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A NEW APPROACH FOR STUDYING ASYMPTOTICS OF EXPONENTIAL MOMENTS OF SUMS OF RANDOM VARIABLES, CONNECTED IN MARKOV CHAIN I

Abstract

Investigation of limit behaviour of sums of dependent random variables one of the most intensively developed directions in up-to-date probability theory, is important not only by the fact that it generalizes classic theory of summation of independent random variables constituting central theoretical nucleur of probability theory, but also in inconsiderable degree by perturbance of various applications to the solution of statistics problems concrete problems of applied fields of probability theory. Potential possibility of very various forms of dependence of random variables forming the sum, makes the investigation subject of this direction very wide and reduces to difficult mathematical problems.

The sums of random variables connected in Markov chain have been most studied after A. A. Markov classic papers. The main results and achievements here are connected with the names of V. I. Romanovsky, A. N. Kolmogorov, T. A. Sarymsakov, S. Kh. Sirazhdinov, A. V. Skorokhod, S. V. Nagayev, V. A. Statulyavichyus, V. S. Korolyuk.

One of the most effective methods for investigating asymptotic behavior of the sum of random variables connected in Markov chain was suggested by S. V. Nagayev ([1]) . Its idea is the following:

Let $\{\eta(n), n \geq 0\}$ be a homogeneous Markov chain with phase space $(X, \mathcal{F}_X)$, definable by transient probability $P(x, A)$, $x \in X$, $A \in F_X$, $f(x)$ be a measurable mapping of X into R^1. Fixing some banach L space of F_X-measuble functions, we determine therein the operator P_λ by the equality

$$P_\lambda g(x) = \int_X e^{i\lambda x} p(x, dy)\, g(y)\,. \tag{1}$$

Under natural assumptions on ergodicity of Markov chain $\{\eta_n, n \geq 0\}$ (e.g. $P_0^n \to \Pi$ $\lambda \to 0$ and $\Pi g(x) = \int_X g(y)\, \pi(dy)$) the investigation of astmptotic behavior of the characteristic function of the sum

$$S(n) = \sum_{k=0}^{n} f(\eta_k)$$

is reduced to investigation as $\lambda \to 0$ of the eigen function $\Lambda(\lambda)$ and eigen

$$\psi(\lambda) = \Pi(\lambda)\, \psi$$

vector where the ψ is a function from L that is identically equals to unit in the following eigen value problem

$$P_\lambda \Pi_\lambda \psi = \Lambda(\lambda)\, \Pi_\lambda \psi, \tag{2}$$

where $\Pi_\lambda \to \Pi$ as $\lambda \to 0$.

Investigation of limit theorems for functionals determined in Markov chains, including for moments of attainment of difficult of access domains is of significant interest alongside with traditional limit theorems for sums of random variables. Such type theorems and related theorems on asymptotic enlargement of markov and semi-markov processes were studied in the papers by V. S. Korolynk, V. V. Anisimov, D. S. Silvestrov, A. F. Turbin and other ([5], [6]) authors.

In the present paper we investigate a more general scheme of summation when a Markov chain η_n $n \geq 0$ and the function $f(x)$ depend on the small parameter $\varepsilon \in (0, \varepsilon_0]$ and we study asymptotic behaviour of exponential moments

$$E_{p_\varepsilon} \exp\{zS_\varepsilon(n)\} = \int_X p_\varepsilon(dx)\, E\left[\exp\{zS_\varepsilon(n)\}|_{\eta_{\varepsilon_0}=x}\right], \tag{3}$$

where

$$S_\varepsilon(n) = \sum_{k=0}^{n} f_\varepsilon(\eta_{\varepsilon k}).$$

Distinctive property of the present paper is that unlike the earlier investigated cases, in (3) z is not necessarily a pure imaginary, but in the general case complex parameter.

Problem statement. Let $\eta_{\varepsilon n}$, $n = 0, 1, ...$ for each $0 < \varepsilon < 1$ (ε is a parameter of the series) be a homogeneous Markov chain with phase space X, F_X, with transient probabilities $p_\varepsilon(x, A)$, initial distribution $p_\varepsilon(A)$; $f_\varepsilon(x) - F_X$ be a measurable, number function.

Let's consider sums of random variables, determined on the chain $\eta_{\varepsilon n}$ $n \geq 0$.

$$S_\varepsilon(n) = \sum_{r=0}^{n} f_\varepsilon(\eta_{\varepsilon r}). \tag{4}$$

In the paper we study uniform by the parameter ε with explicit estimation of remainder terms, expansion of exponential moments $E_{P_\varepsilon} \exp\{zS_\varepsilon(n)\}$, where z is a complex number.

