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ON THE APPLICATION OF THE NET METHOD TO THE SOLUTION OF A PROBLEM FOR A PARABOLIC TYPE LINEAR, LOADED DIFFERENTIAL EQUATION

Abstract

Application of the net method to the solution of a problem for a parabolic type linear, loaded differential equation is investigated. The difference problem approximating the initial problem is constructed, and the method of its solution is given. The validity of the maximum principle from which the uniqueness of the solution of the difference problem follows, is proved by fulfilling some conditions. The comparison theorem by means of which the convergence of the solution of the difference problem to the initial value problem is established, is proved simultaneously. The convergence rate of the difference problem is determined.

Problem statement. It is required to find in the closed domain $\overline{D} = \{0 \le x \le l,\ 0 \le t \le T\}$ a continuous function $u = u(x,t)$ satisfying the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + bu(x,t) + b_1 u(\overline{x},t) + f(x,t), \quad 0 < x < l, 0 < t \le T, \tag{1}$$

the boundary conditions

$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \ \ 0 \le t \le T \tag{2}$$

and initial condition

$$u(x,0) = \phi(x),\ 0 \le x \le l, \tag{3}$$

where a, b, b_1 are real numbers, $\overline{x} \in (0,l)$ is a fixed point, $f(x,t)$, $\mu_1(t)$, $\mu_2(t)$ and $\phi(x)$ are the known continuous functions of own arguments.

The equation (1) is a loaded differential equation of parabolic type. It should be noted that we meet loaded differential equations is studying many phenomens of biology [1].

1. Difference of problem and its error. In a closed domain $\overline{D}$ we introduce a net

$$\overline{\omega}_{h\tau} = \{(x_j, t_n),\ x_j = jh,\ t_n = n\tau,\ j = 0,1,...,N,\ n = 0,1,...,n_0\},$$

with steps $h = l/N$ and $\tau = T/n_0$. By y_j^n we denote the value of the net function y in the node (x_j, t_n) of the net $\overline{\omega}_{h\tau}$. On this net we associate to the problem (1)-(3) the difference problem

$$\begin{gathered} y_{\bar{t},j}^n = a^2 \Lambda \left(\sigma y_j^n + (1-\sigma) y_j^{n-1} \right) + b \frac{y_j^n + y_j^{n-1}}{2} + b_1 \frac{y_{j_0}^n + y_{j_0}^{n-1}}{2} + \varphi_j^n, \\ j = 1,2,...,N-1, n = 1,2,...,n_0, \end{gathered} \tag{1.1}$$

$$y_0^n = \mu_1(t_n), \quad y_N^n = \mu_2(t_n), \quad n = 0, 1, ..., n_0, \tag{1.2}$$

$$y_j^0 = \phi(x_j), \ j = 0, 1, ..., N, \tag{1.3}$$

where

$$j_0 = \overline{x}/h, \qquad \varphi_j^n = f(x_j, t_n - 0,5\tau).$$

Here we use the denotation accepted in [2].

Let $C_k^m(D)$ be a class of functions having k continuous derivatives with respect to x and m continuous derivatives with respect to t in a domain D.

The difference problem (1.1)-(1.3) has an approximation $O\left(h^2+\tau^2\right)$, if $\sigma = 0.5$, $u(x,t) \in C_4^3(D)$ and $O\left(h^2+\tau\right)$, if $\sigma \neq 0.5$, $u(x,t) \in C_4^2(D)$.

