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SOME SPECTRAL PROPERTIES OF A FOURTH ORDER DIFFERENTIAL OPERATOR WITH SPECTRAL PARAMETER IN BOUNDARY CONDITION

Abstract

In the paper we consider the spectral problem

$$y^{(4)}(x) - (q(x)\,y'(x))' = \lambda y(x), \qquad x \in (0,l),$$

$$y(0) = y'(0) = y''(l) = 0,$$

$$Ty(l) = (a\lambda + b)\,y(l),$$

where λ is a spectral parameter, q is an absolutely continuous positive function on interval $[0,\ l]$, $Ty \equiv y''' - qy'$, a , b are real constants with $a > 0$.

The general characteristic of eigenvalues disposition on a real axis (complex plane) is given, the structure of roots subspaces is studied, the oscillation properties of eigenfunctions are investigated, and the asymptotic formulae for eigenvalues and eigenfunctions of this problem are obtained.

Let's consider the spectral problem

$$y^{(4)}(x) - \left(q(x)\,y'(x)\right)' = \lambda y(x), \qquad x \in (0,l), \tag{0.1}$$

$$y(0) = y'(0) = y''(l) = 0, \tag{0.2}$$

$$Ty(l) = (a\lambda + b)\,y(l), \tag{0.3}$$

where λ is a spectral parameter, q is an absolutely continuous positive function on interval $[0,\ l]$, $Ty \equiv y''' - qy'$, a , b are real constants with $a \neq 0$.

Problem (0.1)-(0.3), in case $b = 0$, arises at description of transverse vibrations of pendulum formed of vertically situated homogeneous rod with fixed upper end and the bottom end subjected to the action of tracing force. In particular, the case $a < 0$ describes situation when on the right end of a rod, the additional mass of the quantity $(-a)$ is concentrated. We can find more exact informations on physical meaning of the similar type problems in [1,2].

Boundary value problems with spectral parameter in the boundary condition in different statements were studied in a whole series of papers (see, for example, [3-13]). In [7-13] the basicity in spaces L_p of system of root functions of boundary value problems for ordinary differential operators of the second and fourth orders with spectral parameter in the boundary condition, are investigated.

Note that problem (0.1)-(0.3) in the case $a < 0$ in a more general formulation is considered in [11], and in the case $a > 0$, $b \geq 0$, is considered in [12]. In these papers the oscillation properties of eigenfunctions are investigated, location of eigenvalues on real line is studied, and the basicity in $L_p(0,l)$, $1 < p < \infty$, of the systems of eigenfunctions with one removed eigenfunction was established.

Everywhere hereinafter we assume that the condition $a > 0$ holds.

To study the basis properties in the space $L_p(0,l)$, $1 < p < \infty$, of the systems of root functions of problem (0.1)-(0.3) it has been ascertained conformation between root functions of problem (0.1)-(0.3) and the known system forming basis in $L_p(0,l)$, $1 < p < \infty$, (see, for example, [7-13]). Ascertainment of such conformations real on attraction of asymptotic formulae for eigenfunctions of problem (0.1)-(0.3). To obtain the asymptotic formulae for eigenfunctions it is necessary: general characteristic of location of eigenvalues on real line (complex plane), the structure of roots subspace and oscillation properties of eigenfunctions. Note that reasonings used in [11, 12] don't give full answer to above indicated questions. The present paper is dedicated to detailed investigation of these questions (at this the special attention is given to multiple eigenvalues).

§ 1. Some properties of solutions of problem (0.1), (0.2)

We introduce the boundary condition

$$y(l)\cos\delta - Ty(l)\sin\delta = 0, \quad \delta \in [0,\ \pi/2]. \tag{0.3$'$}$$

Problem (0.1), (0.2), (0.3′) is considered in [14] where, in particular, the folllowing oscillation theorem is proved.

Theorem 1.1(see [14; §5, theorem 5.4 and 5.5]). *The eigenvalues of boundary value problem (0.1), (0.2), (0.3′) are simple and form an infinitely increasing sequence $\{\mu_n(\delta)\}$ such, that $0 \le \mu_1(\delta) < ... < \mu_n(\delta) < ...$. The eigenfunction $\vartheta_n^{(\delta)}(x)$ corresponding to an eigenvalue $\mu_n(\delta)$ has $n-1$ simple zeros in the interval $(0,l)$.*

There holds the following

Lemma 1.1*(see [11; §2, theorem 2.1]). For each fixed $\lambda \in \mathbb{C}$ there exists a unique nontrivial solution $y(x,\lambda)$ of problem (0.1), (0.2) to within constant multiplier .*

Without loosing generality, the solution $y(x,\lambda)$ of problem (0.1), (0,2) for each fixed $x \in [0,l]$ can be considered as entire function of the parameter λ [11; remark 2.1].

Denote: $D_n = (\mu_{n-1}(0), \mu_n(0))$, $n = 1, 2, ...$, where $\mu_0(0) = -\infty$.

Obviously, the eigenvalues $\mu_n(0)$ and $\mu_n(\pi/2)$, $n \in \mathbb{N}$, of boundary value problem (0.1), (0.2), (0.3′) are zeros of the entire functions $y(l,\lambda)$ and $Ty(l,\lambda)$, respectively. Note that the function $F(\lambda) = Ty(l,\lambda)/y(l,\lambda)$ is defined for $\lambda \in D \equiv (\mathbb{C}/\mathbb{R}) \cup \bigcup\limits_{n=1}^{\infty} D_n$ and is a meromorphic function of finite order and the eigenvalues $\mu_n(\pi/2)$ and $\mu_n(0)$, $n \in \mathbb{N}$, of boundary value problem (0.1), (0.2), (0.3′) are zeros and poles of this function, respectively.