The basic conditions under which this problem is solved, are the followings:

$A_{\rho,h}$: There exists an entire positive h_ε and $0 \leq \rho_\varepsilon < 1$ such that

$$\sup_{\substack{x,y\in X\\ A\in F_X}} \left| p_\varepsilon^{(h_\varepsilon)}(x, A) - p_\varepsilon^{(h_\varepsilon)}(y, A) \right| \leq \rho_\varepsilon$$

where a) $\sup\limits_\varepsilon \rho_\varepsilon = \rho < 1$, b) $\sup\limits_\varepsilon h_\varepsilon = h < \infty$;

B_k : There exists $0 < b < \infty$ such that

$$\sup_\varepsilon \sup_x \int_X |f_\varepsilon(y)|^{k+1} e^{b|f_\varepsilon(y)|} p_\varepsilon(x, dy) = L_k < \infty$$

C_k : There exists $0 < c < \infty$ such that

$$\sup_\varepsilon \int_X |f_\varepsilon(x)|^{k+1} e^{c|f_\varepsilon(x)|} p_\varepsilon(dx) = M_k < \infty.$$

It should be noted that by fulfilling the condition $A_{\rho,h}$ for each $0<\varepsilon\leq 1$ there exists such a stationary distribution $\pi_\varepsilon(A)$ that

$$\sup_{\substack{x,y\in X\\ A\in F_X}}\left|p_\varepsilon^{(n)}(x,A)-\pi_\varepsilon(A)\right|\leq\rho^{[n/h_\varepsilon]}. \tag{5}$$

It is clear that the stationary distribution $\pi_\varepsilon(A)$ satisfies the equation

$$\int_X p_\varepsilon(x,A)\,\pi_\varepsilon(dx)=\pi_\varepsilon(A)\,,\quad A\in F_X.$$

In addition, except $A_{\rho,h}$ there exist another conditions providing the existence of the stationary distribution $\pi_\varepsilon(\cdot)$ and uniform convergence of $p_\varepsilon^{(n)}(x,A)$ to $\pi_\varepsilon(A)$ exponentially rapid under unrestricted growth of n to infinity ([3]).

Let X, F_X be arbitrary measurable space, $\mathfrak{M}$ be a space of all restricted complex functions $g(x)$, $x\in X$ measurable with respect to F_X, with the norm $\|g(x)\|=\sup\limits_{x\in X}|g(x)|$; $\mathfrak{M}^*$ be a space of all complex completely additive functions of the set $\mu(A)$, $A\in F_X$, with the norm $\|\mu\|=|\mu|(X)$, where $|\mu|(\cdot)$ is a full variation of $|\mu|(\cdot)$ on X.

We determine the operators P_ε and P_ε^* in the following way:

$$\begin{aligned} P_\varepsilon g(\cdot)&=\int_X g(x)\,p_\varepsilon(\cdot\ dx)\,,\\ P_\varepsilon^*\mu(\cdot)&=\int_X p_\varepsilon(x,\cdot)\,\mu(dx)\,. \end{aligned} \tag{6}$$

The operators P_ε and P_ε^* are determined in the spaces $\mathfrak{M}$ and $\mathfrak{M}^*$ respectively, and $\|P_\varepsilon\|=\|P_\varepsilon^*\|$

In future, a completely additive function of the set $\Phi(x,A)$, $x\in X$, $A\in F_X$ with restricted for fixed x complete variation determining in $\mathfrak{M}$ an operator of type P_ε will be said to be a kernel of this operator.

Let Π_ε be an operator in $\mathfrak{M}$, determined by stationary distribution of profanities $\pi_\varepsilon(A)$, i.e.

$$\Pi_\varepsilon g(\cdot)=\int_X g(x)\,\pi_\varepsilon(dx)\,. \tag{7}$$

Obviously

$$P_\varepsilon\Pi_\varepsilon=\Pi_\varepsilon P_\varepsilon=\Pi_\varepsilon=\Pi_\varepsilon^2 \tag{8}$$

Π_ε projects $\mathfrak{M}$ into one-dimensional space $\mathfrak{M}_1$, generated by the function

$$\psi(\cdot)\equiv 1$$

(see[1]).

It is easy to see that the spectrum of the operator P_ε equals to the sum of the spectra of the operators Π_ε and $P_\varepsilon-\Pi_\varepsilon$ considered in $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively, where $\mathfrak{M}_2$ consists of those elements g, for which $\Pi_\varepsilon g=0$.

It follows from (8) that $(P_\varepsilon-\Pi_\varepsilon)^n=P_\varepsilon^n-\Pi_\varepsilon$.

By $V_n(x,A)$ we denote a complete variation of the measure

$$p_\varepsilon^{(n)}(x,B)-\pi_\varepsilon(B)$$

on the set A. On account of (5)

$$V_n(x,A)\le 2\rho_\varepsilon^{[n/h_\varepsilon]}. \tag{9}$$

Since all the points of λ for which $|\lambda|>\lim\limits_{n\to\infty}\|(P_\varepsilon-\Pi_\varepsilon)^n\|^{1/n}$ belong to resolvent set of the operator $P_\varepsilon-\Pi_\varepsilon$ it follows from (9) that the spectrum of the operator $P_\varepsilon-\Pi_\varepsilon$ lies in the circle of radius ρ^{1/h_ε}, with centre at the point 0 (see [4]). Obviously, the spectrum of the operator Π_ε consists of a unique point 1.

Consequently, the domain external to the circle of radius ρ^{1/h_ε} with center at the point 0, from which the point 1 is removed, is wholly contained in the resolvent set of the operator P_ε.

