong to the classes $W_2^2[-l,2l]$ and $W_2^1[-l,2l]$ respectively, and we consider the function $\upsilon(x,t)$ determined by D'Alambert formula with so continued functions $\varphi(x)$ and $\psi(x)$:

$$\upsilon(x,t)=\frac{1}{2}\left[\varphi(x+t)+\varphi(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t}\psi(y)dy. \tag{41}$$

This function $\upsilon(x,t)$ belongs to the class $\widehat{W}_2^2[Q_l]$ and satisfies the initial conditions for $t=0$

$$\upsilon(x,0)=\varphi(x),\ \upsilon_t(x,0)=\psi(x)\quad for\quad 0\le x\le l. \tag{42}$$

Differentiating equality (41) with respect to t and x, and then assuming $t = l$, we get

$$\upsilon_t(x,l) = \frac{1}{2}\left[\varphi'(x+l) - \varphi'(x-l)\right] + \frac{1}{2}\left[\psi'(x+l) + \psi'(x-l)\right], \qquad (43)$$

$$\upsilon_x(x,l) = \frac{1}{2}\left[\varphi'(x+l) + \varphi'(x-l)\right] + \frac{1}{2}\left[\psi'(x+l) - \psi'(x-l)\right]. \qquad (44)$$

Now, notice that by virtue of equalities (42) the difference $[u(x,t) - \upsilon(x,t)]$ is a solution from $\widehat{W}_2^2[Q_l]$ of problem III with initial conditions, being identically equal to zero for $t = 0$. Therefore, by the above considered special case the three requirements of the form:

$$-\left[u_t(l,l) - \upsilon_t(l,l)\right] + \left[u_x(l,l) - \upsilon_x(l,l)\right] = 0, \qquad (45)$$

$$\left[u_t(0,l) - \upsilon_t(0,l)\right] + \left[u_x(0,l) - \upsilon_x(0,l)\right] = 0, \qquad (46)$$

$$\int_0^l \left[u_t(x,l) - \upsilon_t(x,l)\right] dx - \left[u(0,l) - \upsilon(x,l)\right] - \left[u(l,l) - \upsilon(l,l)\right] = 0. \qquad (47)$$

have been fulfilled for this difference.

In equalities (43) and (44) we assume $x = l$:

$$\upsilon_t(l,l) = \frac{1}{2}\left[\varphi'(2l) - \varphi'(0)\right] + \frac{1}{2}\left[\psi(2l) + \psi(0)\right],$$

$$\upsilon_x(l,l) = \frac{1}{2}\left[\varphi'(2l) + \varphi'(0)\right] + \frac{1}{2}\left[\psi(2l) - \psi(0)\right].$$

From these equalities we get

$$\upsilon_t(l,l) - \upsilon_x(l,l) = \psi(0) - \varphi'(0) = u_t(0,0) - u_x(0,0). \qquad (48)$$

Requirement (30) follows from (45) and (48). Further, in equalities (43) and (44) we assume $x = 0$:

$$\upsilon_t(0,l) = \frac{1}{2}\left[\varphi'(l) - \varphi'(-l)\right] + \frac{1}{2}\left[\psi(l) + \psi(-l)\right],$$

$$\upsilon_x(0,l) = \frac{1}{2}\left[\varphi'(l) + \varphi'(-l)\right] + \frac{1}{2}\left[\psi(l) - \psi(-l)\right].$$

From these equalities we get

$$-\upsilon_t(0,l) - \upsilon_x(0,l) = -\psi(l) - \varphi'(l) = -u_t(l,0) - u_x(l,0). \qquad (49)$$