Lemma 1.2 *(see [11; §3, lemma 3.1]). Let $\lambda \in D$. Then the equality*

$$\frac{dF(\lambda)}{d\lambda} = \left(\int\limits_0^l y^2(x,\lambda)\,dx\right) / y^2(l,\lambda) \tag{1.1}$$

holds.

Lemma 1.3. *The following asymptotic formula*

$$F(\lambda) = \sqrt[4]{\lambda^3}\frac{\cos\sqrt[4]{\lambda}l}{\cos\sqrt[4]{\lambda}l - \sin\sqrt[4]{\lambda}l}\left[1 + 0\left(\frac{1}{\sqrt[4]{\lambda}}\right)\right] \tag{1.2}$$

is true.

Proof. In equation (0.1) we assume $\lambda = \rho^4$. It is known (see [15; ch.II, i.5, theorem 1]), that in any domain T of complex ρ-plane, equation (0.1) has four linealy independent solutions $z_k(x,\rho)$, $k = \overline{1,4}$, which are regular subject to ρ (for sufficiently great $|\rho|$) and satisfy the relations

$$z_k^{(s)}(x,\rho) = (\rho\omega_k)^s e^{\rho\omega_k x}\left[1 + 0\left(\frac{1}{\rho}\right)\right], \ k = \overline{1,4}, s = \overline{0,3}, \tag{1.3}$$

where ω_k, $k = \overline{1,4}$, are fourth order distinct roots from 1.

Using relation (1.3) and taking into account boundary conditions (0.2) we get

$$y(x,\lambda) = \sin\rho x - \cos\rho x - e^{-\rho x} + (\sin\rho l - \cos\rho l)\, e^{\rho(x-l)} + 0\left(\frac{1}{\rho}\right), \tag{1.4}$$

$$Ty(x,\lambda) = \rho^3\left[-\cos\rho x - \sin\rho x + e^{-\rho x} + (\sin\rho l - \cos\rho l)\, e^{\rho(x-l)} + 0\left(\frac{1}{\rho}\right)\right]. \tag{1.5}$$

From (1.4) and (1.5) we find

$$\frac{Ty(l,\lambda)}{y(l.\lambda)} = \rho^3\frac{\cos\rho l}{\cos\rho l - \sin\rho l}\left(1 + 0\left(\frac{1}{\rho}\right)\right). \tag{1.6}$$

Assuming $\rho = \sqrt[4]{\lambda}$ in (1.6), we get (1.2).

Lemma 1.3 is proved.

By property 1 (see [14; §4]) and lemma 1.2 for $\delta \in (0, \pi/2)$ the relations

$$\mu_1\left(\frac{\pi}{2}\right) < \mu_1(\delta) < \mu_1(0) < \mu_2\left(\frac{\pi}{2}\right) < \mu_2(\delta) < \mu_2(0) < \ldots \tag{1.7}$$

are true.

Lemma 1.4. *The expansion*

$$F(\lambda) = F(0) + \sum_{n=1}^{\infty}\frac{\lambda c_n}{\mu_n(0)(\lambda - \mu_n(0))}, \quad \lambda \in D, \tag{1.8}$$

hods; where c_n, $n \in \mathbb{N}$, are some negative numbers.

Proof. It is known (see [16; ch.6; §5]) that the meromorphic function $F(\lambda)$ with simple poles $\mu_n(0)$ allows the representation

$$F(\lambda) = G(\lambda) + \sum_{n=1}^{\infty}\left(\frac{\lambda}{\mu_n(0)}\right)^{s_n}\frac{c_n}{\lambda - \mu_n(0)}, \tag{1.9}$$

where $G(\lambda)$ is an entire function,

$$c_n = \underset{\lambda=\mu_n(0)}{res} \frac{Ty(l,\lambda)}{y(l,\lambda)} = \frac{Ty(l,\mu_n(0))}{\frac{\partial}{\partial\lambda} y(l,\mu_n(0))}, \tag{1.10}$$

and integers s_n, $n = 1, 2, ...$, are chosen so that series (1.9) be uniformly convergent in any finite circle (after truncation of terms having poles in this circle).

By lemma 1.2 we have: $y(l,\lambda)\, Ty(l,\lambda) < 0$ for $\lambda \in \left(-\infty, \mu_1\left(\frac{\pi}{2}\right)\right)$, $y(l,\lambda)\, Ty(l,\lambda) > 0$ for $\lambda \in \left(\mu_1\left(\frac{\pi}{2}\right), \mu_1(0)\right)$. Without loosing generality, we can assume $y(l,\lambda) > 0$ for $\lambda \in (-\infty, \mu_1)$. Then $Ty(l,\lambda) < 0$ for $\lambda \in \left(-\infty, \mu_1\left(\frac{\pi}{2}\right)\right)$, $Ty(l,\lambda) > 0$ for $\lambda \in \left(\mu_1\left(\frac{\pi}{2}\right), \mu_1(0)\right)$. Since the eigenvalues $\mu_n(\pi/2)$ and $\mu_n(0)$, $n \in \mathbb{N}$, are simple zeros of functions $Ty(l,\lambda)$ and $y(l,\lambda)$, respectively, then by (1.7) the relations

$$(-1)^{n+1}\, Ty(l,\mu_n(0)) > 0, \quad (-1)^{n+1} \frac{\partial y(l,\mu_n(0))}{\partial\lambda} < 0, \quad n \in \mathbb{N}, \tag{1.11}$$

are true.