It is easy to see that the resolvent of the operator Π_ε

$$P_\varepsilon(\lambda)=\frac{1}{\lambda-1}\Pi_\varepsilon+\sum_{n=0}^{\infty}\left(P_\varepsilon^n-\Pi_\varepsilon\right)\lambda^{-n-1} \tag{10}$$

for $|\lambda|>\rho^{1/h_\varepsilon},\ \lambda\ne 1$

Now consider in $\mathfrak{M}$ the operator $P_{\varepsilon z}$, that is generated by the kernel $p_\varepsilon(z,x,A)=\int\limits_A e^{zf_\varepsilon(y)}p_\varepsilon(x,dy)$, $A\in F_X$ where $f_\varepsilon(\cdot)$ is a real function, measurable with respect to F_X. Obviously, by fulfilling the conditions $A_{\rho,h}$ and B_k, $k\ge 1$, for all complex z, for which $|z|\le b$, the definition of the operator $P_{\varepsilon z}:\mathfrak{M}\to\mathfrak{M}$ is correct, $P_{\varepsilon z}|_{z=0}=P_\varepsilon$ and if

$$\|P_{\varepsilon z}-P_\varepsilon\|<\frac{1}{\|R_\varepsilon(x)\|}, \tag{11}$$

the series $\sum\limits_{n=0}^{\infty}R_\varepsilon(\lambda)\left[(P_{\varepsilon z}-P)R_\varepsilon(x)\right]^n$ converges and defines for $P_{\varepsilon z}$ resolvent operator

$$R_\varepsilon(\lambda,z)=\sum_{n=0}^{\infty}R_\varepsilon(\lambda)\left[(P_{\varepsilon z}-P_\varepsilon)R_\varepsilon(\lambda)\right]^n. \tag{12}$$

Let J_1 and J_1 be circumferences with centers at the points 1 and 0, respectively of radii $\rho_1=\dfrac{1}{3}\left(1-\rho^{1/h}\right)$ and $\rho_2=\dfrac{1}{3}\left(1+2\rho^{1/h}\right)$, and E be a closed domain external with respect to circles bounded by circumferences J_1 and $J_1$and

$$M_\rho=\sup_\varepsilon\sup_{\lambda\in E}\|R_\varepsilon(\lambda)\|.$$

Let's consider the function

$$M(\rho,h)=\frac{2h\cdot 3^h+3}{1-\rho^{1/h}}.$$

In sequel, we'll essentially use the function $M(\rho,h)$ in obtaining different estimates. Clearly, $M(\rho,h)$ is an increasing function in both arguments ρ,h, and

$$M(\rho,h)\ge 9.$$

Lemma 1. *Let conditions $A_{\rho,h}$ and for some $k \geq 2$ the condition B_k be fulfilled. Then for all complex z, for which*

$$|z| < \min\left\{\frac{1}{4M^2(\rho,h)\,L_k^{1/(k+1)}}, b\right\}$$

and for all $\lambda \in E$ the resolvent of the operator $P_{\varepsilon z}$ exists and determined by the formula (12).

Proof. At first we introduce the denotation:

$$R_{1\varepsilon}(\lambda) = \frac{\Pi_\varepsilon}{\lambda - 1} \quad \text{and } R_{2\varepsilon}(\lambda) = \sum_{n=0}^{\infty}(P_\varepsilon^n - \Pi_\varepsilon)\,\lambda^{-n-1}.$$

Then

$$R_\varepsilon(\lambda) = R_{1\varepsilon}(\lambda) + R_{2\varepsilon}(\lambda) \tag{13}$$

and

$$\|R_\varepsilon(\lambda)\| \leq \|R_{1\varepsilon}(\lambda)\| + \|R_{2\varepsilon}(\lambda)\|. \tag{14}$$

Obviously, in the domain E

$$\|R_{1\varepsilon}(\lambda)\| \leq \frac{3}{1-\rho^{1/h}}. \tag{15}$$

It follows from estimation (9) that

$$\|P_\varepsilon^n - \Pi_\varepsilon\| \leq 2\rho^{[n/h]}.$$

Therefore

$$\|R_{2\varepsilon}(\lambda)\| \leq \frac{1}{|\lambda|}\sum_{n=0}^{\infty}\frac{\|P_\varepsilon^n - \Pi_\varepsilon\|}{|\lambda|^n} \leq$$

$$\leq \frac{2}{|\lambda|}\left\{\sum_{n=0}^{h-1}\frac{\rho^{[n/h]}}{|\lambda|^n} + \sum_{n=h}^{2h-1}\frac{\rho^{[n/h]}}{|\lambda|^n} + \ldots + \sum_{n=rh}^{(r+1)h-1}\frac{\rho^{[n/h]}}{|\lambda|^n} + \ldots\right\} =$$

$$= \frac{2}{|\lambda|}\left[\sum_{n=0}^{h-1}\frac{1}{|\lambda|^n} + \rho\sum_{n=h}^{2h-1}\frac{1}{|\lambda|^n} + \ldots + \rho^r\sum_{n=rh}^{(r+1)h-1}\frac{1}{|\lambda|^n} + \ldots\right] \leq$$

$$\leq \frac{2\left(1 - |\lambda|^h\right)}{(1-|\lambda|)\left(|\lambda|^h - \rho\right)}.$$

Thus, we get that in the domain E

$$\|R_{2\varepsilon}(\lambda)\| \leq \frac{2^h \cdot 3^h}{1-\rho^{1/h}}. \tag{16}$$

It follows from $(14) - (16)$ that

$$\|R_\varepsilon(\lambda)\| \leq \frac{1}{1-\rho^{1/h}}\left(2^h \cdot 3^h + 3\right) \equiv M(\rho,h). \tag{17}$$

