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GLOBAL SOLVABILITY OF THE CAUCHY PROBLEM FOR THE SEMILINEAR HYPERBOLIC EQUATION WITH ANISOTROPIC ELLIPTIC PART

Abstract

In the papers [1]-[8] *the existence of global solutions was investigated for the semilinear dissipative equations. In this paper we investigate global solvability of the Cauchy problem for the semilinear dissipative hyperbolic equations with anisotropic elliptic part.*

We consider the Cauchy problem for the semilinear dissipative hyperbolic equation with anisotropic elliptic pats

$$u_{tt} + u_t + \sum_{k=1}^{3} (-1)^{l_k} D_{x_k}^{2l_k} u = f(u), \quad t > 0, \quad x \in R_3, \tag{1}$$

$$u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x), \quad x \in R_3, \tag{2}$$

where $D_{x_k} = \dfrac{\partial}{\partial x_k}$, $k = 1,2,3$. Here $f(u)$ is continuously differentable function and

$$|f(u)| \le c\,|u|^p, \quad |f'(u)| \le c\,|u|^{p-1}, \quad u \in R, \tag{3}$$

where $c > 0$,

$$p > 1 + \frac{2l_1l_2l_3}{l_1l_2 + l_1l_3 + l_2l_3}, \quad for \quad l_1l_2 + l_1l_3 + l_2l_3 \le 2l_1l_2l_3, \tag{4}$$

$$2 < p < \frac{l_1l_2 + l_1l_3 + l_2l_3}{l_1l_2 + l_1l_3 + l_2l_3 - 2l_1l_2l_3}, \quad for \quad 2l_1l_2l_3 < l_1l_2 + l_1l_3 + l_2l_3 \le 4l_1l_2l_3. \tag{5}$$

Let $W_2^l(R_3)$ is anisotropic Sobolev space with the norm

$$\|u\|_{W_2^1(R_3)} = \left(\sum_{k=1}^{3} \int_{R_3} \left| D_{x_k}^{l_k} u \right|^2 dx + \int_{R_3} |u|^2 \, dx \right)^{\frac{1}{2}},$$

where $l = (l_1, l_2, l_3)$ (see [9]).

The main results of this paper is the following theorem

Theorem 1. *Suppose that the condition (3)-(5) are satisfied. Then there exists a real number $\delta_0 > 0$, such that, if the initial data $(\varphi, \psi) \in \left[W_2^l(R_3) \cap L^1(R_3)\right] \times$ $\times \left[L_2(R_3) \cap L_1(R_3)\right]$ further satisfied $\|\varphi\|_{W_2^l(R_3)} + \|\varphi\|_{L_1(R_3)} + \|\psi\|_{L_2(R_3)} + \|\psi\|_{L_1(R_3)} <$ $< \delta_0$ the problem (1), (2) admits a global solution $u(t,x) \in C\left([0,\infty); W_2^1(R_3)\right) \cap$ $C^1\left([0,\infty); L_2(R_3)\right)$ satisfying the decay property:*

$$\sum_{k=1}^{3} \left\| D_{x_k}^{l_k} u(t,\cdot) \right\|_{L_2(R_3)} \le c(1+t)^{-\left(\frac{1}{2} + \frac{l_1l_2 + l_1l_3 + l_2l_3}{4\; l_1l_2l_3}\right)}, \tag{6}$$

$$\|u(t,\cdot)\|_{L_2(R_3)} \le c(1+t)^{-\frac{l_1l_2 + l_1l_3 + l_2l_3}{4\; l_1l_2l_3}}, \tag{7}$$

$$\|D_t u(t,\cdot)\|_{L_2(R_3)} \le c(1+t)^{-\left(1+\frac{l_1l_2+l_1l_3+l_2l_3}{4\ l_1l_2l_3}\right)}, \tag{8}$$

with $c>0$ some constant $t>0$.

Remark. In case $l_1 = l_2 = l_3 = 1$, the analogous result was received in paper [4].