Taking into account (1.11) in (1.10) we get $c_n < 0$. Asymptotic form (1.2) holds outside of domains $B_n(\varepsilon) = \left\{\lambda \in \mathbb{C} \left| \sqrt[4]{\lambda} - \sqrt[4]{\mu_n(0)} \right| < \varepsilon\right\}$, where $\varepsilon > 0$ is some small number. From asymptotic formula $\sqrt[4]{\mu_n(0)} = \left(n + \frac{1}{4}\right)\pi/l + 0\left(\frac{1}{n}\right)$ (see [11, §6, formula (6.2)]) it follows that for $\varepsilon < \frac{\pi}{4l}$ the domains $B_n(\varepsilon)$ asymptotically don't intersect and contain only one pole $\mu_n(0)$ of the function $F(\lambda)$. Following corresponding reasonings (see [17, chapter VII, §2, i.4]) we see that outside of domains $B_n(\varepsilon)$ the estimation

$$|F(\lambda)| \le M\sqrt[4]{\lambda^3}, \qquad (\, M = const > 0)\,, \tag{1.12}$$

holds, using which in (1.10) we get

$$|c_n| = \left| \frac{1}{2\pi i} \int\limits_{\partial B_n(\varepsilon)} F(\lambda)\, d\lambda \right| = \frac{2}{\pi} \left| \int\limits_{\left|\nu - \sqrt[4]{\mu_n(0)}\right| = \varepsilon} \nu^3 F\left(\nu^4\right) d\nu \right| \le M_1 n^6$$

$$(\, M_1 = const > 0)\,.$$

By asymptotic formula $\sqrt[4]{\mu_n(0)} = \left(n + \frac{1}{4}\right)\pi/l + 0\left(\frac{1}{n}\right)$ the series $\sum\limits_{n=1}^{\infty} |c_n\, \| \mu_n(0)|||^{-2}$ converges. Then according to theorem 2 from [16, chapter 6, §5] in the formula (1.9) we can assume $s_n = 1, n \in \mathbb{N}$.

Let Γ_n be sequence of the expanding circles which are not crossing domains $B_n(\varepsilon)$. Then according to formula (9) [18, chapter V, §13] we have

$$F(\lambda)-F(0)-\sum_{\mu_k(0)\in \text{int}\,\Gamma_n}\frac{\lambda c_k}{\mu_k(0)(\lambda-\mu_k(0))}=\frac{1}{2\pi i}\int_{\Gamma_n}\frac{\lambda F(\xi)}{\xi(\xi-\lambda)}d\xi. \tag{1.13}$$

By (1.12) the right side of equality (1.13) tends to zero as $n\to\infty$. Then from (1.13) we get

$$F(\lambda)=F(0)+\sum_{k=1}^{\infty}\frac{\lambda c_k}{\mu_k(0)(\lambda-\mu_k(0))},$$

whence it follows, that $G(\lambda)\equiv F(0)$.

Lemma 1.4 is proved.

From (1.8) it follows the equalities

$$F'(\lambda)=-\sum_{n=1}^{\infty}\frac{c_n}{(\lambda-\mu_n(0))^2}, \tag{1.14}$$

$$F''(\lambda)=2\sum_{n=1}^{\infty}\frac{c_n}{(\lambda-\mu_n(0))^3}, \tag{1.15}$$

$$F'''(\lambda)=-6\sum_{n=1}^{\infty}\frac{c_n}{(\lambda-\mu_n(0))^4}. \tag{1.16}$$

§2. Structure of root subspace and location of eigenvalues of problem (0.1)-(0.3) on real line (complex plane)

Lemma 2.1. *Problem (0.1)-(0.3) can have at most one pair of adjoint complex eigenvalues..*

Proof. By (0.1) we have

$$(Ty(\mu,x))'\,y(x,\lambda)-(Ty(x,\lambda))'\,y(\mu,x)=(\mu-\lambda)\,y(x,\mu)\,y(x,\lambda).$$

Integrating this equality in the range from 0 to l, using the formula of integration by parts and taking into account (0.2) we obtain

$$y(l,\lambda)\,Ty(l,\mu)-y(l,\mu)\,Ty(l,\lambda)=(\mu-\lambda)\int_0^l y(x,\mu)\,y(x,\lambda)\,dx. \tag{2.1}$$

Let problem (0.1)-(0.3) have two pairs of complex eigenvalues, λ, $\overline{\lambda}$ and μ, $\overline{\mu}$, $\lambda\neq\mu$.

