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CONTACT PROBLEM FOR ELASTIC PIECEWISE-HOMOGENEOUS INFINITE PLATE WITH INFINITE STRINGER

Abstract

The contact problem on distribution of intensity of tangential contact forces operating under stringer is considered.

Problem statement and derivation of system of basic singular integral equations. Let elastic solid plate in the form of thin piecewise-homogeneous infinite plate of small constant thickness h, composed of two semi-infinite plates with different elastic constants coupling among themselves along common rectangular boundary, be reinforced by piecewise-homogeneous infinite stringer of rectangular cross-section of sufficiently small constant thickness h_s and of small width d_s, the part of which $(-\infty, 0)$ is elastic with elasticity modulus $E_s^{(1)}$, the other part $(0, \infty)$ is absolutely rigid. It is supposed, that the piecewise-homogeneous infinite stringer is welded (adhered) to plates, has different elastic characteristics, and is perpendicular to the boundary line of the mentioned semi-infinite plates. It is required to define distribution of intensity of tangential contact forces operating under stringer, when contacting pair is deformed by concentrated force Q_1 operating on elastic part of the stringer at the point $y = -a$ (fig.1).

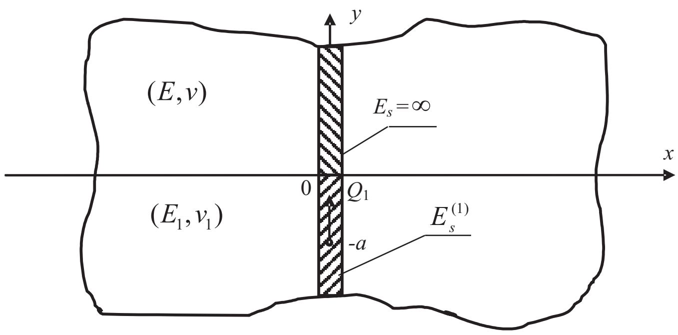

Fig. 1.

Relative to stringer we accept the model of line contact, and for elastic piecewise-homogeneous infinite plate the model of generalized plane stress state is true [1].

According to model accepted above the equilibrium equations of stringer are written in the form

$$\frac{d\upsilon_s^{(1)}(y)}{dy} = 0 \qquad (0 < y < \infty), \tag{1}$$

$$\frac{d\upsilon_s^{(2)}(y)}{dy} = \frac{1}{E_s^{(1)} F_s} \int\limits_{-\infty}^{0} \theta(y-t)\, \tau^{(2)}(t)\, dt - \frac{Q_1}{E_s^{(1)} F_s} \theta(y+a) \tag{2}$$

$$(-\infty < y < 0)$$

and the following conditions hold

$$\int\limits_0^\infty \tau^{(1)}(\eta)\,d\eta = -\widetilde{Q}_0, \qquad \int\limits_{-\infty}^0 \tau^{(2)}(t)\,dt = \widetilde{Q}_0,$$
$$\left.\frac{d\upsilon_s^{(2)}(y)}{dy}\right|_{y=-0} = \frac{Q_1 - \widetilde{Q}_0}{E_s^{(1)}F_s} \tag{3}$$

Here $\upsilon_s^{(1)}(y)$, $\upsilon_s^{(2)}(y)$ are vertical contact displacements of the stringer for $0 < y < \infty$ and $-\infty < y < 0$; $\tau^{(1)}(y)$ and $\tau^{(2)}(y)$ are intensities of tangential contact forces in domains $0 < y < \infty$ and $-\infty < y < 0$, respectively, $\widetilde{Q}_0$ is longitudinal force untill unknown.

For vertical deformations of elastic piecewise-homogeneous infinite plates, when tangential contact forces with the intensities $\tau^{(1)}(y)$ and $\tau^{(2)}(y)$ operate on semi-infinite intervals $(-\infty, 0)$ and $(0, \infty)$, we get

$$h\ell\frac{d\upsilon^{(1)}(0,y)}{dy} = \frac{1}{\pi}\int\limits_0^\infty \left[\frac{1}{\eta - y} - \frac{A_1}{\eta + y} + \frac{A_2\eta(\eta - y)}{(\eta + y)^3}\right]\tau^{(1)}(\eta)\,d\eta +$$
$$+\frac{1}{\pi}\int\limits_{-\infty}^0 \left[\frac{A_3}{t - y} + \frac{A_4 t}{(t - y)^2}\right]\tau^{(2)}(t)\,dt \qquad (0 < y < \infty) \tag{4}$$

$$h\ell_1\frac{d\upsilon^{(2)}(0,y)}{dy} = \frac{1}{\pi}\int\limits_{-\infty}^0 \left[\frac{1}{t - y} - \frac{B_1}{t + y} + \frac{B_2 t(t - y)}{(t + y)^3}\right]\tau^{(2)}(t)\,dt +$$
$$+\frac{1}{\pi}\int\limits_0^\infty \left[\frac{B_3}{\eta - y} + \frac{B_4\eta}{(\eta - y)^2}\right]\tau^{(1)}(\eta)\,d\eta \qquad (-\infty < y < 0) \tag{5}$$

