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# NONLINEAR LONG WAVES IN DISPERSIVE LIQUID ENCLOSED IN VISCO-ELASTIC TUBE

**Abstract**

*In the paper we theoretically study wave flow of dispersive weakly compressible liquid in semi-infinite thin-shelled visco-elastic tube. Using diffusion approximation, inertia effect of relative motion of phases is neglected and mixture velocity is given on the whole. Dispersibility is taken into account by means of "correction" of dynamic viscosity coefficient and mixture density. Evolution equation describing non-linear waves propagation in dispersible liquid allowing for tube reaction, is derived. We consider principal regularities following from numerical integration of Korteweg-de Vries-Burgers (KdVB) modified equation by means of the finite difference method.*

Dispersible liquid is a mixture of solid particles, liquid drops or bubbles (discrete phase) distributed in liquid (continuous carrier phase) [1].

Investigations of dynamics of dispersible liquid cover wide fields of science and engineering and are connected with solution of fundamental problems. Moreover, non-linear wave processes are modelled in cardio-viscular system [2,3]. In the paper [4] pulsating flow of viscous incompressible liquid in visco-elastic tube is considered.

**1.** Let's consider wave flow of dispersible liquid in annular semi-infinite visco-elastic tube. It is assumed that a tube is rigidly fastened in environment so that there is no shift in axial direction along $X$. Accept the following assumptions: 1) strain of the tube is determined by the change of its radius $R(X,t)$; 2) shift of a tube wall $\zeta$ and its thickness $h$ is assumed to be small in comparison with equilibrium radius $R_0$, typical lenghs of waves are significantly greater than $R_0$; 3) density of tube wall is constant and dispersible liquid is weakly compressible (elastic); 4) typical velocity of wave is greater than mean velocity of the flow; 5) hydraulic resistance in the tube is given by Darcy-Weisbach linearized formula.

For describing dispersible liquid flow in visco-elastic axially symmetric tube of variable thickness we use equations of continuity and motion of viscous liquid averaged by cross-section of the tube [5].

$$\frac{\partial(\rho s)}{\partial t}+\frac{\partial(\rho s u)}{\partial X}+\frac{2s\rho}{R}\left(\frac{\partial\zeta}{\partial t}+u\frac{\partial\zeta}{\partial X}\right)=0 \tag{1.1}$$

$$\frac{\partial(\rho s u)}{\partial t}+\frac{\partial\left[(1+\beta)\rho s u^2\right]}{\partial X}=-s\frac{\partial P}{\partial X}+\mu s\frac{\partial^2 u}{\partial X^2}-2a\rho s u. \tag{1.2}$$

We accept rheological equation of the tube as follows [6]

$$\left(1+\sum_{l=1}^{2}b_l\frac{D^l}{Dt^l}\right)\Delta p=\frac{h}{R^2(X,t)}\left(a_0+\sum_{l=1}^{2}a_l\frac{D^l}{Dt^l}\right)\zeta. \tag{1.3}$$

Equations (1.1)-(1.3) close state equation of barotropic dispersible liquid

$$\rho = \rho\left(p\right). \tag{1.4}$$

Here $u$ is a mean axial component of velocity along the section, $s = \pi R^2\left(X,t\right)$ is a cross-section area of the tube. $\rho = \left(1-\alpha\right)\rho_f + \varepsilon\rho_s$, $\alpha$ is volume concentration of dispersion phase, $\rho_f$, $\rho_s$ are true densities of liquid and dispersion phases, $p\left(X,t\right)$ is a mean hydradynamical pressure along the section, $p_0\left(X\right)$, $u_0 = const$ are averaged stationary pressure and velocity before perturbation, respectively, $\mu$ is dynamic viscosity of the mixture that depending on the form of discrete particles and concentration of $\alpha$ will be given below [1,4], $\beta$ is Koriolis coefficient for non-uniform distribution of velocities, $2a = \lambda u/4R \approx const$, $\lambda$ is hydraulic resistence coefficient. For a round tube of radius $R$, as is known, $2a = 8\mu/\rho R$ [5], $b_1 = \theta_1 + \theta_*$, $b_2 = \theta_1\theta_*$, $\left(1-\nu^2\right)a_0 = E_2$, $a_1 = \left(E_1+E_2\right)\theta_1 + \left(E_2+E_*\right)\theta_*$, $a_2 = \theta_1\theta_* E_2 + \left(E_1+E_2\right)\theta_1\theta_*$, $\theta_1 = \mu_1/E_1$, $\theta_* = \mu_*/E_*$, $E_1$, $E_2$, $E_*$ are Young's longitudinal elasticity modules, $\nu$ is a Poisson ratio, $\mu_*$ is dynamical vicosity of particles.