Requirement (31) follows from (46) and (49). Finally,

$$-\int_0^l \upsilon_t(x,l)dx + \upsilon(0,l) + \upsilon(l,l) = -\frac{1}{2}\int_0^l \left[\varphi'(x+l) - \varphi'(x-l)dx\right] -$$

$$-\frac{1}{2}\int_0^l \left[\psi(x+l) + \psi(x-l)\right] dx + \frac{1}{2}\left[\varphi(l) + \varphi(-l)\right] + \frac{1}{2}\int_{-l}^l \psi(y)dy +$$

$$+\frac{1}{2}\left[\varphi(2l)+\varphi(0)\right]+\frac{1}{2}\int\limits_{0}^{2l}\psi(y)dy=\frac{1}{2}\left[-\varphi(2l)+\varphi(l)+\varphi(0)-\varphi(-l)\right]-$$

$$-\frac{1}{2}\int\limits_{-l}^{2l}\psi(y)dy-\frac{1}{2}\int\limits_{-l}^{0}\psi(y)dy+\frac{1}{2}\left[-\varphi(2l)+\varphi(l)+\varphi(0)+\varphi(-l)\right]+\frac{1}{2}\int\limits_{-l}^{l}\psi(y)dy+$$

$$+\frac{1}{2}\int\limits_{0}^{2l}\psi(y)dy=\int\limits_{0}^{l}\psi(y)dy+\varphi(0)+\varphi(l)=\int\limits_{0}^{l}u_t(y,0)dy+u(0,0)+u(l,0).$$

follows from relations (43) and (41)

Validity of requirement (32) is established by the comparison of the last equality with (47). Theorem 3 is proved.

§2. Basic theorem

Theorem 4 (basic). *In order for the preassigned four functions $\varphi(x)\in W_2^2[0,l]$, $\psi(x)\in W_2^1[0,l]$, $\varphi_1(x)\in W_2^2[0,l]$, $\psi_1(x)\in W_2^1[0,l]$ there exist the boundary controls $\mu(t)$ and $\nu(t)$ from the classes $W_2^2[0,l]$ and $W_2^1[0,l]$ respectively, providing satisfaction by the solution $u(x,t)$ from $W_2^2[Q_l]$ of the mixed problem*

$$u_{tt}-u_{xx}=0 \quad in \quad Q_l, \tag{50}$$

$$u(0,t)\equiv\mu(t),\ u_x(l,t)\equiv\nu(t) \quad for \quad 0\le t\le l, \tag{51}$$

$$u(x,0)\equiv\varphi(x),\ u_t(x,0)=\psi(x) \quad for \quad 0\le x\le l, \tag{52}$$

to the conditions

$$u(x,l)\equiv\varphi_1(x),\ u_t(x,l)=\psi_1(x) \quad for \quad 0\le x\le l, \tag{53}$$

and subjected to the agreement conditions with the functions $\varphi(x)$ and $\psi(x)$ for $t=0$ and with the functions $\varphi_1(x)$ and $\psi_1(x)$ for $t=l$ it is necessary and sufficient that the three requirements

$$\psi(0)-\varphi'(0)-\psi_1(l)+\varphi_1'(l)=0, \tag{54}$$

$$\psi(l)+\varphi'(l)-\psi_1(l)-\varphi_1'(0)=0, \tag{55}$$

$$\int\limits_{0}^{l}\psi(x)dx+\varphi(0)+\varphi(l)+\int\limits_{0}^{l}\psi_1(x)dx-\varphi_1(0)-\varphi_1(l)=0 \tag{56}$$

be fulfilled.

In fulfilling the indicated three requirements the boundary controls $\mu(t)$ and $\nu(t)$ have the form:

$$\mu(t)=\frac{1}{2}\int\limits_{0}^{t}\psi(x)dx+\frac{1}{2}\varphi(t)+\frac{1}{2}\varphi(0)+$$

$$+\frac{1}{2}\int\limits_{l-t}^{l}\psi_1(x)dx-\frac{1}{2}\varphi_1(l)+\frac{1}{2}\varphi_1(l-t),$$

$$\nu(t) = \frac{1}{2}\psi_1(t) + \frac{1}{2}\varphi_1'(t) - \frac{1}{2}\psi(l-t) + \frac{1}{2}\varphi'(l-t).$$

Proof: Necessity of fulfilment of three requirements (54)-(56) is proved in theorem 3. Show that in fulfilling these three requirements, the boundary controls $\mu(t)$ and $\nu(t)$ from classes $W_2^2[0,l]$ and $W_2^1[0,l]$, respectively, providing satisfaction of the solution from $W_2^2[Q_l]$ of the problem (50)-)52) to the conditions (53) and agreed with $\varphi(x)$ and $\psi(x)$ for $t=0$ and with $\varphi_1(x)$ and $\psi_1(x)$ for $t=l$ exist in an obvious analytical form.