Hence, it is clear that $\sup_{\varepsilon}\sup_{\lambda\in E}\|R_\varepsilon(\lambda)\|\equiv M_\rho\le M(\rho,h)$ and since

$$M(\rho,h)\ge 9,$$

then naturally

$$\frac{1}{2M^2(\rho,h)}<\frac{1}{M_\rho}. \tag{18}$$

Further we determine the operators $P_\varepsilon^{(r)}:\mathfrak{M}\to\mathfrak{M}$, $r=1,2,...,k$ in the following way:

$$P_\varepsilon^{(r)}g(\cdot)=\int\limits_X[f_\varepsilon(x)]^r g(x)\,p_\varepsilon(\cdot,dx). \tag{19}$$

Obviously, by fulfilling the conditions B_k, definition of the operators $P_\varepsilon^{(r)}$, $r=\overline{1,k}$ is correct and since (Kolmogorov-Chepten equation)

$$\int\limits_X|f_\varepsilon(x)|\,p_\varepsilon(\cdot,dx)\le\left(\int\limits_X|f_\varepsilon(x)|^2\,p_\varepsilon(\cdot,dx)\right)^{1/2}\le...$$

$$...\le\left(\int\limits_X|f_\varepsilon(x)|^{k+1}\,p_\varepsilon(\cdot,dx)\right)^{1/(k+1)}$$

$$\le\left(\int\limits_X|f_\varepsilon(x)|^{k+1}\,e^{b|f_\varepsilon(x)|}p_\varepsilon(\cdot,dx)\right)^{1/(k+1)}\le L_k^{1/(k+1)},$$

we get that it is uniform with respect to ε

$$\left\|P_\varepsilon^{(r)}\right\|\le L_k^{r/(k+1)},\quad r=\overline{1,k}. \tag{20}$$

Since, for z indicated in the theorem, definition of the operator $P_{\varepsilon z}$ is correct, we expand $\exp\{zf_\varepsilon(x)\}$ in series and get:

$$P_{\varepsilon z}=P_\varepsilon+\frac{P_\varepsilon^{(1)}}{1!}z+...+\frac{P_\varepsilon^{(k)}}{k!}z^k+T_{\varepsilon z}(z), \tag{21}$$

where the operator $T_{\varepsilon z}(z):\mathfrak{M}\to\mathfrak{M}$ is determined in the following way:

$$T_{\varepsilon z}(z)\,g(\cdot)=\int\limits_X\int\limits_0^{zf_\varepsilon(x)}(e^u-1)\frac{[f_\varepsilon(y)\,z-u]^{k-1}}{(k-1)!}du\,g(y)\,p_\varepsilon(\cdot,dy). \tag{22}$$

Here integration is conducted with respect to any line connecting the points 0 and $zf_\varepsilon(x)$.

Obviously,

$$\|T_{\varepsilon k}(z)\|\le\sup_x\int\limits_X\int\limits_0^{|zf_\varepsilon(x)|}|e^u-1|\frac{[f_\varepsilon(y)\,z-u]^{k-1}}{(k-1)!}du\,p_\varepsilon(x,dy), \tag{23}$$

where integration is conducted with respect to straight line connecting the points 0 and $|zf_\varepsilon(x)|$.

Since for any $|e^u - 1| \le |u| e^u$ it follows from (23) that:

$$\|T_{\varepsilon k}(z)\| \le \frac{L_k}{(k-1)!} |z|^{k+1}. \tag{24}$$

Thus, from $(18), (20)$ and (24) and taking into account that $M(\rho, h) \ge 9$ and for z indicated in the lemma

$$|z| L_k^{r/(k+1)} \le \frac{1}{4M^2(\rho,h)} \le \frac{1}{324},$$

we get:

$$\|P_{\varepsilon z} - P_\varepsilon\| \le \frac{1}{4M^2(\rho,h)} \left\{ 1 + \frac{1}{2!}\frac{1}{324} + \ldots \right.$$

$$\left. \ldots + \frac{1}{k!}\left(\frac{1}{324}\right)^{k-1} + \frac{1}{(k-1)!}\left(\frac{1}{324}\right)^{k} \right\} < \frac{1}{2M^2(\rho,h)}.$$

Hence, allowing for inequality (18) we have:

$$\|P_{\varepsilon z} - P_\varepsilon\| < \frac{1}{M_\rho} \le \frac{1}{\|R_\varepsilon(\lambda)\|},$$

i.e. inequality (11) is fulfilled, that is equivalent to the statement of the lemma.

Remark 1. S. V. Nagayev in the paper [1] and lately many authors in the case of $z = it$ use the estimation

$$M_\rho \le \frac{1}{1 - \rho^{1/h}} \left[2^h \cdot 3^h \left(1 + 2\rho^{1/h}\right) + 3 \right] \equiv \widehat{M}(\rho, h).$$

Lemma 1 affirms that by fulfilling the conditions $A_{\rho,h}$ and for some $k \ge 2$ B_k for all z, for which

$$|z| < \min\left\{ \frac{1}{4M^2(\rho,h) L_k^{1/(k+1)}}, b \right\},$$

the circumferences J_1 and J_2 lie on a resolvent set of the operator $P_{\varepsilon z}$ and therefore for such z we can define the operator-projectors

$$\Pi_{\varepsilon z} = \frac{1}{2\pi i}\int_{J_1} R_\varepsilon(\lambda, z)\, d\lambda \quad \text{and} \quad \Pi'_{\varepsilon z} = \frac{1}{2\pi i}\int_{J_2} R_\varepsilon(\lambda, z)\, d\lambda \tag{25}$$

(Here in definition of integral we mean limit in the norm).