**Fig.1. Rheological scheme of summation of elastic and viscous stresses.**

Rheological model corresponding to these coefficients in shown is figure 1. Besides, preservation of nonlinear summands by Oldroid in the derivatives $D^l/Dt^l$ and in the mass and pulse balance equations allows to get evolution equation of long nonlinear waves. Let before perturbation flow be stationary and uniform

$$\frac{dp_0}{dX} + 2a\rho_0 u_0 = 0. \tag{1.5}$$

**2.** When there is no damping the system of equations (1.1)-(1.4) has the solution: $f = f\left(c_s^{-1}X - t\right)$, where $f$ is an arbitrary argument function. It is natural to suppose that in the case of small damping and non-linearity the form of the function $f$ will slowly variable with distance from the entrance, i.e. $f = f\left(\eta X, c_s^{-1}X - t\right)$. Introducing new variables [7,8]

$$x = \eta X, \qquad \tau = c_s^{-1}X - t, \;\; \eta << 1, \tag{2.1}$$

and substututung them into the equation (1.1)-(1.3), we have

$$-\frac{\partial(\rho s)}{\partial\tau}+\mu\frac{\partial(\rho s u)}{\partial x}+c_s^{-1}\frac{\partial(\rho s u)}{\partial\tau}+\frac{2s\rho}{R}\left(-\frac{\partial\zeta}{\partial\tau}+\eta u\frac{\partial\zeta}{\partial X}+c_s^{-1}u\frac{\partial s}{\partial\tau}\right)=0. \quad (2.2)$$

$$-\frac{\partial(\rho s u)}{\partial\tau}+\eta\frac{\partial\left[(1+\beta)\rho s u^2\right]}{\partial X}+c_s^{-1}\frac{\partial\left[(1+\beta)\rho s u^2\right]}{\partial\tau}=$$

$$=-s\left(\eta\frac{\partial p}{\partial X}+c_s^{-1}\frac{\partial p}{\partial\tau}\right)+\mu s\left(\eta^2\frac{\partial^2 u}{\partial X^2}+2\eta c_s^{-1}\frac{\partial^2 u}{\partial X\partial\tau}+c_s^{-2}\frac{\partial^2 u}{\partial\tau^2}\right)-2a\rho s u; \quad (2.3)$$

$$\left[1+\sum_{l=1}^{2}b_l\prod_{q=1}^{l}\left(-\frac{\partial}{\partial\tau}+\eta u\frac{\partial}{\partial X}+c_s^{-1}u\frac{\partial}{\partial\tau}\right)^q\right]\Delta p=$$

$$=\frac{h}{R^2}\left[a_0+\sum_{l=1}^{2}a_l\prod_{q=1}^{l}\left(-\frac{\partial}{\partial\tau}+\eta u\frac{\partial}{\partial X}+c_s^{-1}u\frac{\partial}{\partial\tau}\right)^q\right]\zeta. \quad (2.4)$$

Using the assumptions of the estimation $[X]\sim C_s[\tau]$, $|\partial\tau| >> \eta\,|u\partial/\partial x|$, $|\partial/\partial\tau| >> c_s^{-1}\,|u\partial/\partial t|$ we simplify the rheological equation (2.4)

$$\left[1+\eta\sum_{l=1}^{2}(-1)^l\,\frac{b_l}{\eta}\frac{\partial^l}{\partial\tau^l}\right]\Delta p=\frac{h}{R^2}\left(a_0+\eta\sum_{l=1}^{2}(-1)^l\,\frac{a_l}{\eta}\frac{\partial^l}{\partial\tau^l}\right)\zeta. \quad (2.5)$$

In order to study evolution of perturbations in the approximation of long waves of small amplitude it is convenient to look for the solution of (2.2), (2.3) and (2.5) in the form of series by the power $\eta$ [7,8]

$$\zeta=\sum_{k=1}^{\infty}\eta^k\zeta_k,\qquad u=u_0+\sum_{k=1}^{\infty}\eta^k u_k, \quad (2.6)$$

$$\Delta p=\sum_{k=1}^{\infty}\eta^k\Delta p_k,\ \rho=\rho_0+\left(\frac{\partial\rho}{\partial p}\right)_{p_0}\Delta p=\rho_0+\frac{h a_0}{R_0^2 c_f^2}\sum_{k=1}^{\infty}\eta^k\zeta_k.$$