By substitution $\nu_1 = u_1, \ \ \nu_2 = u_t$ we can reduce problem (1), (2) to the Cauchy problem

$$w' = Aw + F(w), \tag{9}$$

$$w(0) = w_0 \tag{10}$$

in the Hilbert space $H = W_2^l(R_3) \times L_2(R_3)$, where $w = (\nu_1, \nu_2)$,

$$A = \begin{pmatrix} 0 & I \\ -\sum\limits_{k=1}^{3}(-1)^{l_k} D_{x_k}^{2l_k} & -I \end{pmatrix}, \ \ D(A) = W_2^{2l}(R_3) \times W_2^l(R_3);$$

$F(w) = (0, f(\nu_1))$, I-identity operator in $L_2(R_3)$.

Lemma 1. *The operator $A - J$ generates a strongly continuous contraction semi-group in the space H, where*

$$J = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Proof. From definitions of Hilbert space H and linear operator A it, follows that

$$\begin{gathered} < Aw, w >_H = \sum_{k=1}^{3} \int\limits_{R_3} D_{x_k}^{l_k}\nu_2 D_{x_k}^{l_k}\nu_1 dx + \int\limits_{R_3} \nu_2\nu_1 dx - \\ - \int\limits_{R_3} \sum_{k=1}^{3} (-1)^{l_k} D_{x_k}^{2l_k}\nu_1\nu_2 dx - \int\limits_{R_3} \nu_2\nu_2 dx \le \int\limits_{R_3} \nu_1^2 dx. \end{gathered} \tag{11}$$

Consider the equation

$$Aw - w = F, \tag{12}$$

where $F = (f_1,f_2) \in H$. From (12) we get

$$\nu_2 - \nu_1 = f_1, \tag{13}$$

$$-\sum_{k=1}^{3} (-1)^{l_k} D_{x_k}^{l_k}\nu_1 - 2\nu_2 = f_2. \tag{14}$$

From (13), (14) follows that

$$\widehat{\nu}_1 = \frac{1}{|\xi|_l + 2}\left(\widehat{f}_2 + 2\widehat{f}_1\right), \quad \widehat{\nu}_2 = \widehat{\nu}_1 + \widehat{f}_1,$$

where $\widehat{\nu}_2 = F[\nu_1]$, $i = 1{,}2$, F is Fourier transformation, $|\xi|_l = \sum\limits_{k=1}^{N} \xi_k^{2l_k}$.

Then form of the definition the norm in the functional space $W_2^l(R_3)$, follows that

$$\|\nu_1\|_{W_2^l}^2 = \int\limits_{R_3} (1 + |\xi|_l) \, |\widehat{\nu}_1(\xi)|^2 \, d\xi \le \left(c\,\|f_2\|_{L_2(R_3)} + 2\,\|f_1\|_{L_2(R_3)}\right) \le c_1 \|F\|_H\,,$$

$$\|\nu_2\|_{L_2(R_3)} \le c\,\|F\|_H$$

Thus $A-J$ is dissipative and invertable operator, i.e., $A-J$ is maximal dissipative operator. It is well known that maximal dissipative operator generates a continuous contraction semi-group (see [12]).

Using the embedding theorem for the anisotropic space and taking into account (3), (5) we obtain the following Lemma.

Lemma 2. *The nonlinear operator $F(w)$ is acting from H to H and satisfieds the local Lipchitz condition in the following sense: for arbitrary w^1,$w^2 \in H$*

$$\left\|F(w^1) - F(w^2)\right\|_H \leq c(r) \cdot \left\|w^1 - w^2\right\|_H ,$$

where $r = \left\|w^1\right\|_H + \left\|w^2\right\|_H$, $c(r) \in C(R_+)$.

From Lemma 1 and 2 the condition of local solvability theorem is satisfied for the Cauchy problem (9), (10). Therefore the following theorem is true.

Theorem 2. *Suppose that $f(u)$ is continuously differentable and in case $2l_1l_2l_3 \leq l_1l_2 + l_1l_3 + l_2l_3$ the condition (3) is satisfied, where $p \in [1, \infty)$, for $2l_1l_2l_3 = l_1l_2+l_1l_3+l_2l_3$ and $p \in \left[1, \dfrac{l_1l_2 + l_1l_3 + l_2l_3}{l_1l_2 + l_1l_3 + l_2l_3 - 2l_1l_2l_3}\right]$, for $l_1l_2+l_1l_3+l_2l_3 > 2l_1l_2l_3$.*

Then for each $(\varphi, \psi) \in H$ there exists $T_0 = T(\varphi, \psi)$ such that the problem (1), (2) has a unique solution

$$u(\cdot) \in C\left([0, T_0]\,; W_2^l(R_3)\right) \cap C^1\,[0, T_0]\,;\ L_2(R_3)).$$

Moreover u can be unequaly continued to a maximal solution which defined in interval $[0.T')$, and at least one of the following statements is valid

(i) $T' = \infty$

(ii) $\lim\limits_{t\to T'-0} E(u,t) = +\infty$,

where $E(u,t) = \|u(t,\cdot)\|_{W_2^l(R_3)} + \|u(t,\cdot)\|_{L_2(R_3)}$.