2. Solution of the difference problem. We rewrite the difference problem (1.1)-(1.3) in the form

$$-\frac{a^2\sigma}{h^2}y_{j-1}^n + \left(\frac{1}{\tau} + \frac{2a^2\sigma}{h^2} - \frac{b}{2}\right)y_j^n - \frac{a^2\sigma}{h^2}y_{j+1}^n - \frac{b_1}{2}y_{j_0}^n = F_j^{n-1},$$
$$j = 1, 2, ..., N-1, \qquad n = 1, 2, ..., n_0, \tag{2.1}$$

$$y_0^n = \mu_1(t_n), \quad y_N^n = \mu_2(t_n), \quad n = 0, 1, ..., n_0, \tag{2.2}$$

$$y_j^0 = \phi(x_j), \quad j = 0, 1, ..., N, \tag{2.3}$$

where

$$F_j^{n-1} = \frac{a^2(1-\sigma)}{h^2}\left(y_{j-1}^{n-1} - 2y_j^{n-1} + y_{j+1}^{n-1}\right) + \frac{1}{\tau}y_j^{n-1} + \frac{b}{2}y_j^{n-1} + \frac{b_1}{2}y_{j_0}^{n-1} + \varphi_j^n,$$
$$j = 1, 2, ..., N-1, \quad n = 1, 2, ..., n_0.$$

Let's consider the following three-point difference problems with respect to u_j^n and ν_j^n:

$$\left(\frac{1}{\tau} + \frac{2a^2\sigma}{h^2} - \frac{b}{2}\right)u_1^n - \frac{a^2\sigma}{h^2}u_2^n = F_1^{n-1} + \frac{a^2\sigma}{h^2}\mu_1(t_n),$$
$$-\frac{a^2\sigma}{h^2}u_{j-1}^n + \left(\frac{1}{\tau} + \frac{2a^2\sigma}{h^2} - \frac{b}{2}\right)u_j^n - \frac{a^2\sigma}{h^2}u_{j+1}^n = F_j^{n-1}, \quad j = 2, 3, ..., N-2, \tag{2.4}$$
$$-\frac{a^2\sigma}{h^2}u_{N-2}^n + \left(\frac{1}{\tau} + \frac{2a^2\sigma}{h^2} - \frac{b}{2}\right)u_{N-1}^n = F_{N-1}^{n-1} + \frac{a^2\sigma}{h^2}\mu_2(t_n);$$

$$\left(\frac{1}{\tau} + \frac{2a^2\sigma}{h^2} - \frac{b}{2}\right)\nu_1^n - \frac{a^2\sigma}{h^2}\nu_2^n = \frac{b_1}{2},$$
$$-\frac{a^2\sigma}{h^2}\nu_{j-1}^n + \left(\frac{1}{\tau} + \frac{2a^2\sigma}{h^2} - \frac{b}{2}\right)\nu_j^n - \frac{a^2\sigma}{h^2}\nu_{j+1}^n = \frac{b_1}{2}, \quad j = 2, 3, ..., N-2, \tag{2.5}$$
$$-\frac{a^2\sigma}{h^2}\nu_{N-2}^n + \left(\frac{1}{\tau} + \frac{2a^2\sigma}{h^2} - \frac{b}{2}\right)\nu_{N-1}^n = \frac{b_1}{2}.$$

It should be noted that the solution of the last two difference problems may be found, for example, by the known transfer method.

Lemma. *If u_j^n satisfies the problem (2.4) and ν_j^n - the problem (2.5), the function*

$$y_j^n = u_j^n + y_{j_0}^n\nu_j^n, \quad j = 1, 2, ..., N-1, \tag{2.6}$$

satisfies the problem (2.1)-(2.2).

We note that for proving the lemma in difference equations (2.1) at first we have to take into account the conditions (2.2) and then to substitute the expression y_j^n defined by the equality (2.6) into these equations.

Using the lemma, for each value n, beginning with the first one, we can find the solution of the problem (2.1)-(2.3) by means of the following algorithm:

1) we find the solutions of the problems (2.4) and (2.5) by the transfer method;

2) assuming $j = j_0$ in the equality (2.6), we find $y_{j_0}^n = u_{j_0}^n / \left(1 - \nu_{j_0}^n\right)$;

3) we find the solution of the problem (2.1)-(2.3) allowing for the found value of $y_{j_0}^n$ by the equality (2.6).

Remark. For $n = 1$ at the right hand sides of difference equations (2.4) we take into account the values y_j^0, $j = 0, 1, ..., N$ determined by the equality (2.3). For $n > 1$ at the right hand sides of these equations the values y_j^{n-1}, $j = 0, 1, ..., N$ found at the previous value of n, are taken into account.