Denote that the following contact conditions hold on mount line of stringer with semi-infinite plates

$$\frac{d\upsilon_s^{(1)}(y)}{dy} = \frac{d\upsilon^{(1)}(0,y)}{dy} \quad (0 < y < \infty), \tag{6}$$

$$\frac{d\upsilon_s^{(2)}(y)}{dy} = \frac{d\upsilon^{(2)}(0,y)}{dy} \quad (-\infty < y < 0). \tag{7}$$

On the basis of contact conditions (6) and (7), allowing (1), (2), (4) and (5) with regard to tangential contact forces with intensities $\tau^{(1)}(y)$ $(0 < y < \infty)$ and $\tau^{(2)}(y)$ $(-\infty < y < 0)$, we obtain the following system of first order singular integral equations with fixed singularity:

$$\frac{1}{\pi}\int\limits_0^\infty \left[\frac{1}{\eta - y} - \frac{A_1}{\eta + y} + \frac{A_2\eta(\eta - y)}{(\eta + y)^3}\right]\tau^{(1)}(\eta)\,d\eta +$$

$$+\frac{1}{\pi}\int\limits_{-\infty}^{0}\left[\frac{A_3}{t-y}+\frac{A_4 t}{(t-y)^2}\right]\tau^{(2)}(t)\,dt=0,\qquad (0<y<\infty)\,, \tag{8}$$

$$\frac{1}{\pi}\int\limits_{-\infty}^{0}\left[\frac{1}{t-y}-\frac{B_1}{t+y}+\frac{B_2 t\,(t-y)}{(t+y)^3}\right]\tau^{(2)}(t)\,dt+$$

$$+\frac{1}{\pi}\int\limits_{0}^{\infty}\left[\frac{B_3}{\eta-y}+\frac{B_4\eta}{(\eta-y)^2}\right]\tau^{(1)}(\eta)\,d\eta=\lambda_1\int\limits_{-\infty}^{0}\theta\,(y-t)\,\tau^{(2)}(t)\,dt-\lambda_1 Q_1\theta\,(a+y)$$

$$(-\infty<y<0)\,. \tag{9}$$

Thus solution of considered contact problem for piecewise-homogeneous infinite plate reinforced by piecewise-homogeneous infinite stringer, is reduced to solution of system of singular integral equations (8), (9) under condition (3).

Solution of system of singular integral equations (8), (9). Solution of system of first order singular integral equations (8) and (9) under condition (3) is searched in the class of the functions:

$$\Omega_0^{(k)}=\left\{\tau^{(k)}(y)\,|\tau^{(k)}(y)\underset{y\to 0}{=}O\left(y^{-\rho}\right);\tau^{(k)}(y)=O\left(|y|^{-1-\delta}\right);\;k=1,2\right\}. \tag{10}$$

Further, in the system of equations (8), (9), we make change of variables: $y=e^{\upsilon}$, $\eta=e^{u}$ for y, $\eta\in(0,\infty)$ and $y=-e^{\upsilon}$, $t=-e^{\omega}$ for $y,t\in(-\infty,0)$ and obtain:

$$\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\left[\frac{1}{1-e^{\upsilon-u}}-\frac{A_1}{1+e^{\upsilon-u}}+\frac{A_2\left(1-e^{\upsilon-u}\right)}{\left(1+e^{\upsilon-u}\right)^3}\right]\tau_+(u)\,du-$$

$$-\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\left[\frac{A_3}{1+e^{\upsilon-\omega}}+\frac{A_4}{\left(1+e^{\upsilon-\omega}\right)^2}\right]\tau_-(\omega)\,d\omega,\quad(-\infty<\upsilon<\infty)\,; \tag{11}$$

$$-\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\left[\frac{1}{1-e^{\upsilon-\omega}}-\frac{B_1}{1+e^{\upsilon-\omega}}+\frac{B_2\left(1-e^{\upsilon-\omega}\right)}{\left(1+e^{\upsilon-\omega}\right)^3}\right]\tau_-(\omega)\,d\omega+$$

$$+\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\left[\frac{B_3}{1+e^{\upsilon-u}}+\frac{B_4}{\left(1+e^{\upsilon-u}\right)^2}\right]\tau_+(u)\,du=$$

$$=\lambda_1\int\limits_{-\infty}^{\infty}\theta\,(\omega-\upsilon)\,e^{\omega}\tau_-(\omega)\,d\omega-\lambda_1 Q_1\theta\,(\ln a-\upsilon)\,,\quad(-\infty<\upsilon<\infty)\,, \tag{12}$$