Obviously

$$\Pi_{\varepsilon z}|_{z=0} = \frac{1}{2\pi i}\int_{J_1} R_\varepsilon(\lambda)\, d\lambda = \Pi_\varepsilon \tag{26}$$

and $\Pi_{\varepsilon z} + \Pi'_{\varepsilon z} = I$, where I is a unit operator.

Introduce the denotation:

$$\langle g, \mu \rangle = \int_X g(x)\, \mu(dx), \quad g \in \mathfrak{M}, \ \ \mu \in \mathfrak{M}^*,$$

$$\Lambda_\varepsilon(z)=\frac{\langle P_{\varepsilon z}\Pi_{\varepsilon z}\psi,\pi_\varepsilon\rangle}{\langle \Pi_{\varepsilon z}\psi,\pi_\varepsilon\rangle},\quad \psi\equiv 1. \tag{27}$$

Obviously, $\Lambda_\varepsilon(z)$ is an eigen function of the operator $P_{\varepsilon z}$.

In sequel we'll essentially use the following

Lemma 2. *Let the condition $A_{\rho,h}$ and for some $k\geq 2$ the condition B_k be fulfilled. Then for all z, for which*

$$|z|<\min\left\{\frac{1}{4M^2(\rho,h)\,L_k^{1/(k+1)}},b\right\},$$

a) $\langle \Pi_{\varepsilon z}\psi,\pi_\varepsilon\rangle\neq 0$;
b) $\Lambda_\varepsilon(z)$ *is analytic;*
c) $P^n_{\varepsilon z}=\Lambda^n_\varepsilon(z)\,\Pi_{\varepsilon z}+P^n_{\varepsilon z}\Pi'_{\varepsilon z}$
and
d) $\|P^n_{\varepsilon z}\Pi'_{\varepsilon z}\|\leq\dfrac{18}{17}M(\rho,h)\left(\dfrac{1+2\rho^{1/h}}{3}\right)^n$
uniformly with respect to ε.

The proof of this lemma (statements a)-c)) in general case belongs to S. V. Nagayev ([1], [2]), estimation g) in the case $z=it$, (i.e. for a characteristic function) was obtained by A. E. Skvortsov ([7]).

Therefore we cite only principal moments of this proof.

Notice that therewith unlike the case $z=it$ for complex z the estimation d) holds in the vicinity of zero and appropriate boundaries are shown in this lemma.

In the course of the proof of lemma 1 we showed that for the z indicated in the lemma

$$\|P_{\varepsilon z}-P_\varepsilon\|<\frac{1}{2M^2(\rho,h)}. \tag{28}$$

It follows from the definition of the operators $\Pi_{\varepsilon z}$ and Π_ε that:

$$\|\Pi_{\varepsilon z}-\Pi_\varepsilon\|\leq\frac{1}{2\pi}\int\limits_{J_1}\|R_\varepsilon(\lambda,z)-R_\varepsilon(\lambda)\|\,d\lambda\leq$$

$$\leq\frac{1}{3}\cdot\frac{M(\rho,h)}{2M(\rho,h)-1}. \tag{29}$$

Hence, allowing for $M(\rho,h)\geq 9$ we get:

$$\|\Pi_{\varepsilon z}-\Pi_\varepsilon\|<1. \tag{30}$$

By fulfilling this inequality (see [1], [2]) the sub-space $\mathfrak{M}_{1\varepsilon}(z)$ into which $\Pi_{\varepsilon z}$ projects $\mathfrak{M}$, will be one-dimensional.

Let $\psi_\varepsilon(z)$ be an element generating $\mathfrak{M}_{1\varepsilon}(z)$. Obviously:

$$P_{\varepsilon z}\Pi_{\varepsilon z}\psi_\varepsilon(z)=\Pi_{\varepsilon z}P_{\varepsilon z}\psi_{\varepsilon z}=\Lambda_\varepsilon(z)\,\psi_\varepsilon(z).$$

On the other hand we can choose $\psi_\varepsilon(z)$ so that $\psi_\varepsilon(z)=\Pi_{\varepsilon z}\psi$, where $\psi\equiv 1$.

Then, it is clear that

$$\langle P_{\varepsilon z}\Pi_{\varepsilon z},\pi_\varepsilon\rangle=\Lambda_\varepsilon(z)\,\langle \Pi_{\varepsilon z}\psi,\pi_\varepsilon\rangle \tag{31}$$

and hence

$$\Lambda_\varepsilon(z) = \frac{\langle P_{\varepsilon z}\Pi_{\varepsilon z}\psi, \pi_\varepsilon\rangle}{\langle \Pi_{\varepsilon z}\psi, \pi_\varepsilon\rangle}.$$

Further, since $P_{\varepsilon z}^n = P_{\varepsilon z}^n \Pi_{\varepsilon z} + P_{\varepsilon z}^n \Pi'_{\varepsilon z}$, then taking into account

$$P_{\varepsilon z}^n \Pi_{\varepsilon z} = \Lambda_{\varepsilon z}^n(z)\,\Pi_{\varepsilon z}$$

we get c).