Now let's investigate the behavior solution of corresponding linear problems.

In order to solve it, we consider the following Cauchy problem

$$u_{tt} + u_t + \sum_{k=1}^{3} (-1)^{l_k} D_{x_k}^{2l_k} u = 0, \quad t > 0, \quad x \in R_3, \tag{15}$$

$$u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x), \quad x \in R_3, \tag{16}$$

For each $\varphi \in W_2^l(R_3)$, $\psi \in L_2(R_3)$ the problem (15)-(16) has a unique solution

$$u(\cdot) \in C\left([0,\infty)\,;\quad W_2^l(R_3)\right) \cap C^1\left([0,\infty)\,;\quad L_2(R_3)\right).$$

Using the Fourier transformation, Plancherel theorem and the Hausdorf-Young inequality we have the following result:

Theorem 3. *Let $\varphi \in W_2^{m_1l}(R_3) \cap L_r(R_3)$, $\psi \in W_2^{(m_1-1)l}(R_3) \cap L_r(R_3)$, $1 \leq r \leq 2$, where $\lambda l = (\lambda l_1, \lambda l_2, \lambda l_3)$, $\lambda \in R$. Then the weak solution of the problem (15)-(16), has the decay estimates*

$$\begin{aligned}\left\|D_t^i D_x^{\alpha} u(t,\cdot)\right\|_{L_2(R_3)} &\leq ce^{-t}\left[\|\varphi\|_{W_2^{m_1l}(R_3)} + \|\psi\|_{W_2^{(m_1-1)l}(R_3)}\right] + \\ &+c(1+t)^{-\left(i+\frac{\alpha_1l_2l_3+l_1\alpha_2l_3+l_1l_2\alpha_3}{2l_1l_2l_3}+\frac{(2-r)}{r}\frac{(l_1l_3+l_1l_3+l_2l_3)}{l_1l_2l_3}\right)}\left(\|\varphi\|_{L_r(R_3)}+\|\psi\|_{L_r(R_3)}\right),\end{aligned} \tag{17}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_k = 1, 2, ..., k = 1, 2, 3;$ $i = 1, 2, ...,$ $m_1 = \frac{\alpha_1}{l_1} + \frac{\alpha_2}{l_2} + \frac{\alpha_3}{l_3} + i$, $c > 0$ *doesn't depend form* $t \geq 0$, φ *and* ψ.

Proof of theorem 1. The solutions of the probelm (1)-(2) can be represented as the following form:

$$u(t,x) = u_0(t,x) + \int_0^t u_1(t-\tau, x) * f(u(\tau, x))d\tau. \tag{18}$$

where $*$ is convolution to x, variable $u_0(t,x)$ is solution of the problem (15)-(16) and $u_1(t,x) = F^{-1}[u_1](t,x)$

Here F^{-1} is inverse Fourye transformation, $\widehat{u}_1(t,\xi)$ is solution of the following Cauchy problem

$$\widehat{u}_{1_{tt}}(t,\xi) + \widehat{u}_{1_t}(t,\xi) + |\xi|_l \,\widehat{u}_1(t,\xi) = 0, \ t > 0, \ \xi \in R_3,$$