in this case equilibrium conditions of stringer (3) have the form:

$$\int\limits_{-\infty}^{\infty}\tau_+(u)\,e^{u}du=-\widetilde{Q}_0;\qquad\int\limits_{-\infty}^{\infty}\tau_-(\omega)\,e^{\omega}d\omega=\widetilde{Q}_0; \tag{13}$$

where

$$\tau_+(u) = \tau^{(1)}(e^u), \qquad \tau_-(\omega) = \tau^{(2)}(-e^\omega).$$

Applying Fourier transform on variable υ $(-\infty < \upsilon < \infty)$ to the system singular integral equations (11), (12), we get the following system of functional-difference equations with respect to Fourier transformants of the functions $\tau_+(u)$ and $\tau_-(\omega)$:

$$R_{11}(\alpha)\bar{\tau}_+(\alpha) - R_{12}(\alpha)\bar{\tau}_-(\alpha) = 0, \tag{14}$$

$$\alpha R_{21}(\alpha)\bar{\tau}_+(\alpha) - \alpha R_{22}(\alpha)\bar{\tau}_-(\alpha) = \lambda_1\bar{\tau}_-(\alpha - i) - \lambda_1 Q_1 a^{i\alpha}. \tag{15}$$

$$(-1 < Jm\alpha < -\rho)$$

under the boundary conditions

$$\bar{\tau}_+(-i) = -\widetilde{Q}_0; \qquad \bar{\tau}_-(-i) = \widetilde{Q}_0 \tag{16}$$

Solving the system of functional-difference equations (14), (15) relative to $\bar{\tau}_+(\alpha)$ and $\bar{\tau}_-(\alpha)$, we have

$$\bar{\tau}_+(\alpha) = R(\alpha)\bar{\tau}_-(\alpha) \qquad \left(\widetilde{C}(\alpha) = \frac{1}{\alpha}C(\alpha)\right), \tag{17}$$

$$\bar{\tau}_-(\alpha) + \lambda_1\widetilde{C}(\alpha)\bar{\tau}_-(\alpha - i) = \lambda_1 Q_1 a^{i\alpha}\widetilde{C}(\alpha) \tag{18}$$

$$(-1 < Jm\alpha < -\rho).$$

Here $\bar{\tau}_\pm(\alpha) = F[\tau_\pm(\upsilon)]$ are Fourier transformants of the function $\tau_\pm(\upsilon)$.

Thus, the problem has been reduced to solution of nonhomogeneous functional difference equation (18) under boundary condition (16), determined in the band $(-1 < Jm\alpha < -\rho)$. It follows from (18) that $\bar{\tau}_-(\alpha)$ is regular in the band $(-2 < Jm\alpha < -\rho)$, i.e. $\delta = 1$.

Solution of difference equation (18) under availability of second condition (16) has the form

$$\bar{\tau}_-(\alpha) = \frac{\lambda_1 Q_1}{2i\tau_0(\alpha)}\int\limits_{i\tau-\infty}^{i\tau+\infty}[cth\pi(\alpha - s) + cth\pi s]\,\lambda_1^{-i(\alpha-s)}a^{is}\tau_0(s)\,\widetilde{C}(s)\,ds +$$

$$+\frac{\widetilde{Q}_0\lambda_1^{1-i\alpha}}{\tau_0(\alpha)} + \lambda_1 Q_1 a^{i\alpha}\widetilde{C}(\alpha) \tag{19}$$

$$(0 < Jm(\alpha - s) < 1 - \rho, \qquad -1 < \tau < -\rho);$$

at this, according to (17) we'll have for $\bar{\tau}_+(\alpha)$

$$\bar{\tau}_+(\alpha) = \frac{\lambda_1 Q_1 R(\alpha)}{2i\tau_0(\alpha)}\int\limits_{i\tau-\infty}^{i\tau+\infty}[cth\pi(\alpha - s) + cth\pi s]\,\lambda_1^{-i(\alpha-s)}a^{is}\tau_0(s)\,\widetilde{C}(s)\,ds +$$

$$+\frac{\widetilde{Q}_0\lambda_1^{1-i\alpha}}{\tau_0(\alpha)}R(\alpha) + \lambda_1 Q_1 a^{i\alpha}\widetilde{C}(\alpha)\,R(\alpha) \tag{20}$$

$$(0 < Jm(\alpha - s) < 1 - \rho, \qquad -1 < \tau < -\rho);$$

Now, applying inverse Fourier transform to (19) and (20), and passing to prior variables, we obtain