Since $\langle \Pi_{\varepsilon z}\psi, \pi_\varepsilon\rangle = \langle (\Pi_{\varepsilon z} - \Pi_\varepsilon)\,\psi, \pi_\varepsilon\rangle > +1$ we get that for the z indicated in the lemma, $\langle \Pi_{\varepsilon z}\psi, \pi_\varepsilon\rangle \neq 0$ i.e. a) is true.

Hence, considering that (see [1]) $\langle P_{\varepsilon z}\Pi_{\varepsilon z}\psi, \pi_\varepsilon\rangle$ and $\langle \Pi_{\varepsilon z}\psi, \pi_\varepsilon\rangle$ are analytic, we get statement b)

For obtaining estimation d) we notice that (see [1])

$$P_{\varepsilon z}^n \Pi'_{\varepsilon z} = \frac{1}{2\pi i}\int\limits_{J_2} \lambda^n R_\varepsilon(\lambda, z)\,d\lambda \tag{32}$$

and therefore

$$\left\| P_{\varepsilon z}^n \Pi'_{\varepsilon z} \right\| \leq \sup_{\varepsilon}\sup_{\lambda\in J_2} \| R_\varepsilon(\lambda, z) \| \cdot \left(\frac{1+2\rho^{1/h}}{3}\right)^n. \tag{33}$$

It is easily obtained from (12) that

$$\sup_{\substack{\lambda\in J_2\\ \varepsilon}} \| R_\varepsilon(\lambda, z) \| \leq M(\rho, h) \cdot \frac{1}{1 - \dfrac{1}{2M(\rho, h)}} \leq \frac{18}{17} M(\rho, h). \tag{34}$$

Estimation d) follows from $(33), (34)$.

The goal of the next step is the expansion of the resolvent operator $R_\varepsilon(\lambda, z)$, projector $\Pi_{\varepsilon z}$ and some related functionals in powers of the argument z. To this end we introduce the denotation

$$(x_1 + x_2 + ... + x_k)^r \equiv \sum_{r_1+r_2+...+r_k=r} \frac{r!}{r_1! r_2! ... r!} x_1^{r_1} x_2^{r_2} ... x_k^{r_k} ... \tag{35}$$

$$\left(a_0 + a_2\varepsilon + ... + a_k\varepsilon^k\right)^r \equiv \sum_{s=0}^{rk} \sum_{\substack{r_0+r_1+...+r_k=r\\ r_1+...+kr_k=s}} \frac{r!}{r_0! r_1! ... r_k!} a_0^{r_0} a_1^{r_1} ... a_k^{r_k} \varepsilon^s, \tag{36}$$

where $\sum\limits_{r_1+...+r_k=r}$ and $\sum\limits_{\substack{r_0+r_1+...+r_k=r\\ r_1+...+kr_k=s}}$ mean the summation over all entire non-negative solutions of equations $r_1 + r_2 + ... + r_k = r$ and systems of equations:

$$\begin{cases} r_0 + r_1 + ... + r_k = r \\ r_1 + 2r_2 + ... + kr_k = s \end{cases},$$

$$B_\varepsilon^{(r)}(\lambda) = R_\varepsilon(\lambda)\,P_\varepsilon^{(r)}, \quad r = \overline{1, k}$$

$$B_\varepsilon^{(k+1)}(\lambda, z) = (k+1)! R_\varepsilon(\lambda)\,T_{\varepsilon k}(z) \cdot z^{-k-1}; \tag{37}$$

$$R_\varepsilon^{(s)}(\lambda)=\sum_{r=1}^{s}\sum_{\substack{s_1+...+s_r=s\\ 1\le s_1,...,s_k\le k}}\frac{1}{s_1!...s_k!}B_\varepsilon^{(s_1)}(\lambda)...B_\varepsilon^{(s_r)}(\lambda)R_\varepsilon(\lambda), \tag{38}$$

$$R_\varepsilon^{(k+1)}(\lambda,z)=\sum_{s=k+1}^{\infty}\sum_{r=1}^{s}\sum_{\substack{s_1+...+s_r=s\\ 1\le s_1,...,s_k\le k+1}}\frac{1}{s_1!...s_k!}B_\varepsilon^{(s_1)}(\lambda)...B_\varepsilon^{(s_r)}(\lambda)R_\varepsilon(\lambda)z^s$$

$$D(u)=\begin{cases} eM(\rho,h)(1+u)^{1/(k+1)}, & \text{if} \quad u<1\\ eM(\rho,h)(1+u)^{1/(k+1)}, & \text{if} \quad u\ge 1\end{cases} \tag{39}$$

Theorem 1. *Let the condition $A_{\rho,h}$ and for some $k\ge 2$ the condition B_k be fulfilled. Then for all complex z for which*

$$|z|<\min\left\{\frac{1}{4M^2(\rho,h)L_k^{1/(k+1)}},\frac{e}{3D(L_k)},b\right\}$$

it holds

$$R_\varepsilon(\lambda,z)=R_\varepsilon(\lambda)+\sum_{s=1}^{k}R_z^{(s)}(\lambda)z^s+R_z^{(k+1)}(\lambda,z), \tag{40}$$

and uniformly with respect to ε

$$\left\|R_z^{(s)}(\lambda)\right\|\le\frac{26}{23}M(\rho,h)[D(L_k)]^s \tag{41}$$

$$\left\|R_\varepsilon^{(k+1)}(\lambda,z)\right\|\le 17M(\rho,h)[D(L_k)]^{k+1}|z|^{k+1}. \tag{42}$$