Taking into account (0.3) in (2.1) we have

$$ay(l,\mu)\,y(l,\lambda)=\int_0^l y(x,\mu)\,y(x,\lambda)\,dx. \tag{2.2}$$

$$a\overline{y\left(l,\mu\right)}y\left(l,\lambda\right)=\int\limits_{0}^{l}\overline{y\left(x,\ \mu\right)}y\left(x,\lambda\right)dx. \tag{2.3}$$

Multiplying the both hand sides of equation (0.1) by the function $\overline{y\left(x,\ \lambda\right)}$ and integrating the obtained equality in range from 0 to l, using formula of integration by parts, and also taking into account (0.2), we get

$$b\left|y\left(l,\lambda\right)\right|^{2}+\int\limits_{0}^{l}\left\{\left|y''\left(x,\lambda\right)\right|^{2}+q\left(x\right)\left|y'\left(x,\ \lambda\right)\right|^{2}\right\}dx=$$

$$=\lambda\left[\int\limits_{0}^{l}\left|y\left(x,\ \lambda\right)\right|^{2}dx-a\left|y\left(l,\lambda\right)\right|^{2}\right]. \tag{2.4}$$

Since $\lambda\in\mathbb{C}\backslash\mathbb{R}$, from (2.4) it follows the equality

$$\int\limits_{0}^{l}\left|y\left(x,\ \lambda\right)\right|^{2}dx=a\left|y\left(l,\lambda\right)\right|^{2}. \tag{2.5}$$

Similarly, we get

$$\int\limits_{0}^{l}\left|y\left(x,\ \mu\right)\right|^{2}dx=a\left|y\left(l,\mu\right)\right|^{2}. \tag{2.6}$$

By theorem 1.1 $y\left(l,\mu\right)\neq0$, $y\left(l,\lambda\right)\neq0$ for $\lambda,\mu\in\mathbb{C}\backslash\mathbb{R}$.

Equalities (2.2), (2.3), (2.5), (2.6) can be rewrittten in the following form

$$\int\limits_{0}^{l}\frac{y\left(x,\mu\right)}{y\left(l,\mu\right)}\frac{y\left(x,\lambda\right)}{y\left(l,\lambda\right)}dx=a, \tag{2.2$'$}$$

$$\int\limits_{0}^{l}\overline{\frac{y\left(x,\mu\right)}{y\left(l,\mu\right)}}\frac{y\left(x,\lambda\right)}{y\left(l,\lambda\right)}dx=a, \tag{2.3$'$}$$

$$\int\limits_{0}^{l}\left|\frac{y\left(x,\lambda\right)}{y\left(l,\lambda\right)}\right|^{2}dx=a, \tag{2.5$'$}$$

$$\int\limits_{0}^{l}\left|\frac{y\left(x,\mu\right)}{y\left(l,\mu\right)}\right|^{2}dx=a, \tag{2.6$'$}$$

Summing equalities (2.2$'$) and (2.3$'$) we obtain

$$2\int\limits_{0}^{l}\frac{y\left(x,\lambda\right)}{y\left(l,\lambda\right)}\operatorname{Re}\frac{y\left(x,\mu\right)}{y\left(l,\mu\right)}dx=2a. \tag{2.7}$$

Summing $(2.5')$ and $(2.6')$ and subtracting (2.7) from the obtained equality we find

$$\int_0^l \left\{ \left(\operatorname{Re} \frac{y(x,\mu)}{y(l,\mu)} - \operatorname{Re} \frac{y(x,\lambda)}{y(l,\lambda)} \right)^2 + \operatorname{Im}^2 \frac{y(x,\mu)}{y(l,\mu)} + \right.$$

$$\left. + \operatorname{Im}^2 \frac{y(x,\lambda)}{y(l,\lambda)} - 2i \operatorname{Re} \frac{y(x,\mu)}{y(l,\mu)} \operatorname{Im} \frac{y(x,\lambda)}{y(l,\lambda)} \right\} dx = 0,$$

whence it follows that $\operatorname{Im} \dfrac{y(x,\lambda)}{y(l,\lambda)} = 0$, $\operatorname{Im} \dfrac{y(x,\mu)}{y(l,\mu)} = 0$, which by (0.1) contradict the conditions $\lambda \in \mathbb{C} \backslash \mathbb{R}$, $\mu \in \mathbb{C} \backslash \mathbb{R}$.

Lemma 2.1 is proved.

Lemma 2.2. *Let $\lambda \in \mathbb{R}$ be eigenvalue of boundary value problem (0.1)-(0.3) and $F'(\lambda) \leq a$. Then problem (0.1)-(0.3) has no complex eigenvalues.*

Proof. By (1.1) we have

$$\int_0^l \left(\frac{y(x,\lambda)}{y(l,\lambda)} \right)^2 dx \leq a. \tag{2.8}$$

If $\mu \in \mathbb{C} \backslash \mathbb{R}$ is eigenvalue of problem (0.1)-(0.3), then equalities $(2.2')$, $(2.3')$, $(2.6')$, (2.7) are hold. From $(2.6')$, (2.7), (2.8) we get

$$\int_0^l \left\{ \left(\operatorname{Re} \frac{y(x,\mu)}{y(l,\mu)} - \frac{y(x,\lambda)}{y(l,\lambda)} \right)^2 + \operatorname{Im}^2 \frac{y(x,\mu)}{y(l,\mu)} \right\} dx < 0, \quad \text{if} \quad F'(\lambda) < a,$$

$$\int_0^l \left\{ \left(\operatorname{Re} \frac{y(x,\mu)}{y(l,\mu)} - \frac{y(x,\lambda)}{y(l,\lambda)} \right)^2 + \operatorname{Im}^2 \frac{y(x,\mu)}{y(l,\mu)} \right\} dx = 0, \quad \text{if} \quad F'(\lambda) = a.$$

From the second relation it follows that $\operatorname{Im} \dfrac{y(x,\mu)}{y(l,\mu)} = 0$, which by (0.1) contradicts the condition $\mu \in \mathbb{C} \backslash \mathbb{R}$. The obtained contradictions prove lemma 2.2.