We'll look for the solution from $W_2^2[Q_l]$ of problem III considered for $T=l$, i.e. of the problem

$$u_{tt} - u_{xx} = 0 \quad in \quad Q_l, \tag{57}$$

$$u(x,0) \equiv \varphi(x), \; u_t(x,0) \equiv \psi(x) \quad for \quad 0 \le x \le l, \tag{58}$$

$$u(x,l) \equiv \varphi_1(x), \; u_t(x,l) = \psi_1(x) \quad for \quad 0 \le x \le l, \tag{59}$$

in the following form:

$$u(x,l) = F(x+t) + G(t+l-x) \quad for \quad 0 \le x \le l, \quad 0 \le t \le l, \tag{60}$$

where $F(x)$ and $G(x)$ are two functions belonging to the class $W_2^2[0,2l]$. In order to express the functions $F(x)$ and $G(x)$ for all the values of x from the segment $0 \le x \le 2l$ by $\varphi(x)$, $\psi(x)$, $\varphi_1(x)$ and $\psi_1(x)$ we differentiate (60) with respect to t.

We get

$$u_t(x,l) = F'(x+t) + G'(t+l-x). \tag{61}$$

Then we assume in (60) and (61) at first $t=0$ and then $t=l$ and using conditions (58)-(59) and arrive at the following relations:

$$F(x) + G(l-x) = \varphi(x), \tag{62}$$

$$F'(x) + G'(l-x) = \psi(x), \tag{63}$$

$$F(l+x) + G(2l-x) = \varphi_1(x), \tag{64}$$

$$F'(l+x) + G'(2l-x) = \psi_1(x), \tag{65}$$

valid for all x from the segment $[0,l]$. Differentiating relations (62) and (64) with respect to x, we get

$$F'(x) - G'(l-x) = \varphi'(x) \quad for \quad 0 \le x \le l, \tag{66}$$

$$F'(l+x) - G'(2l-x) = \varphi_1'(x) \quad for \quad 0 \le x \le l. \tag{67}$$

Half-sum and half-difference of relations (63) and (66) reduce to the equalities

$$F'(x) = \frac{1}{2}\psi(x) + \frac{1}{2}\varphi'(x), \tag{68}$$

$$G'(l-x) = \frac{1}{2}\psi(x) - \frac{1}{2}\varphi'(x), \tag{69}$$

determining the functions $F(x)$ and $G(x)$ on the segment $0 \le x \le l$ by the functions $\varphi(x)$ and $\psi(x)$, half-sum and half-difference of relations (65) and (67) reduce to the equalities

$$F'(l+x) = \frac{1}{2}\psi_1(x) + \frac{1}{2}\varphi_1'(x), \tag{70}$$

$$G'(2l-x)=\frac{1}{2}\psi_1(x)-\frac{1}{2}\varphi_1'(x), \tag{71}$$

determining the functions $F(x)$ and $G(x)$ on the segment $0\le x\le 2l$ by the functions $\varphi_1(x)$ and $\psi_1(x)$. Let's establish expressions for the functions $F(t)$ and $G(t)$ themselves on the segment $0\le t\le l$ and $l\le t\le 2l$. Integrating (68) with respect to x in the ranges from 0 to t, for any t from the segment $[0,l]$ we get

$$F(t)=F(0)+\frac{1}{2}\int\limits_0^t\psi(x)dx+\frac{1}{2}\varphi(t)-\frac{1}{2}\varphi(0), \tag{72}$$

and integrating (69) with respect to x in the ranges from $l-t$ to l, for any t from the segment $[0,l]$ we get

$$G(t)=G(0)+\frac{1}{2}\int\limits_{l-t}^{l}\psi(x)dx-\frac{1}{2}\varphi(l)+\frac{1}{2}\varphi(l-t). \tag{73}$$

Further, integrating (70) with respect to x in the ranges from $t-l$ to l, for any t from the segment $[l,2l]$ we get

$$F(t)=F(2l)-\frac{1}{2}\int\limits_{t-t}^{l}\psi_1(x)dx-\frac{1}{2}\varphi_1(l)+\frac{1}{2}\varphi_1(t-l). \tag{74}$$