Proof. By fulfilling the conditions of the theorem the conditions of lemma 1 are fulfilled as well. Therefore, the resolvent of the operator $P_{\varepsilon z}$ exists and determined as

$$R_\varepsilon(\lambda,z)=\sum_{r=0}^{\infty}[R_\varepsilon(\lambda)(P_{\varepsilon z}-P_\varepsilon)]^r R_\varepsilon(\lambda).$$

Here, considering expansion (21) we have:

$$R_\varepsilon(\lambda,z)=R_\varepsilon(\lambda)+\sum_{r=1}^{\infty}\left\{R_\varepsilon(\lambda)\left[\sum_{l=1}^{k}\frac{P_\varepsilon^{(l)}}{l!}z^l+T_{\varepsilon k}(z)\right]\right\}^r R_\varepsilon(\lambda).$$

Considering denotation (37), hence we get:

$$R_\varepsilon(\lambda,z)=R_\varepsilon(\lambda)+\sum_{r=1}^{\infty}\left[\sum_{l=1}^{k+1}\frac{B_\varepsilon^{(l)}(\lambda)}{l!}z^l\right]^r R_\varepsilon(\lambda). \tag{43}$$

Allowing for identity (35) we have:

$$\left[\sum_{l=1}^{k+1}\frac{B_\varepsilon^{(l)}(\lambda)}{l!}z^l\right]^r=\sum_{1\le s_1,...,s_r\le k+1}\frac{1}{s_1!s_2!...s_r!}B_\varepsilon^{(s_1)}(\lambda)B_\varepsilon^{(s_2)}(\lambda)...$$

$$...B_\varepsilon^{(s_r)}(\lambda)R_\varepsilon(\lambda)z^{s_1+s_2+...+s_r}. \tag{44}$$

From $(43), (44)$ allowing for identity (36) and regrouping in powers of z we get:

$$R_\varepsilon(\lambda, z) = R_\varepsilon(\lambda) + \sum_{s=1}^{k}\sum_{r=1}^{s}\sum \frac{1}{s_1!s_2!...s_r!} B_\varepsilon^{(s_1)}(\lambda) B_\varepsilon^{(s_2)}(\lambda) ... B_\varepsilon^{(s_r)}(\lambda) R_\varepsilon(\lambda) z^s +$$

$$+ \sum_{s=k+1}^{\infty}\sum_{r=1}^{s}\sum \frac{1}{s_1!s_2!...s_r!} B_\varepsilon^{(s_1)}(\lambda) B_\varepsilon^{(s_2)}(\lambda) ... B_\varepsilon^{(s_r)}(\lambda) R_\varepsilon(\lambda) z^s, \tag{45}$$

where the sign $\sum$ means summation over entire non-negative solutions of the equation

$$\begin{cases} s_1 + s_2 + ... + s_r = s \\ 1 \leq s_1, s_2, ..., s_r \leq k+1 \end{cases}$$

Integral solutions of this equation for $1 \leq r \leq s \leq k$ are of the form of $1 \leq s_1, s_2, ..., s_r \leq k$. Therefore, considering (38) from (45) we get expansion for the resolvent of the operator $P_{\varepsilon z}$.

For obtaining the estimations $(41), (42)$ we notice that in the course of the proof of lemma 1 we showed that in the domain $E \quad \|R_{2\varepsilon}(\lambda)\| \leq M(\rho, h)$. Therefore, for $1 \leq s \leq k$ we have:

$$\left\|R_z^{(s)}(\lambda)\right\| \leq \sum_{r=1}^{s} \sum_{\substack{s_1+...+s_r \ =s \\ 1\leq s_1,...,s_k \leq k}} \frac{1}{s_1!...s_k!} \left\|B_\varepsilon^{(s_1)}(\lambda)\right\| ... \left\|B_\varepsilon^{(s_r)}(\lambda)\right\| \|R_\varepsilon(\lambda)\| \leq$$

$$\leq M(\rho, h) \sum_{r=1}^{\infty} M^r(\rho, h) \sum_{\substack{r_0+r_1+...+r_k=r \\ r_1+...+kr_k=s}} \frac{r!}{r_0!r_1!...r_k!} \left(\frac{\left\|P_\varepsilon^{(1)}\right\|}{1!}\right)^{r_1} \times$$

$$\times \left(\frac{\left\|P_\varepsilon^{(2)}\right\|}{2!}\right)^{r_2} ... \left(\frac{\left\|P_\varepsilon^{(k)}\right\|}{k!}\right)^{r_k}.$$

Here, considering the estimation (20), allowing for the inequalities

$$eM(\rho, h) - 1 \geq \left(e - \frac{1}{9}\right) M(\rho, h), \qquad \frac{e}{e - \dfrac{1}{9}} \leq \frac{26}{23} \tag{46}$$

we get the estimation (41).

Estimation (42) is obtained in the similar way.

The theorem is proved.

Using expansion (40) we can get expansion in powers of z for the operator $\Pi_{\varepsilon z}$, as well. To this end we introduce the denotation:

$$\begin{gathered} \Pi_\varepsilon^{(s)} = \tfrac{1}{2\pi i} \int_{J_1} R_\varepsilon^{(s)}(\lambda, z)\, d\lambda, \quad s = 1, 2, ..., k \\ \Pi_\varepsilon^{(k+1)}(z) = \tfrac{1}{2\pi i} \int_{J_1} R_\varepsilon^{(k+1)}(\lambda, z)\, d\lambda. \end{gathered} \tag{47}$$

The existence of these integrals for z indicated in theorem 1 follows from the estimations (41) and (42).