If μ is real eigenvalue of problem (0.1)-(0.3), then by (1.7) we have $y(l,\mu) \neq 0$.

Lemma 2.3. *Let $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$, be eigenvalues of problem (0.1)-(0.3) and $F'(\lambda_1) \leq a$. Then $F'(\lambda_2) > a$.*

Proof. Let $F'(\lambda_2) \leq a$. By (1.1) and (2.2) we have

$$\int_0^l \left(\frac{y(x,\lambda_1)}{y(l,\lambda_1)} \right)^2 dx \leq a, \quad \int_0^l \left(\frac{y(x,\lambda_2)}{y(l,\lambda_2)} \right)^2 dx \leq a,$$

$$\int_0^l \frac{y(x,\lambda_1)}{y(l,\lambda_1)} \frac{y(x,\lambda_2)}{y(l,\lambda_2)} dx = a.$$

Hence we get

$$\int_0^l \left\{ \frac{y(x,\lambda_1)}{y(l,\lambda_1)} - \frac{y(x,\lambda_2)}{y(l,\lambda_2)} \right\}^2 dx < 0, \quad \text{if} \quad F'(\lambda_1) < a \quad \text{or} \quad F'(\lambda_2) < a,$$

$$\int_0^l \left\{ \frac{y(x,\lambda_1)}{y(l,\lambda_1)} - \frac{y(x,\lambda_2)}{y(l,\lambda_2)} \right\}^2 dx = 0, \quad \text{if} \quad F'(\lambda_1) = F'(\lambda_2) = a. \tag{2.9}$$

From (2.9) it follows, that $\dfrac{y(x,\lambda_1)}{y(l,\lambda_1)} = \dfrac{y(x,\lambda_2)}{y(l,\lambda_2)}$, $x \in [0,l]$. Therefore

$$y(l,\lambda_2)\, y(x,\lambda_1) = y(l,\lambda_1)\, y(x,\lambda_2).$$

Since $\lambda_1 \neq \lambda_2$ and $y(l,\lambda_2) \neq 0$, then by (0.1) $y(x,\lambda_1) \equiv 0$. The obtained contradictions prove lemma 2.3.

Note, that the eigenvalues of problem (0.1)-(0.3) are the roots of the equation

$$G(\lambda) \equiv Ty(l,\lambda) - (a\lambda + b)\, y(l,\lambda) = 0. \tag{2.10}$$

If μ is eigenvalue of problem (0.1)-(0.3), we have $y(l,\mu) \neq 0$, and therefore, each root (taking into account multiplicity) of equation (2.10) is a root of the equation

$$F(\lambda) = a\lambda + b. \tag{2.11}$$

Lemma 2.4. *Problem (0.1)-(0.3) can have at most one multiple real eigenvalue. Multiplicities of real eigenvalues of problem (0.1)-(0.3) don't exceed three.*

Proof. Problem (0.1)-(0.3) has multiple real eigenvalue μ just in case, when $F'(\mu) = a$. Then by lemma 2.3 for remaining eigenvalues there holds $F'(\lambda) > a$. Therefore, problem (0.1)-(0.3) can have only one multiple real eigenvalue.

From (1.16) we get $F'''(\lambda) > 0$, $\forall \lambda \in D_n$, $n \in \mathbb{N}$, whence it follows, that multiplicity of real eigenvalue of problem (0.1)-(0.3) doesn't exceed three.

Lemma 2.4 is proved.

Theorem 2.1. *One of the statements holds:*

(i) all eigenvalues of problem (0.1)-(0.3) are real, at that D_1 contains algebraically two (either two simple, or one double) eigenvalues, and D_n, $n = 2,3,...$, contains one simple eigenvalue;

(ii) all eigenvalues of problem (0.1)-(0.3) are real, at that D_1 doesn't contain eigenvalues, meanwhile there exists natural number n_b ($n_b \geq 2$) such that D_{n_b} contains algebraically three (either tree simple, or one simple and one double, or one triple) eigenvalues, and D_n, $n = 2,3,...$, $n \neq n_b$, contains one simple eigenvalue;

(iii) problem (0.1)-(0.3) has one pair adjoint complex eigenvalues, at that D_1 doesn't contain eigenvalues, and D_n, $n = 2,3,...$, contains one simple eigenvalue.