$$\widehat{u}_1(0,\xi) = 0, \ \widehat{u}_{1_t}(0,\xi) = 1, \ \xi \in R_3.$$

Using the Theorem 3, from (18) we have

$$\begin{aligned}
&\|u(t,\cdot)\|_{L_2(R_3)} \leq ce^{-t}\left[\|\varphi\|_{L_2(R_3)} + \|\psi\|_{L_2(R_3)}\right] + \\
&+c(1+t)^{-\frac{l_1l_2+l_1l_3+l_2l_3}{4\ l_1l_2l_3}}\left[\|\varphi\|_{L_1(R_3)} + \|\psi\|_{L_1(R_3)}\right] + \\
&+c\int_0^t \left[e^{-(t-\tau)}\|f(u)\|_{L_3(R_3)} + (1+t-\tau)^{-\frac{l_1l_2+l_1l_3+l_2l_3}{4\ l_1l_2l_3}} + \right.\\
&\left.+(1+t-\tau)^{-\frac{l_1l_2+l_1l_3+l_2l_3}{4\ l_1l_2l_3}}\|f(u)\|_{L_1(R_3)}\right]d\tau \leq c(1+t)^{-\frac{l_1l_2+l_1l_3+l_2l_3}{4\ l_1l_2l_3}} \times \\
&\times\left[\|\varphi\|_{L_2(R_3)} + \|\psi\|_{L_2(R_3)} + \|\varphi\|_{L_1(R_3)} + \|\psi\|_{L_1(R_3)}\right] + \\
&+c\int_0^t (1+t-\tau)^{-\frac{l_1l_2+l_1l_3+l_2l_3}{4\ l_1l_2l_3}}\left[\|u\|^p_{L_{2p}(R_3)} + \|u\|^p_{L_p(R_3)}\right]d\tau.
\end{aligned} \tag{19}$$

On the other hand as well from theorem 3 it follows that

$$\begin{aligned}
&\sum_{k=1}^3 \left\|D_{x_k}^{l_k} u(t,\cdot)\right\|_{L_2(R_3)} \leq c(1+t)^{-\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4\ l_1l_2l_3}\right)} \times \\
&\times\left[\|\varphi\|_{W_2^l(R_3)} + \|\psi\|_{L_2(R_3)} + \|\varphi\|_{L_1(R_3)} + \|\psi\|_{L_1(R_3)}\right] + \\
&+c\int_0^t (1+t-\tau)^{-\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4\ l_1l_2l_3}\right)}\left(\|u\|^p_{L_{2p}(R_3)} + \|u\|^p_{L_p(R_3)}\right)d\tau
\end{aligned} \tag{20}$$

Further we will use the following multiplicative inequality of Galiardo-Nirenberg type (see [9])

$$\|u\|_{L_q(R_3)} \leq \|u\|^{\gamma_0}_{L_2(R_3)} \cdot \prod_{j=1}^3 \left\|D_{x_j}^{l_j} u\right\|^{\gamma_j}_{L_2(R_3)}, \tag{21}$$

where

$$\gamma_j = l_j^{-1}\left(\frac{1}{2} - \frac{1}{q}\right), \ j = 1,2,3 \quad \gamma_0 = 1 - \left(\frac{1}{2} - \frac{1}{q}\right)\frac{l_1l_2+l_1l_3+l_2l_3}{l_1l_2l_3}, \tag{22}$$

Taking into account (21), (22) for (19), (20) we have the following inequality

$$\|u(t,\cdot)\|_{L_2(R_3)} \le c(1+t)^{-\frac{l_1l_2+l_1l_3+l_2l_3}{4\ l_1l_2l_3}} \Big[\|\varphi\|_{L_2(R_3)} + \|\psi\|_{L_2(R_3)}$$

$$+\|\varphi\|_{L_1(R_3)} + \|\psi\|_{L_1(R_3)}\Big] + c\int_0^t (1+t-\tau)^{-\frac{l_1l_2+l_1l_3+l_2l_3}{4\ l_1l_2l_3}} \times$$

$$\times\left(\|u\|^{p\gamma_0}_{L_2(R_3)} \cdot \prod_{j=1}^{3}\left\|D^{l_j}_{x_j}u\right\|^{p\gamma_j}_{L_2(R_3)} + \|u\|^{p\gamma_0^1}_{L_2(R_3)} \cdot \prod_{j=1}^{3}\left\|D^{l_j}_{x_j}u\right\|^{p\gamma_j^1}_{L_2(R_3)}\right)d\tau,$$

$$\sum_{k=1}^{3}\left\|D^{l_k}_{x_k}u(t,x)\right\|_{L_2(R_3)} \le c(1+t)^{-\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4\ l_1l_2l_3}\right)} \Big[\|\varphi\|_{W_2^l(R_3)} + \|\psi\|_{L_2(R_3)} +$$