It follows from (2.6)

$$R=R_0+\sum_{k=1}^{\infty}\eta^k\zeta_k,\ \ s=\pi\left[R_0^2+2\eta R_0\zeta_1+\eta^2\left(\zeta_1^2+2R_0\zeta_2\right)+...\right], \quad (2.7)$$

where

$$\beta=\beta_0+\sum_{k=1}^{\infty}\eta^k\beta_k,\quad c_f=\sqrt{(\partial\rho/\partial p)_{p_0}}.$$

Substituting (2.6) and (2.7) in (2.2), (2.3) and (2.5) allowing for (1.5) and equating the coefficients for $\eta$, we have relations in the first approximation

$$\begin{array}{l}\left(4R_0\rho_0+c_f^{-2}ha_0\right)\left(1-c_s^{-1}u_0\right)\zeta_1-c_s^{-1}R_0^2\rho_0u_1=0\\ \left[c_s^{-1}ha_0-u_0\left(2R_0\rho_0+c_f^{-2}ha_0\right)\left(1-c_s^{-1}\left(1+\beta_0\right)u_0\right)\right]\zeta_1-\\ \qquad -R_0^2\rho_0\left[1-2c_s^{-1}\left(1+\beta_0\right)u_0\right]u_1=0.\end{array} \quad (2.8)$$

Condition of non-trivial solution of the system (2.8) gives dispersive relation by which phase velocity of linear waves propagation in dispersive liquid with reaction of the tube is found. If we take into account supposition 4, the variance relation is simplified and the expression for the velocity takes the form

$$c_s = \sqrt{\frac{ha_0}{4R_0\rho_0 + c_f^{-2}ha_0}}. \tag{2.9}$$

Hence it follows that phase velocity $c_s$ is inversely proportional to square root of density $\rho_0 = (1-\alpha_0)\,\rho_f^0 + \alpha_0\rho_s^0$.

Let's find evolution equation for describing perturbations in flowing dispersing liquid in visco-elastic tube. After substitution of (2.6), (2.7) into (2.3)-(2.5), equating the expressions at the same degrees of $\eta^2$ we get the second approximation

$$\left(4R_0\rho_0 + c_f^{-2}ha_0\right)\left(1-\frac{u_0}{c_s}\right)\frac{\partial\zeta_2}{\partial\tau} - c_s^{-1}R_0^2\rho_0\frac{\partial u_2}{\partial\tau} =$$

$$= N_0\frac{\partial\zeta_1^2}{\partial\tau} + N_1\frac{\partial\zeta_2}{\partial x}\left[c_s^{-1}ha_0 - u_0\left(2R_0\rho_0 + c_f^{-2}ha_0\right)\left(1-\frac{u_0\,(1+\beta_0)}{c_s}\right)\right]\frac{\partial\zeta_2}{\partial\tau} -$$

$$-R_0^2\rho_0\left[1-\frac{2u_0\,(1+\beta_0)}{c_s}\right]\frac{\partial u_2}{\partial\tau} = N_2\zeta_1^2 + N_3\frac{\zeta_1}{\eta} - N_4\frac{\partial\zeta_1}{\partial x} + N_5\frac{\partial\zeta_1}{\partial\tau} +$$

$$+\frac{\mu N_1}{c_s\rho_0}\left(1-\frac{u_0}{c_s}\right)\left(2\frac{\partial}{\partial x} + \frac{1}{\eta c_s}\frac{\partial}{\partial\tau}\right)\frac{\partial\zeta_1}{\partial\tau} + \frac{h}{\eta c_s}\sum_{l=1}^{2}(-1)^{l+1}s_{l+1}\frac{\partial^{l+1}\zeta_1}{\partial\tau^{l+1}}, \tag{2.10}$$

where

$$N_0 = \rho_0\left(10 + \frac{4ha_0}{c_f^2R_0\rho_0} + \frac{(ha_0)^2}{c_f^4R_0^2\rho_0^2}\right)\left(1-\frac{u_0}{c_s}\right), \tag{2.11}$$