3.Convergence of the difference problem. We rewrite the problem (1.1)-(1.3) in the form:

$$\frac{a^2\sigma}{h^2} y_{j-1}^n - \left(\frac{2a^2\sigma}{h^2} + \frac{1}{\tau} - \frac{b}{2}\right) y_j^n + \frac{a^2\sigma}{h^2} y_{j+1}^n + \frac{b_1}{2} y_{j_0}^n + \frac{a^2(1-\sigma)}{h^2} y_{j-1}^{n-1} -$$

$$- \left(\frac{2a^2(1-\sigma)}{h^2} - \frac{1}{\tau} - \frac{b}{2}\right) y_j^{n-1} + \frac{a^2(1-\sigma)}{h^2} y_{j+1}^{n-1} + \frac{b_1}{2} y_{j_0}^{n-1} = -\varphi_j^n, \tag{3.1}$$
$$j = 1, 2, ..., N-1, \quad n = 1, 2, ..., n_0,$$

$$y_0^n = \mu_1(t_n), \quad y_N^n = \mu_2(t_n), \quad n = 0, 1, ..., n_0, \tag{3.2}$$

$$y_j^0 = \phi(x_j), \ j = 0, 1, ..., N. \tag{3.3}$$

Let a parameter σ, the coefficients b, b_1 and the steps h, τ of the net $\overline{\omega}_{h\tau}$ entering into the difference equation (3.1) satisfy the conditions

$$0 \le \sigma \le 1, \ b_1 > 0, \ b + b_1 \le 0, \ \tau \le 2h^2 / \left(4a^2(1-\sigma) - bh^2\right). \tag{3.4}$$

Theorem 1 (The maximum principle). *Let the conditions (3.4) be fulfilled. If* $\varphi_j^n \le 0$ $\left(\varphi_j^n \ge 0\right)$, $j = 1, 2, ..., N-1$, $n = 1, 2, ..., n_0$, *the solution* y_j^n *of the problem (3.1)-(3.3) differ from the constant can't accept the greatest positive (the greatest negative) value for* $j = 1, 2, ..., N-1$, $n = 1, 2, ..., n_0$.

Proof. Let's prove the first part of the theorem. Let $\varphi_j^n \le 0$, $j = 1, 2, ..., N-1$, $n = 1, 2, ..., n_0$ and there exist a node (x_m, t_k), $1 \le m \le N-1$, $1 \le k \le n_0$ where the solution of the problem (3.1)-(3.3) accepts the greatest positive value:

$$y_m^k = \max_{\overline{\omega}_{h\tau}} y_j^n = M > 0.$$

Not losing generality, we can consider that even if in one of neighboring nodes $(x_{m\pm1}, t_k)$, $(x_{m\pm1}, t_{k-1})$ and (x_m, t_{k-1}) the value of the net function y_j^n is less than y_m^k.

Consider the equation (3.1) for $j = m$ and $n = k$. Under our assumptions we have:

$$-\varphi_m^k = \frac{a^2\sigma}{h^2} y_{m-1}^k - \left(\frac{2a^2\sigma}{h^2} + \frac{1}{\tau} - \frac{b}{2}\right) y_m^k + \frac{a^2\sigma}{h^2} y_{m+1}^k + \frac{b_1}{2} y_{j_0}^k + \frac{a^2(1-\sigma)}{h^2} y_{m-1}^{k-1} -$$

$$-\left(\frac{2a^2(1-\sigma)}{h^2} - \frac{1}{\tau} - \frac{b}{2}\right) y_m^{k-1} + \frac{a^2(1-\sigma)}{h^2} y_{m+1}^{k-1} + \frac{b_1}{2} y_{j_0}^{k-1} < (b + b_1) M \le 0,$$

i.e. $\varphi_m^k > 0$ that contradicts the condition $\varphi_m^k \le 0$.

The first part of the theorem is proved. We can prove the second part of the theorem in a similar way.