Finally integrating (71) with respect to x in the ranges from 0 to $2l-t$, for any t from the segment $[l,2l]$ we get

$$G(t)=G(2l)-\frac{1}{2}\int\limits_0^{2l-t}\psi_1(x)dx+\frac{1}{2}\varphi_1(2l-t)-\frac{1}{2}\varphi_1(0). \tag{75}$$

Using the relations (62) and (64) we express $G(0)$ by $F(0)$, and $G(2l)$ by $F(2l)$. We determine the values of $F(x)$ and $G(l-x)$ from relations (72) and (73)

$$F(x)=F(0)+\frac{1}{2}\int\limits_0^x\psi(\xi)d\xi+\frac{1}{2}\varphi(x)-\frac{1}{2}\varphi(0),$$

$$G(l-x)=G(0)+\int\limits_x^l\psi(\xi)d\xi-\frac{1}{2}\varphi(l)-\frac{1}{2}\varphi(x).$$

Substituting these two relations into (62) we find

$$F(0)+\frac{1}{2}\int\limits_0^x\psi(\xi)d\xi+\frac{1}{2}\varphi(x)-\frac{1}{2}\varphi(0)+$$
$$+G(0)+\frac{1}{2}\int\limits_x^l\psi(\xi)d\xi-\frac{1}{2}\varphi(l)+\frac{1}{2}\varphi(x)=\varphi(x),$$

whence

$$G(0) = -F(0) + \frac{1}{2}\varphi(0) + \frac{1}{2}\varphi(l) - \frac{1}{2}\int\limits_0^l \psi(\xi)d\xi. \tag{76}$$

We find the values of $F(l+x)$ and $G(2l-x)$ from the relations (74) and (75) in a similar way:

$$F(l+x) = F(2l) - \frac{1}{2}\int\limits_x^l \psi_1(\xi)d\xi - \frac{1}{2}\varphi_1(l) + \frac{1}{2}\varphi_1(x),$$

$$G(2l-x) = G(2l) - \frac{1}{2}\int\limits_0^x \psi_1(\xi)d\xi + \frac{1}{2}\varphi_1(x) - \frac{1}{2}\varphi_1(0).$$

Substituting the last two relations into (64), we get

$$F(2l) - \frac{1}{2}\int\limits_x^l \psi_1(\xi)d\xi - \frac{1}{2}\varphi_1(l) + \frac{1}{2}\varphi_1(x) + G(2l) -$$
$$-\frac{1}{2}\int\limits_0^x \psi_1(\xi)d\xi + \frac{1}{2}\varphi_1(x) - \frac{1}{2}\varphi_1(0) = \varphi_1(x),$$

whence

$$G(2l) = -F(2l) + \frac{1}{2}\varphi_1(l) + \frac{1}{2}\varphi_1(0) + \frac{1}{2}\int\limits_0^l \psi_1(\xi)d\xi. \tag{77}$$

Now, from the continuity condition of the functions $F(x)$ at the point $x = l$, we express $F(2l)$ by $F(0)$. Equating the right hand sides of (72) and (74) taken for $t = l$ we get

$$F(0) + \frac{1}{2}\int\limits_0^l \psi(\xi)d\xi + \frac{1}{2}\varphi(l) - \frac{1}{2}\varphi(0) =$$
$$= F(2l) - \frac{1}{2}\int\limits_0^l \psi_1(\xi)d\xi - \frac{1}{2}\varphi_1(l) + \frac{1}{2}\varphi_1(0).$$

From the last equality and relation (56) we find

$$F(2l) = F(0) + \varphi_1(l) - \varphi(0). \tag{78}$$

Finally,

$$G(2l) = -F(0) + \varphi(0) - \frac{1}{2}\varphi_1(l) + \frac{1}{2}\varphi_1(0) + \frac{1}{2}\int\limits_0^l \psi_1(\xi)d\xi. \tag{79}$$

follows from (77) and (78).