$$+\|\varphi\|_{L_1(R_3)} + \|\psi\|_{L_1(R_3)}\Big] + c\int_0^t (1+t-\tau)^{-\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4\ l_1l_2l_3}\right)} \times$$

$$\times\left(\|u\|^{p\gamma_0}_{L_2(R_3)} \cdot \prod_{j=1}^{3}\left\|D^{l_j}_{x_j}u\right\|^{p\gamma_j}_{L_2(R_3)} + \|u\|^{p\gamma_0^1}_{L_2(R_3)} \cdot \prod_{j=1}^{3}\left\|D^{l_j}_{x_j}u\right\|^{p\gamma_j^1}_{L_2(R_3)}\right)d\tau,$$

where $\gamma_0 = 1-\left(\dfrac{1}{2}-\dfrac{1}{p}\right)\cdot\dfrac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}$, $\gamma_j = l_j^{-1}\left(\dfrac{1}{2}-\dfrac{1}{p}\right)$, $j=1,2,3,$
$\gamma_0^1 = 1-\frac{1}{2}\left(1-\dfrac{1}{p}\right)$, $\gamma_j^1 = 2l_j^{-1}\left(1-\dfrac{1}{p}\right)$, $j=1,2,3,$

From here by denoting

$$y_0(t) = \|u(t,\cdot)\|_{L_2(R_3)}\cdot(1+t)^{\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}}, \tag{23}$$

$$y_k(t) = \left\|D^{l_k}_{x_k}u(t,\cdot)\right\|_{L_2(R_3)}\cdot(1+t)^{\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}\right)} \tag{24}$$

we have

$$y_0(t) \le cE_0 + c(1+t)^{\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}} \cdot \int_0^t (1+t-\tau)^{-\frac{l_1l_2+l_1l_3+l_2l_3}{4\ l_1l_2l_3}} \times$$

$$\times(1+\tau)^{-p\left[\frac{\gamma_0(l_1l_2+l_1l_3+l_2l_3)}{4\ l_1l_2l_3}+\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}\right)\sum\limits_{k=1}^{3}\gamma_j\right]} \cdot \prod_{j=0}^{3} y_j^{p\gamma_j}(\tau)d\tau, \tag{25}$$

$$\begin{gathered}\sum_{k=1}^{3} y_k(t) \le cE_0 + c(1+t)^{\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}\right)} \cdot \int_0^t (1+t-\tau)^{-\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}\right)} \times \\ \times(1+\tau)^{-p\left[\frac{\gamma_0(l_1l_2+l_1l_3+l_2l_3)}{4l_1l_2l_3}+\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}\right)\sum\limits_{k=1}^{3}y_j\right]} \cdot \prod_{j=0}^{3} y_j^{p\gamma_j}(\tau)d\tau,\end{gathered} \tag{26}$$

where

$$E_0 = \|\varphi\|_{W_2^l(R_3)} + \|\varphi\|_{L_1(R_3)} + \|\psi\|_{L_2(R_3)} + \|\psi\|_{L_1(R_3)}$$

Using the Young inequality from (25), (26) we obtain that

$$z(t) \le cE_0 + c(1+t)^{\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}} \cdot \int_0^t (1+t-\tau)^{-\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}} \times$$

$$\times(1+\tau)^{p\theta_0}z^p(\tau)d\tau+c(1+t)^{\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}\right)}\times$$
$$\times\int\limits_0^t(1+t-\tau)^{-\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}\right)}\cdot(1+\tau)^{p\theta_1}z^p(\tau)d\tau, \tag{27}$$

where $z(t)=\sum\limits_{k=0}^{3}\gamma_k(t)$,

$$\theta_0=\frac{\gamma_0^1\left(l_1l_2+l_1l_3+l_2l_3\right)}{4l_1l_2l_3}+\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}\right)\sum_{j=1}^{3}y_j=$$
$$=\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3},$$

$$\theta_1=\frac{\gamma_0^1\left(l_1l_2+l_1l_3+l_2l_3\right)}{4l_1l_2l_3}+\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}\right)\sum_{j=1}^{3}y_j=$$
$$=\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}+\left(1-\frac{1}{p}\right)\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}.$$