Proof. Let's fix a number a. On the basis of lemma 1.2 the function $F(\lambda)$ increases on the interval D_1. By (1.2) we have $\lim\limits_{\lambda \to -\infty} F(\lambda) = -\infty$, $\lim\limits_{\lambda \to \mu_1(0)-0} F(\lambda) = +\infty$. From (1.15) it follows that $F''(\lambda) > 0$, $\lambda \in D_1$, therefore, the function $F(\lambda)$

is convex on the interval D_1. Since $F(0)<0$, in case of $b>0$, equation (2.11) has exactly two solutions $\lambda_1\in(-\infty,0)$ and $\lambda_2\in(0,\mu_1(0))$, at this $F'(\lambda_1)-a<0$ and $F'(\lambda_2)-a>0$. Then by Cauchy theorem there exists (a unique) point $\tilde{\lambda}$ such that $F'\left(\widetilde{\lambda}\right)=a$. Denote $b_a=F'\left(\widetilde{\lambda}\right)-a\widetilde{\lambda}$. By virtue of convexity of the function $F(\lambda)$ in the interval D_1, equation (2.11) for $b>b_a$ has two simple roots $\lambda_1<\lambda_2$, for $b=b_a$ has one double root $\lambda_1=\widetilde{\lambda}$ (at this the functions $y_1(x)=y(x,\lambda_1)$ and $y^{[1]}(x)=\dfrac{\partial y(x,\lambda_1)}{\partial\lambda}$ form a chain of eigen and associated functions), and for $b<b_a$ has no roots (see, also [12]).

By lemma 1.1 equation (2.11) has at least one solution λ_n^* in the interval D_n, $n=2,3,....$

Let $b>0$. Multiplying both sides of equation (0.1) (supposing $\lambda=\lambda^*$ in (0.1)) by the function $y(x,\lambda_n^*)$ and integrating the obtained equality in the range from 0 to l, using formula of integration by parts, and also taking into account (0.2) we get

$$by^2(l,\lambda_n^*)+\int\limits_0^l\left[y''^2(x,\lambda_n^*)+q(x)\,y'^2(x,\lambda_n^*)\right]dx=$$

$$=\lambda_n^*\left[\int\limits_0^l y^2(x,\lambda_n^*)\,dx-ay^2(l,\lambda_n^*)\right], \tag{2.12}$$

whence it follows that

$$\int\limits_0^l y^2(x,\lambda_n^*)\,dx-ay^2(l,\lambda_n^*)>0.$$

Since $y(l,\lambda_n^*)\neq 0$, hence we get

$$\left(\int\limits_0^l y^2(x,\lambda_n^*)\,dx/y^2(l,\lambda_n^*)\right)-a>0,$$

and according to (1.1)

$$\left.\frac{d}{d\lambda}\left(F(\lambda)-(a\lambda+b)\right)\right|_{\lambda=\lambda_n^*}>0.$$

Thus, the function $F(\lambda)-(a\lambda+b)$, $\lambda\in D_n$, $n=2,3,...$, possesses the value zero steadily increasing. Therefore, equation (2.11) in the interval D_n, $n=2,3,...$, has a unique solution $\lambda_{n+1}=\lambda_n^*$.

Let $b<0$ be any fixed number. By lemma 2.3 either $F'(\lambda_n^*)>a$, $n=2,3,...$, or there exists $n_b\in\mathbb{N}$ $(n_b\geq 2)$ such that $F'\left(\lambda_{n_b}^*\right)\leq a$ and $F'(\lambda_n^*)>a$, $n=2,3,...$, $n\neq n_b$. By (1.2) there exists sufficiently great natural number N $(N>n_b)$ such that the inequalities

$$aR_N+b>0,$$

$$|F(\lambda)-(a\lambda+1)|>|1-b|, \quad \lambda\in S_{R_N}, \tag{2.13}$$

are fulfilled; where $R_N=\mu_N\left(\frac{\pi}{2}\right)+\delta_0$, δ_0 is sufficiently small positive number, $S_{R_N}=\{z\in\mathbb{C}: \ |z|=R_N\}$.

We have

$$\Delta_{S_{R_N}}\arg\left(F(\lambda)-(a\lambda+b)\right)=\Delta_{S_{R_N}}\arg\left(F(\lambda)-(a\lambda+1)\right)+$$
$$+\Delta_{S_{R_N}}\arg\left(1+\frac{1-b}{F(\lambda)-(a\lambda+1)}\right), \tag{2.14}$$

where $\Delta_{S_{R_N}}\arg f(z)=\frac{1}{i}\int\limits_{S_{R_N}}\frac{f'(z)}{f(z)}dz$ (see [18, ch.IV,§10]).

By (2.13) $\left|\frac{1-b}{F(\lambda)-(a\lambda+1)}\right|<1$, $\lambda\in S_{R_N}$, therefore, the point $\omega=(1-b)/(F(\lambda)-(a\lambda+1))$ doesn't go out of circle $\{|\omega|<1\}$. Therefore vector $w=1+\omega$ can't turn around the point $\omega=0$, and the second summand in (2.14) equals zero. Thus,

$$\Delta_{S_{R_N}}\arg\left(F(\lambda)-(a\lambda+b)\right)=\Delta_{S_{R_N}}\arg\left(F(\lambda)-(a\lambda+1)\right). \tag{2.15}$$