$$N_1 = c_sR_0\rho_0\left(4 + \frac{ha_0}{c_f^2R_0\rho_0}\right),$$

$$N_2 = -2ac_s\rho_0\left[\left(4 + \frac{ha_0}{c_f^2R_0\rho_0}\right)\left(2 + \frac{ha_0}{c_f^2R_0\rho_0}\right)\left(1-\frac{u_0}{c_s}\right) + \left(1 + \frac{2ha_0}{c_f^2R_0\rho_0}\right)\frac{u_0}{c_s}\right],$$

$$N_3 = -2ac_sR_0\rho_0\left(4 + \frac{ha_0}{c_f^2R_0\rho_0} - \frac{u_0}{c_s}\right),$$

$$N_4 = ha_0 + u_0c_s\,(1+\beta_0)\,R_0\rho_0\left[\left(8 + \frac{2ha_0}{c_f^2R_0\rho_0}\right) - \frac{u_0}{c_s}\left(6 + \frac{ha_0}{c_f^2R_0\rho_0}\right)\right],$$

$$N_5 = \frac{ha_0}{c_sR_0} - \frac{N_1}{R_0}\left[2 + 4\beta_0 + \frac{ha_0}{c_f^2R_0\rho_0}\left(\beta_0 + \frac{(1+\beta_0)\,u_0}{c_s}\right)\right] +$$

$$+u_0\rho_0\left(1 + \frac{2ha_0}{c_f^2R_0\rho_0}\right)\left(1 - \frac{(1+\beta_0)\,u_0}{c_s}\right),$$

$$s_{l+1} = (a_l - b_l a_0)\,\Gamma_{2-l}, \quad \Gamma_{2-l} = 1, \quad 2 - l \geq 0.$$

According to supposition 4 the coefficients (2.11) of the system of equations (2.10) for $u_0/c_s << 1$ are significantly simplified.

Variance relation in the first approximation corresponds to the principal determination of (2.10). Solvability condition of a system of equations in the second approximation gives the desired nonlinear evolution equation

$$\frac{\partial\zeta_1}{\partial X} + A_1\zeta_1\frac{\partial\zeta_1}{\partial\tau} - \frac{A_2}{\eta}\sum_{l=1}^{2}(-1)^{l+1}\,s_{l+1}\frac{\partial^{l+1}\zeta_1}{\partial\tau^{l+1}} -$$

$$-A_4\frac{\partial}{\partial\tau}\left(\frac{\partial}{\partial X} + \frac{1}{2\eta c_s}\frac{\partial}{\partial\tau}\right)\zeta_1 + A_6\frac{\zeta_1}{\eta} + A_7\zeta_1^2 = 0, \tag{2.12}$$

where

$$b = c_s R_0\left[4 + \frac{ha_0}{c_f R_0\rho_0} + \frac{ha_0}{c_s^2 R_0\rho_0} - (1+\beta_0)\left(6 + \frac{ha_0}{c_f^2 R_0\rho_0}\right)\frac{u_0^2}{c_s^2}\right],$$

$$A_1 = \frac{2}{b}\left(10 + \frac{4ha_0}{c_f^2 R_0\rho_0} + \frac{h^2 a_0^2}{c_f^4 R_0^2\rho_0^2}\right)\left(1 - \frac{2u_0\,(1+\beta_0)}{c_s}\right)\left(1 - \frac{u_0}{c_s}\right) -$$

$$-\frac{u_0}{c_s}\left(1 - \frac{u_0\,(1+\beta_0)}{c_s}\right)\left(1 + \frac{2ha_0}{c_f^2 R_0\rho_0}\right) + \left(4 + \frac{ha_0}{c_f^2 R_0\rho_0}\right)\times$$

$$\times\left(2\,(1+2\beta_0) + \frac{ha_0}{c_f^2 R_0\rho_0}\left(\beta_0 + \frac{u_0\,(1+\beta_0)}{c_s}\right)\right)\left(1 - \frac{u_0}{c_s}\right) - \frac{ha_0}{c_f^2 R_0\rho_0}, \tag{2.13}$$

$$B_1 = \frac{h}{bc_s^2\rho_0}, \quad B_2 = \frac{\mu R_0}{bc_s\rho_0}\left(1 - \frac{u_0}{c_s}\right)\left(4 + \frac{ha_0}{c_f^2 R_0\rho_0}\right).$$