Let $E(t)=\sup\limits_{0\le s\le t}z(s)$, then from (27) it follows that

$$E(t)\le cE_0+c\left[(1+t)^{\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}}\cdot\int\limits_0^t(1+t-\tau)^{-\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}}\times\right.$$
$$\times(1+\tau)^{-p\theta_0}d\tau+(1+t)^{\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}\right)}$$
$$\left.\int\limits_0^t(1+t-\tau)^{-\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}\right)}\cdot(1+\tau)^{-p\theta_1}d\tau\right]E^p(t). \tag{28}$$

By the conditions (4), (5)

$$p\theta_0>1,\ p\theta_0\ge\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3},$$

$$p\theta_1>1,\ p\theta_1\ge\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}.$$

Therefore in base of Sigel's lemma (see [14]) the following inequality is true

$$(1+t)^{\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}}\cdot\int\limits_0^t(1+t-\tau)^{\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}}\cdot(1+\tau)^{p\theta_0}d\tau\le c, \tag{29}$$

$$(1+t)^{\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}\right)}\cdot\int\limits_0^t(1+t-\tau)^{-\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}\right)}\cdot(1+\tau)^{p\theta_1}d\tau\le c, \tag{30}$$

where $c>0$ doesn't depend on $t>0$ (see [14])

Taking into account (29), (30) from (28) we get

$$E(t)\le cE_0+c_1E^p(t).$$

From here it follows that for sufficiently small E_0

$$E(t)\le M,\quad t\ge 0, \tag{31}$$

where M is first root of the equation $c_1x^p - x + cE_0 = 0$. From (23), (24) and (31) it follows that

$$\sum_{k=1}^{3} \left\| D_{x_k}^{l_k} u(t,\cdot) \right\|_{L_2(R_3)} \le c(1+t)^{-\left(\frac{1}{2}+\frac{l_1l_2+l_1l_3+l_2l_3}{4\ l_1l_2l_3}\right)}, \tag{32}$$

$$\|u(t,\cdot)\|_{L_2(R_3)} \le c(1+t)^{-\frac{l_1l_2+l_1l_3+l_2l_3}{4\ l_1l_2l_3}}. \tag{33}$$

By returning back to (18) and using the estimation (17) for the $\alpha = (0,0,0)$, $i = 1$ we obtain that

$$\|D_t u(t,\cdot)\|_{L_2(R_3)} \le c(1+t)^{-\left(1+\frac{l_1l_2+l_1l_3+l_2l_3}{4\ l_1l_2l_3}\right)} E_0 +$$

$$+c\int_0^t (1+t-\tau)^{-\left(1+\frac{l_1l_2+l_1l_3+l_2l_3}{4\ l_1l_2l_3}\right)} \left[\|f(u)\|_{L_2(R_3)} + \|f(u)\|_{L_1(R_3)} \right]. \tag{34}$$

By repeating the above mentioned procedure (18)-(34) we have the following estimation

$$\|D_t u(t,\cdot)\|_{L_2(R_3)} \le c(1+t)^{-\left(1+\frac{l_1l_2+l_1l_3+l_2l_3}{4l_1l_2l_3}\right)}.$$

Thus, for the sufficiently small E_0 we have the a' priory estimation

$$E(u,t) \le c,\ t > 0.$$

Therefore for the sufficiently small E_0 the Cauchy problem (1), (2) has a global weak solution.

Now, we reduce the result on nonexistence of global solutions to semilinear hyperbolic equation with anisotropic elliptic part.

Suppose that

$$|f(u)| \ge c\,|u|^p,\ u \in R, \tag{35}$$

where $c > 0$,

$$1 < p \le 1 + \frac{2l_1l_2l_3}{l_1l_2 + l_1l_3 + l_2l_3}. \tag{36}$$

Using the methot of test function we prove the following theorem about the nonexistence of global solutions.

Theorem 4. *Suppose that condition (35), (36) are satisfied,* $\varphi(x) \in L_{1,l_{oc}}(R_3)$, $\psi \in L_{1,l_{oc}}(R_3)$ *and* $\lim\limits_{R\to\infty} \int\limits_{|\xi|_l<R} (\varphi(x) + \psi(x))dx \ge 0$,

Then the Cauchy problem (1); (2) doesn't nontrivial weak solution, which defined in region $R_+ \times R_3$.

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