$$\tau^{(1)}(y) = \frac{\lambda_1 \widetilde{Q}_0}{2\pi} \int\limits_{i\tau'-\infty}^{i\tau'+\infty} \left[\lambda_1^{-i\alpha} - F_1^*(\alpha, a)\right] \frac{R(\alpha)}{\tau_0(\alpha)} e^{-i\alpha \ln y} d\alpha +$$

$$+\frac{\lambda_1 Q_1}{2\pi} \int\limits_{i\tau'-\infty}^{i\tau'+\infty} \widetilde{C}(\alpha) R(\alpha) e^{-i\alpha \ln\left(\frac{y}{a}\right)} d\alpha \qquad \left(0 < y < \infty, \quad -1 < \tau' < -\rho\right), \qquad (21)$$

$$\tau^{(2)}(y) = \frac{\lambda_1 \widetilde{Q}_0}{2\pi} \int\limits_{i\tau'-\infty}^{i\tau'+\infty} \left[\lambda_1^{-i\alpha} F_1^*(\alpha, a)\right] \frac{e^{-i\alpha \ln(-y)}}{\tau_0(\alpha)} d\alpha +$$

$$+\frac{\lambda_1 Q_1}{2\pi} \int\limits_{i\tau'-\infty}^{i\tau'+\infty} \widetilde{C}(\alpha) e^{-i\alpha \ln\left(-\frac{y}{a}\right)} d\alpha \qquad \left(-\infty < y < 0, \quad -1 < \tau' < -\rho\right). \qquad (22)$$

Here

$$F_1^*(\alpha, t) = \frac{iQ_1}{2\widetilde{Q}_0} \int\limits_{i\tau-\infty}^{i\tau+\infty} \left[cth\pi(\alpha - s) + cth\pi s\right] \lambda_1^{-i(\alpha - s)} \tau_0(s) \widetilde{C}(s) e^{is \ln t} ds \qquad (23)$$

$$(0 < Jm(\alpha - s) < 1 - \rho, \quad 0 < t < \infty, \quad -1 < \tau < -\rho)$$

$$\tau_0(\alpha) = \frac{ish\frac{\pi\alpha}{2}}{\Gamma(i\alpha)} \Phi_0(\alpha), \qquad \left(\begin{array}{c} \alpha = \sigma + i\tau \\ s = \sigma_0 + i\tau \end{array}\right),$$

$$\Phi_0(\alpha) = \frac{1}{H(\alpha)} \exp\left\{\frac{1}{2} \int\limits_{i\tau-\infty}^{i\tau+\infty} \left[cth\pi(\alpha - s) + cth\pi s\right] \ln H(s) ds\right\}$$

$$(0 < Jm(\alpha - s) < 1 - \rho, \qquad -1 < \tau < -\rho),$$

$$H(\alpha) = \frac{(ch\pi\alpha - 1)\left(ch\pi\alpha - A_1 - A_2(\alpha + i)^2\right)}{D(\alpha)} \qquad (-1 < Jm\alpha < 0).$$

So, contact tangential forces, with intensities $\tau^{(1)}(y)$ and $\tau^{(2)}(y)$, are completely defined and have the form (21) and (22).

Further, taking into account, that there holds the asymptotic representation

$$\widetilde{C}(\alpha) \simeq \frac{th\pi\sigma}{\sigma}; \quad \text{as} \quad |\sigma| \to \infty \qquad (24)$$

from (22) we get

$$\widetilde{C}(y, a) \equiv \frac{1}{2\pi} \int\limits_{i\tau'-\infty}^{i\tau'+\infty} \widetilde{C}(\alpha) e^{-i\alpha \ln\left(-\frac{y}{a}\right)} d\alpha = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} C(\sigma) e^{-i\sigma \ln\left(-\frac{y}{a}\right)} + r_*(y, a) =$$

$$= \frac{1}{\pi} \ln \left| \frac{\sqrt{-y} + \sqrt{a}}{\sqrt{-y} - \sqrt{a}} \right| + r_* (y, a) \qquad (-\infty < y < 0), \tag{25}$$

where $r_* (y, a)$ is continuous function on variable y in interval $(-\infty, 0)$. Obviously, that $\widetilde{C} (y, a)$ for $y = -a$ has logarithmic singularity. Hence it follows that the tangential contact force $\tau^{(2)} (y)$ at the point $y = -a$ has logarithmic singularity specified by the concentrated force Q_1.

Note, that in the case of infinite plate ($E = E_1$, $\nu = \nu_1$), $\tau_0 (\alpha)$ at the point $\alpha = 0$ has twofold zero.

Then, we can easily see from (21) and (22), that tangential contact forces with intensities $\tau^{(1)} (y)$ $(0 < y < \infty)$ and $\tau^{(2)} (y)$ $(-a < y < 0)$ at the point $y = 0$ have logarithmic singularity specified by nonhomogeneity (piecewise-homogeneity) of infinite stringer.

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