Corollary 1. *Let the conditions (3.4) be fulfilled. If* $\varphi_j^n \le 0$ $\left(\varphi_j^n \ge 0\right)$, $j = 1, 2, ..., N-1$, $n = 1, 2, ..., n_0$, *the solution of the problem (3.1)-(3.3) is not positive (not negative).*

Corollary 2. *Let the conditions (3.4) be fulfilled. Then the problem (3.1)-(3.3) for* $\varphi_j^n = 0$, $j = 1, 2, ..., N-1$, $n = 1, 2, ..., n_0$, $y_0^n = 0$, $y_N^n = 0$, $n = 0, 1, ..., n_0$, $y_j^0 = 0$, $j = 0, 1, ..., N$, *has only a trivial solution* $y_j^n = 0$, $j = 0, 1, ..., N$, $n = 0, 1, ..., n_0$, *and so the problem (3.1)-(3.3) is uniquely solvable for any* φ_j^n, $\mu_1(t_n)$, $\mu_2(t_n)$ *and* $\phi(x_j)$.

Theorem 2 (Comparison theorem). *Let* y_j^n *be a solution of the problem (3.1)-(3.3),* $\widetilde{y}_j^n$ *be a solution of the problem in substituting of the function* φ_j^n, $\mu_1(t_n)$, $\mu_2(t_n)$ *and* $\phi(x_j)$, *respectively for* $\widetilde{\varphi}_j^n$, $\widetilde{\mu}_1(t_n)$, $\widetilde{\mu}_2(t_n)$ *and* $\widetilde{\phi}(x_j)$ *in (3.1)-(3.3). If the conditions (3.4) are fulfilled and* $\left|\varphi_j^n\right| \le \widetilde{\varphi}_j^n$, $j = 1, 2, ..., N-1$, $n = 1, 2, ..., n_0$, $|\mu_1(t_n)| \le \widetilde{\mu}_1(t_n)$, $|\mu_2(t_n)| \le \widetilde{\mu}_2(t_n)$, $n = 0, 1, ..., n_0$, $|\phi(x_j)| \le \widetilde{\phi}(x_j)$, $j = 0, 1, ..., N$, *then* $\left|y_j^n\right| \le \widetilde{y}_j^n$, $j = 0, 1, ..., N$, $n = 0, 1, ..., n_0$.

Using the comparison theorem we get an estimation for solving the problem (3.1)-(3.3). To this end we introduce the auxiliary function

$$w_j^n = K e^{t_n}, \quad j = 0, 1, ..., N, n = 0, 1, ..., n_0, \tag{3.5}$$

where $K > 0$ is a constant.

For w_j^n, allowing for $e^{\tau} - 1 = \tau \cdot e^{\eta\tau}$, $0 < \eta < 1$, we have:

$$\frac{a^2\sigma}{h^2} w_{j-1}^n - \left(\frac{2a^2\sigma}{h^2} + \frac{1}{\tau} - \frac{b}{2}\right) w_j^n + \frac{a^2\sigma}{h^2} w_{j+1}^n + \frac{b_1}{2} w_{j_0}^n + \frac{a^2(1-\sigma)}{h^2} w_{j-1}^{n-1} -$$

$$-\left(\frac{2a^2(1-\sigma)}{h^2} - \frac{1}{\tau} - \frac{b}{2}\right) w_j^{n-1} + \frac{a^2(1-\sigma)}{h^2} w_{j+1}^{n-1} + \frac{b_1}{2} w_{j_0}^{n-1} < -K, \tag{3.6}$$

$$j = 1, 2, ..., N-1, \; n = 1, 2, ..., n_0,$$

$$w_0^n \ge K, \quad w_N^n \ge K, \quad n = 0, 1, ..., n_0, \tag{3.7}$$

$$w_j^0 = K, \quad j = 0, 1, ..., N. \tag{3.8}$$

Let

$$K = \max\left\{\max_{\overline{D}} |f(x,t)|, \max_{0 \le t \le T} (|\mu_1(t)|, |\mu_2(t)|), \max_{0 \le x \le l} |\phi(x)|\right\}. \tag{3.9}$$

Then we compare (3.1)-(3.3) with (3.6)-(3.8) and by the comparison theorem we get the validity of the following theorem.