Be convinced that the function $G(x)$ is continuous at the point $x = l$, i.e. the values of $G(l)$ determined from relations (73) and (75) coincide between themselves. Really, from the equality (73) taken for $t = l$, allowing for relation (76) we get

$$G(l) = -F(0) + \frac{1}{2}\varphi(0) + \frac{1}{2}\varphi(l) - \frac{1}{2}\int_0^l \psi(\xi)d\xi +$$

$$+\frac{1}{2}\int_0^l \psi(\xi)d\xi - \frac{1}{2}\varphi(l) + \frac{1}{2}\varphi(0) = -F(0) + \varphi(0),$$

and from equality (75) taken for $t = l$, allowing for relation (79) we have:

$$G(l) = -F(0) + \varphi(0) - \frac{1}{2}\varphi_1(l) + \frac{1}{2}\varphi_1(0) + \frac{1}{2}\int_0^l \psi_1(\xi)d\xi -$$

$$-\frac{1}{2}\int_0^l \psi_1(\xi)d\xi + \frac{1}{2}\varphi_1(l) - \frac{1}{2}\varphi_1(0) = -F(0) + \varphi(0).$$

Add to this fact that, the values of the derivative $F'(l)$ determined from the equalities (68) and (70) coincide between themselves by the relation (55),the values of the derivative $G'(l)$, determined from the equalities (69) and (71), coincide between themselves by the relation (54). Now, we can confirm that the functions $F(t)$ and $G(t)$ determined by the relations (72)-(75) provided that the constants $G(0)$, $F(2l)$ and $G(2l)$ are expressed by $F(0)$ by means of equalities (76), (78) and (79), belong to the class $W_2^2[0, 2l]$.

From belongness of the functions $F(t)$ and $G(t)$ to the class $W_2^2[0, 2l]$ it follows that the function $u(x, t)$ determined by equality (60) is a solution from $\widehat{W_2^2}[Q_l]$ of the problem (57)-(59).

To complete the proof of the basic theorem we calculate in an obvious analytical form the desired boundary controls $\mu(t)$ and $\nu(t)$. From equality (60) we have

$$\mu(t) = u(0, t) = F(t) + G(l + t), \tag{80}$$

$$\nu(t) = u_x(0, t) = F'(l + t) - G'(t). \tag{81}$$

Determine the values for $G(l + t)$ from equality (75)

$$G(l + t) = G(2l) - \frac{1}{2}\int_0^{l-t} \psi_1(x)dx + \frac{1}{2}\varphi_1(l - t) - \frac{1}{2}\varphi_1(0).$$

Using equality (79) for $G(2l)$ we find

$$G(l + t) = -F(0) + \varphi(0) - \frac{1}{2}\varphi_1(l) + \frac{1}{2}\varphi_1(0) + \frac{1}{2}\int_0^l \psi_1(x)dx - \frac{1}{2}\int_0^{l-t} \psi_1(x)dx +$$

$$+\frac{1}{2}\varphi_1(l - t) - \frac{1}{2}\varphi_1(0) = -F(0) + \varphi(0) - \frac{1}{2}\varphi_1(l) + \frac{1}{2}\int_{l-t}^l \psi_1(x)dx + \frac{1}{2}\varphi_1(l - t).$$

Putting to the right hand side of (80) the values $G(l+t)$ from the last equality and $F(t)$ from equality (72), we get

$$\mu(t) = F(0) + \frac{1}{2}\int_0^t \psi(x)dx + \frac{1}{2}\varphi(t) - \frac{1}{2}\varphi(0) - F(0) + \varphi(0) -$$

$$-\frac{1}{2}\varphi_1(l) + \frac{1}{2}\int_{l-t}^{l} \psi_1(x)dx + \frac{1}{2}\varphi_1(l-t) = \frac{1}{2}\int_0^t \psi(x)dx + \frac{1}{2}\varphi(t) + \frac{1}{2}\varphi(0) +$$

$$+\frac{1}{2}\int_{l-t}^{l} \psi_1(x)dx + \frac{1}{2}\varphi_1(l-t) - \frac{1}{2}\varphi_1(t).$$

So

$$\mu(t) = \frac{1}{2}\int_0^t \psi(x)dx + \frac{1}{2}\varphi(t) + \frac{1}{2}\varphi(0) + \frac{1}{2}\int_{l-t}^{l} \psi_1(x)dx + \frac{1}{2}\varphi_1(l-t) - \frac{1}{2}\varphi_1(l).$$