By the argument principle (see [18; ch.IV, §10, theorem 1]) we have

$$\frac{1}{2\pi}\Delta_{S_{R_N}}\arg\left(F(\lambda)-(a\lambda+1)\right)=$$
$$\sum_{\lambda_n^{(1)}\in \mathrm{int}S_{R_N}}\alpha\left(\lambda_n^{(1)}\right)-\sum_{\mu_n(0)\in \mathrm{int}S_{R_N}}\beta\left(\mu_n(0)\right), \tag{2.16}$$

where $\alpha\left(\lambda_n^{(1)}\right)$ and $\beta\left(\mu_n(0)\right)$ are multiplicity of zero $\lambda_n^{(1)}$ and pole $\mu_n(0)$ of the function $F(\lambda)-(a\lambda+1)$, respectively. Obviously, that $\sum\limits_{\lambda_n^{(1)}\in \mathrm{int}S_{R_N}}\alpha\left(\lambda_n^{(1)}\right)=N$ and $\sum\limits_{\mu_n(0)\in \mathrm{int}S_{R_N}}\beta\left(\mu_n(0)\right)=N-1$. Then from (2.16) we obtain

$$\frac{1}{2\pi}\Delta_{S_{R_N}}\arg\left(F(\lambda)-(a\lambda+1)\right)=1. \tag{2.17}$$

From (2.17) and (2.15) it follows the validity of equality

$$\frac{1}{2\pi}\Delta_{S_{R_N}}\arg\left(F(\lambda)-(a\lambda+b)\right))=1.$$

Using argument principle again, from the last equality we get,

$$\sum_{\lambda_n\in \mathrm{int}S_{R_N}}\alpha(\lambda_n)-\sum_{\mu_n(0)\in \mathrm{int}S_{R_N}}\beta\left(\mu_n(0)\right)=1,$$

whence it follows

$$\sum_{\lambda_n\in \mathrm{int}S_{R_N}}\alpha(\lambda_n)=N, \tag{2.18}$$

where λ_n, $n \in \mathbb{N}$, are eigenvalues of problem (0.1)-(0.3). From the above mentioned reasonings, by (2.18) we have

$$\sum_{\lambda_s \in \operatorname{int} S_{R_N}} \alpha(\lambda_s) = n, \quad n = N, N+1, ..., \tag{2.19}$$

where $R_n = \mu_n\left(\frac{\pi}{2}\right) + \delta_0$, $S_{R_n} = \{z \in \mathbb{C}: \ |z| = R_n\}$, and therefore problem (0.1)-(0.3) in the interval D_n for $n = N$, $N+1$,..., has only one simple eigenvalue.

From equality (2.19) it follows, that in case $b_a \le b \le 0$ problem (0.1)-(0.3) in the interval D_n, $n = 2, 3, ...$, has one simple eigenvalue: λ_{n+1} for $b_a < b \le 0$; λ_n for $b = b_a$.

Let $b < b_a$. Remind, that at this problem (0.1)-(0.3) has no eigenvalues in the interval D_1. Consider two cases.

Case 1. For all real eigenvalues of problem (0.1)-(0.3) the inequalities $F'(\lambda_n) > a$, $\lambda_n \in \bigcup\limits_{k=2}^{\infty} D_k$ are fulfilled. Then problem (0.1)-(0.3) in every interval D_k, $k = 2, 3, ..., N-1$, has one simple eigenvalue. (Indeed, if problem (0.1)-(0.3) in the interval D_k has two different eigenvalues $\lambda_k^* < \lambda_k^{**}$, then under relations $\lim\limits_{\lambda \to \mu_{k-1}(0)+0} F(\lambda) = -\infty$, $\lim\limits_{\lambda \to \mu_{k-1}(0)-0} F(\lambda) = +\infty$ and $F'(\lambda) > 0$, $\lambda \in D_k$, we get that $F'(\lambda_k^{**}) < a$ $(F'(\lambda_k^*) > a)$, which contradicts the condition $F'(\lambda_k^{**}) > a$). Hence, problem (0.1)-(0.3) in the interval $(0, S_{R_n})$, $n \ge N$, has $n-2$ simple eigenvalues, and hence, by (2.19) problem (0.1)-(0.3) in the circle $S_{R_n} \subset \mathbb{C}$ has one pair of simple complex (adjoint) eigenvalues. In this case location of eigenvalues will be in the following form: λ_1, $\lambda_2 \in \mathbb{C}$, $\lambda_2 = \overline{\lambda}_1$, $\lambda_n \in D_{n-1}$, $n = 3, 4, ...$.

Case 2. $F'\left(\lambda_{n_b}^*\right) \le a$, $F'(\lambda_n) > a$, $\lambda_n \in \mathbb{R}$, $\lambda_n \ne \lambda_{n_b}^*$. By lemma 2.2 problem (0.1)-(0.3) has no complex eigenvalues. From above mentioned reasoning it follows that in each interval D_n, $n \ne n_b$, $n = 2, 3, ...$, problem (0.1)-(0.3) has one simple eigenvalue λ_n^*.

Let $F'\left(\lambda_{n_b}^*\right) = a$, $F''\left(\lambda_{n_b}^*\right) \ne 0$, i.e. the eigenvalue $\lambda_{n_b}^*$ be double. Then from (2.19) it follows, that the interval D_{n_b} besides the eigenvalue $\lambda_{n_b}^*$, contains one more simple eigenvalue $\lambda_{n_b}^{**}$, at this either $\lambda_{n_b-1} = \lambda_{n_b}^*$, $\lambda_{n_b} = \lambda_{n_b}^{**}$ or $\lambda_{n_b-1} = \lambda_{n_b}^{**}$, $\lambda_{n_b} = \lambda_{n_b}^*$. Moreover, $\lambda_n \in D_{n+1}$, $n = 1, 2, .., n_b - 2$, $\lambda_n \in D_n$, $n = n_b + 1$, $n_b + 2, ...$.