Theorem 3. *Let the conditions (3.4) be fulfilled. Then, for the solution of the problem (3.1)-(3.3) it holds the estimation*

$$\left|y_j^n\right| \leq K \cdot e^T, \quad j = 0,1,...,N, \quad n = 0,1,...,n_0, \tag{3.10}$$

where$K$ is detemined by the equality (3.9).

Let $u\left(x_j, t_n\right)$ be a value of the exact solution of the problem (1)-(3) in the node $\left(x_j, t_n\right)$ of the net $\overline{\omega}_{h\tau}$, and y_j^n be a solution of the difference problem (3.1)-(3.3). On the net $\overline{\omega}_{h\tau}$ we define the net function z_j^n by the equality $z_j^n = y_j^n - u\left(x_j, t_n\right)$. If we substitute the expression $y_j^n = z_j^n + u\left(x_j, t_n\right)$ obtained from this equality into (3.1)-(3.3) then for z_j^n we get:

$$\frac{a^2\sigma}{h^2} z_{j-1}^n - \left(\frac{2a^2\sigma}{h^2} + \frac{1}{\tau} - \frac{b}{2}\right) z_j^n + \frac{a^2\sigma}{h^2} z_{j+1}^n + \frac{b_1}{2} z_{j_0}^n + \frac{a^2\left(1-\sigma\right)}{h^2} z_{j-1}^{n-1} -$$

$$- \left(\frac{2a^2\left(1-\sigma\right)}{h^2} - \frac{1}{\tau} - \frac{b}{2}\right) z_j^{n-1} + \frac{a^2\left(1-\sigma\right)}{h^2} z_{j+1}^{n-1} + \frac{b_1}{2} z_{j_0}^{n-1} = -\psi_j^n, \tag{3.11}$$

$$j = 1,2,...,N-1, \; n = 1,2,...,n_0,$$

$$z_0^n = 0, \quad z_N^n = 0, \quad n = 0,1,...,n_0, \tag{3.12}$$

$$z_j^0 = 0, \quad j = 0,1,...,N. \tag{3.13}$$

where

$$\psi_j^n = a^2\left(\sigma - 0.5\right)\tau \frac{\partial^3 u\left(x_j, \bar{t}_n\right)}{\partial x^2 \partial t} + O\left(h^2 + \tau^2\right), \quad j = 1,2,...,N, n = 1,2,...,n_0$$

is the approximation error of the difference scheme (3.1)-(3.3) to the solution of the equation (1).

By theorem 3, for the solution of the problem (3.11)-(3.13) we have:

$$\left|z_j^n\right| \leq K \cdot e^T, \quad j = 0,1,...,N, \quad n = 0,1,...,n_0,$$

where

$$K = \max_{\substack{1\leq j\leq N-1 \\ 1\leq n\leq n_0}} \left|\psi_j^n\right|.$$

Theorem 4. *Let the conditions (3.4) be fulfilled. If the solution of the problem (1)-(3) - the function $u = u\left(x,t\right)$ satisfies the condition $u \in C_4^3\left(D\right)$, then as $h \to 0$ and $\tau \to 0$ the solution of the differnce problem (3.1)-(3.3) converges to the solution of the problem (1)-(3). And it holds the estimation*

$$\left|y_j^n - u\left(x_j, t_n\right)\right| \leq K_1\left(h^2 + \tau^2\right) \quad \text{if} \;\; \sigma = 0.5$$

$$\left|y_j^n - u\left(x_j, t_n\right)\right| \leq K_2\left(h^2 + \tau\right) \quad \text{if} \;\; \sigma \neq 0.5.$$

Here K_1 and K_2 are positive constants whose values depend on maximums of modules on partial derivatives of the function $u\left(x,t\right)$ with respect to t up to the third, with respect to x up to the fourth orders in the domain D.

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Received July 2, 2007; Revised October 22, 2007.