$$A_3 = -N_3/\rho_0 c_s, \;\; A_2 = -N_2/\rho_0 c_s.$$

Introducing a new variable $w = \eta A_1\zeta_1 = A_1\zeta$ and using the previous coordinate $X = x/\eta$ and also Boussinesq approximation $\partial/\partial x \approx c_s^{-1}\partial/\partial\tau$ and substituting them into the equation (2.12) we have

$$\frac{\partial w}{\partial X} + w\frac{\partial w}{\partial\tau} - B_1\sum_{l=1}^{2}(-1)^{l+1}\,s_{l+1}\frac{\partial^{l+1}w}{\partial\tau^{l+1}} - \frac{3}{2}B_2\frac{\partial^2 w}{\partial\tau^2} + \frac{A_2}{A_1}w^2 + A_3 w = 0. \tag{2.14}$$

Pressure perturbations, density of dispersive liquid and flow velocity in the tube are subjected to similar equations.

**3.** For ideal liquid non-linear evolution equation (2.14) passes to Korteveg-de Vries-Burgers equation [8]

$$\frac{\partial w}{\partial X} + w\frac{\partial w}{\partial\tau} + \Omega^{-2}\frac{\partial^3 w}{\partial\tau^3} = \Gamma^{-1}\frac{\partial\beta}{\partial\tau}, \tag{3.1}$$

where

$$\Gamma^{-1} = \frac{h\left(a_1 - b_1 a_0\right)}{b c_s^2 \rho_0}, \quad \Omega^{-2} = \frac{h\left(a_2 - b_2 a_0\right)}{b c_s^2 \rho_0}.$$

Detailed chart of numerical solutions for different pulses in the plane $\Omega$, $\Omega\Gamma^{-1}$ is represented in the paper [9]. It is shown that for $\Omega\Gamma^{-1} < 0,4$ dispersing properties are prevalent, and for $\Omega > 12$ the solution may contain solutions. For $\Omega < \sqrt{12}$ typical solution is a wave packet that for $\Omega < 1,4$ is described by a quasilear solution.

The obtained modified equation (2.14) whose form indicates the accounting of additional dissipation of dispersing liquid and hydraulic resistance of the tube wall is

$$\frac{\partial w}{\partial X} + w\frac{\partial w}{\partial \tau} + \Omega^{-2}\frac{\partial^3 w}{\partial \tau^3} + \frac{A_2 w^2}{A_1} + A_3 w = \Gamma_2^{-1}\frac{\partial^2 w}{\partial \tau^2}, \quad \Gamma_2^{-1} = \Gamma^{-1} + \frac{3}{2}B_2. \tag{3.2}$$

By comparison of separate members of (3.2) we can conclude that relaxation and dissipation effects smooth out the wave front.

Let a profile of impute pulse has the form of Gauss distribution

$$w\left(0,\tau\right) = 10^{-2} e^{-(1/4)\tau^2}. \tag{3.3}$$

By numerical method we research the solution of the equation (3.2) corresponding to the condition (3.3) provided its sufficient smoothness and convergence to zero with its derivatives as $|\tau| \to \infty$.

$$w\left(X,\infty\right) = w\left(X,-\infty\right) = 0, \quad \left(\partial w/\partial \tau\right)_{\tau\to\infty} = 0. \tag{3.3}$$

For numerical realization of the problem (3.2)-(3.4) we accept the following data:

$$\nu = 4\cdot 10^{-6}\mathrm{m}^2/\mathrm{s}, \quad \rho_0 = 1\cdot 10^3\mathrm{kg/m}^3, \quad R_0 = 0,1\mathrm{m}, \quad h = 0,01\mathrm{m}$$

$$c_f = \beta_0 = 1/3, \quad E_2 = 10^8\mathrm{n/m}^2, \quad u_0 = 0,1\mathrm{m}, \quad g = 9,8\mathrm{m/sec}^2,$$

$$\theta_1 = 10^{-3}\mathrm{sec}, \quad \theta_* = 10^{-2}\mathrm{sec}.$$

Profiles of waves for different sections of the tube: $X = 0,1;\ 0,4;\ 0,5$ are shown in figure 2. It is seen from the figure that at the left part steepness of the profile of front is reinforced, at the right part profile steepness decreases. Such deformation of wave profile is connected with boundary conditions and with influence of KdV nonlinearity. Wave amplitude essentially depends on concentration and it decreases according increase of its values.

**Fig.2. Wave propagation along the axis $X$, curve 1 corresponds to $X = 0.1$ m, 2. - $X = 0.3$ m, 3.- $X = 0.5$ m.**

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