We determine the values of $F'(l+t)$ and $G'(t)$ from equality (70) and (69) in a similar way:

$$F'(l+t) = \frac{1}{2}\psi_1(t) + \frac{1}{2}\varphi_1'(t), \quad G'(t) = \frac{1}{2}\psi(l-t) - \frac{1}{2}\varphi'(l-t).$$

Putting to the right hand side of (81) the values of $F'(l+t)$ and $G'(t)$ determined from the last equalities, we get

$$\nu(t) = \frac{1}{2}\left[\psi_1(t) + \varphi_1'(t) - \psi(l-t) + \varphi'(l-t)\right].$$

The basic theorem is proved.

§3. Corollaries from the basic theorem

Let's formulate two important statements. One of them follows from the basic theorem for $\varphi_1(x) \equiv 0$ and $\psi_1(x) \equiv 0$, and the second one for $\varphi(x) \equiv 0$ and $\psi(x) \equiv 0$

Theorem on full dampening of vibration process: In order for the preassigned initial functions $\varphi(x) \in W_2^2[0,l]$ and $\psi(x) \in W_2^1[0,l]$ there exist the boundary controls $\mu(t)$ and $\nu(t)$ from $\overline{W}_2^2[0,l]$ and $\overline{W}_2^1[0,l]$ respectively providing satisfaction of the solution $u(x,t)$ from the class $\widehat{W}_2^2\,[Q_l]$ of the mixed problem (50)-(52) to conditions of full dampening $u(x,l) \equiv 0,\ u_t(x,l) \equiv 0$ (for $0 \le x \le$ $\le l$) and subjected to the conditions of agreement with the functions $\varphi(x)$ and $\psi(x)$ for $t = 0$, it is necessary and sufficient that the three requirements:

$$\psi(0) - \varphi'(0) = 0,\ \psi(l) + \varphi'(l) = 0,\ \int_0^l \psi(x)dx + \varphi(0) + \varphi(l) = 0.$$

be fulfilled.

In fulfilling these three requirements the boundary controls $\mu(t)$ and $\nu(t)$ have the form

$$\mu(t)=\frac{1}{2}\int\limits_0^t \psi(x)dx+\frac{1}{2}\varphi(t)+\frac{1}{2}\varphi(0),\ \ \nu(t)=-\frac{1}{2}\psi(l-t)+\frac{1}{2}\varphi'(l-t).$$

Theorem on obtaining arbitrary permutation and arbitrary velocity of initially-resting system by means of boundary controls. In order for preassigned functions $\varphi_1(x)\in W_2^2[0,l]$ and $\psi_1(x)\in W_2^1[0,l]$ there-exist boundary controls $\mu(t)$ and $\nu(t)$ from the classes $W_2^2[0,l]$ and $W_2^1[0,l]$, respectively providing satisfaction of the solution of $u(x,t)$ from $\widehat{W_2^2}\,[Q_l]$ of the mixed problem (50)-(52) with $\varphi(x)\equiv 0$ and $\psi(x)\equiv 0$ to the conditions $u(x,l)=\varphi_1(x),\ u(x,l)=\psi_1(x)$ (for $0\leq x\leq l$) and subjected to the agreement conditions with the functions $\varphi_1(x)$ and $\psi_1(x)$ for $t=l$ it is necessary and sufficient that the three requirements:

$$-\psi_1(l)+\varphi_1'(l)=0,\ \ \psi_1(0)+\varphi_1'(0)=0,\ \ \int\limits_0^l \psi_1(x)dx-\varphi_1(0)-\varphi_1(l)=0.$$

be fulfilled.

In fulfilling these three requirements the boundary controls $\mu(t)$ and $\nu(t)$ have the form:

$$\mu(t)=\frac{1}{2}\int\limits_{l-t}^l \psi_1(x)dx-\frac{1}{2}\varphi_1(l)+\frac{1}{2}\varphi_1(l-t),\ \ \nu(t)=\frac{1}{2}\left[\psi_1(t)+\varphi_1'(t)\right].$$

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