Let $F'\left(\lambda_{n_b}^*\right) = a$, $F''\left(\lambda_{n_b}^*\right) = 0$. By (1.16) $F'''\left(\lambda_{n_b}^*\right) \ne 0$. Hence, $\lambda_{n_b}^*$ is triple eigenvalue of problem (0.1)-(0.3), at this the functions $y\left(x, \lambda_{n_b}^*\right)$, $\frac{\partial}{\partial \lambda} y\left(x, \lambda_{n_b}^*\right)$, $\frac{1}{2}\frac{\partial^2}{\partial \lambda^2} y\left(x, \lambda_{n_b}^*\right)$ form a chain of eigen and associated functions. Then from (2.19) it follows, that in the interval D_{n_b} problem (0.1)-(0.3) has unique triple eigenvalue $\lambda_{n_b-1} = \lambda_{n_b}^*$. At this $\lambda_n \in D_{n+1}$, $n \in \mathbb{N}$.

Let $F'\left(\lambda_{n_b}^*\right) < a$, i.e. the eigenvalue $\lambda_{n_b}^*$ be simple. Then by (2.19) we have that the interval D_{n_b} besides the eigenvalue $\lambda_{n_b}^*$ there contains two more simple eigenvalues $\lambda_{n_b}^{**}$ and $\lambda_{n_b}^{***}$. Without loosing generality, we can consider that $\lambda_{n_b}^{**} < \lambda_{n_b}^{***}$. Then it is obvious that $\lambda_{n_b}^{**} < \lambda_{n_b}^* < \lambda_{n_b}^{***}$, at this $\lambda_{n_b-1} = \lambda_{n_b}^{**}$, $\lambda_{n_b} = \lambda_{n_b}^*$, $\lambda_{n_b+1} = \lambda_{n_b}^{***}$. Moreover, $\lambda_n \in D_{n+1}$, $n = 1, 2, .., n_b - 2$, $\lambda_n \in D_{n-1}$, $n = n_b + 2$,

$n_b+3,....$

Theorem 2.1 is proved.

§3. Oscillation properties of eigenfunctions and asymptotic formulae of eigenvalues and eigenfunctions of problem (0.1) - (0.3)

From theorems 3.1 [11; §3] and 2.1 it follows directly

Theorem 3.1. *Let* $\{\lambda_n\}_{n=1}^{\infty}$ *be sequence of eigenvalues of problem (0.1)-(0.3):* $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, *if there holds statement (i) or (ii) of theorem 2.1;* $\lambda_1, \lambda_2 \in \mathbb{C}/\mathbb{R}$, $\lambda_2 = \overline{\lambda}_1$, $\lambda_3 < \lambda_4 < ... < \lambda_n < ...$, *if there holds statement (iii) of theorem 2.1. Then eigenfunction* $y_n(x)$, $n \in \mathbb{N}$, *corresponding to eigenvalue* λ_n, *posesses the following oscillation properties:*

(a) if D_1 *contains two simple eigenvalues* λ_1, λ_2, *then* $y_1(x)$ *has no zeros in the interval* $(0,l)$, *and* $y_n(x)$, $n = 2,3,...$, *has* $n-2$ *simple zeros in the interval* $(0,l)$;

(b) if D_1 *contains one double eigenvalue* λ_1, *then* $y_n(x)$, $n = 1,2,...$, *has* $n-1$ *simple zeros in the interval* $(0,l)$;

(c) if D_{n_b} *contains three simple eigenvalues, then*$y_n(x)$, $n = 1,2,...n_b-1$, *has* n *simple zeros,* $y_{n_b}(x)$, $y_{n_b+1}(x)$ *has* n_b-1 *simple zeros,* $y_n(x)$, $n = n_b+2, n_b+3,...$, *has* $n-2$ *simple zeros in the interval* $(0,l)$;

(d) if D_{n_b} *contains one double eigenvalue, then* $y_n(x)$, $n = 1,2,...n_b-1$, *has* n *simple zeros,* $y_{n_b}(x)$ *has* n_b-1 *simple zeros,* $y_n(x)$, $n = n_b+1,...$, *has* $n-1$ *simple zeros in the interval* $(0,l)$;

(e) if D_{n_b} *contains one triple eigenvalue, then* $y_n(x)$, $n = 1,2,...$, *has* n *simple zeros in the interval* $(0,l)$.

Theorem 3.2. *The following asymptotic formulae*

$$\sqrt[n]{\lambda_n} = (n+\nu)\frac{\pi}{l} + O\left(\frac{1}{n}\right), \tag{3.1}$$

$$y_n(x) = \sin(n+\nu)\frac{\pi}{l}x - \cos(n+\nu)\frac{\pi}{l}x - e^{-(n+\nu)\frac{\pi}{l}x} + O\left(\frac{1}{n}\right), \tag{3.2}$$

where

$$\nu = \begin{cases} -\dfrac{3}{4}, & \textit{if eigenvalues of problem (0.1)-(0.3) are simple,} \\ \dfrac{1}{4}, & \textit{if problem (0.1)-(0.3) has double eigenvalue,} \\ \dfrac{5}{4}, & \textit{if problem (0.1)-(0.3) has triple eigenvalue.} \end{cases}$$

are true.

Proof of theorem 3.2 is holded on scheme of proof of theorem 6.1 from [11] using theorem 2.1 and 3.1.

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