Corollary 1. *By fulfilling the conditions of theorem* 1 *for the indicated* z *it holds the expansion*

$$\Pi_{\varepsilon z}=\Pi_{\varepsilon}+\sum_{s=1}^{k}\Pi_{\varepsilon}^{(s)}z^{s}+\Pi_{\varepsilon}^{(k+1)}(z), \tag{48}$$

moreover, uniformly with respect to ε

$$\left\|\Pi_{\varepsilon}^{(s)}\right\|\leq\frac{26}{69}M(\rho,h)\left[D(L_{k})\right]^{s} \tag{49}$$

$$\left\|\Pi_{\varepsilon}^{(k+1)}(z)\right\|\leq 6M(\rho,h)\left[D(L_{k})\right]^{k+1}|z|^{k+1}. \tag{50}$$

Proof. Really, for z indicated in theorem 1 as it follows from lemma 1 the circumference J_1 lies on a resolvent set of the operator $P_{\varepsilon z}$, and therefore we can define the projector

$$\Pi_{\varepsilon z}=\frac{1}{2\pi i}\int\limits_{J_{1}}R_{\varepsilon}(\lambda,z)\,d\lambda.$$

Here, considering (40) we get expansion (48).

Taking into account that the radius of the circumference $J_1\to\rho_1<\frac{1}{3}$, we get estimations (49), (50) from (41), (42). Using expansion (40) we can obtain expansion for the functional

$$a_{\varepsilon z}\equiv\langle P_{\varepsilon z}\Pi_{\varepsilon z}\psi,\pi_{\varepsilon}\rangle. \tag{51}$$

To this end we denote:

$$\begin{aligned} a_{r\varepsilon}&=<\tfrac{1}{2\pi i}\int\limits_{J_{1}}\lambda R_{\varepsilon}^{(r)}(\lambda,z)\,\psi d\lambda,\pi_{\varepsilon}> \qquad r=\overline{1,k}\\ a_{k+1,\varepsilon}(z)&=<\tfrac{1}{2\pi i}\int\limits_{J_{1}}\lambda R_{\varepsilon}^{(k+1)}(\lambda,z)\,\psi d\lambda,\pi_{\varepsilon}> \end{aligned} \tag{52}$$

Existence of these integrals by fulfilling the conditions of theorem 1 follow from the estimations (41), (42).

Corollary 2. *By fulfilling the conditions of theorem* 1 *for the indicated* z *it holds the expansion:*

$$a_{\varepsilon z}=1+\sum_{r=1}^{k}a_{r\varepsilon}z^{r}+a_{k+1,\varepsilon}(z), \tag{53}$$

moreover, uniformly with respect to ε

$$|a_{r\varepsilon}|\leq\frac{26}{207}M(\rho,h)\left[D(L_{k})\right]^{r}, \tag{54}$$

$$|a_{k+1,\varepsilon}(z)|\leq 2M(\rho,h)\left[D(L_{k})\right]^{k+1}|z|^{k+1}. \tag{55}$$

Proof. Considering that

$$P_{\varepsilon z}\Pi_{\varepsilon z}=\frac{1}{2\pi i}\int\limits_{J_1}\lambda R_{\varepsilon}(\lambda,z)\,d\lambda \tag{56}$$

for the z, indicated in the theorem, J_1 belongs to the resolvent set of the operator $P_{\varepsilon z}$ and therefore that integral is defined.

Taking into account expansion (40) in (56), we have:

$$a_{r\varepsilon}=<\frac{1}{2\pi i}\int\limits_{J_1}\lambda R_{\varepsilon}(\lambda)\,\psi d\lambda,\pi_{\varepsilon}>+\sum_{r=1}^{k}<\frac{1}{2\pi i}\int\limits_{J_1}\lambda R_{\varepsilon}^{(r)}(\lambda)\,\psi d\lambda,\pi_{\varepsilon}>z^r+$$

$$+<\frac{1}{2\pi i}\int\limits_{J_1}\lambda R_{\varepsilon}^{(k+1)}(\lambda,z)\,\psi d\lambda,\pi_{\varepsilon}>. \tag{57}$$

It is easy to show that

$$<\frac{1}{2\pi i}\int\limits_{J_1}\lambda R_{\varepsilon}(\lambda)\,\psi d\lambda,\pi_{\varepsilon}>=1. \tag{58}$$

Thus, from (57), (58) allowing for (52) we get (53).

Estimates (51), (55) are obtained from the inequalities $|a_{r\varepsilon}|\leq\frac{1}{9}\left\|R_{\varepsilon}^{(r)}(\lambda)\right\|$, $|a_{k+1,\varepsilon}(z)|\leq\frac{1}{9}\left\|R_{\varepsilon}^{(k+1)}(\lambda)\right\|$ and from the estimations (41) (42).

In a similar way, it is easy to get expansion for the functional

$$b_{\varepsilon z}\equiv\langle\Pi_{\varepsilon z}\psi,\pi_{\varepsilon}\rangle, \tag{59}$$

where $\psi(\cdot)\equiv 1$, $\pi_{\varepsilon}(\cdot)$ is a stationary measure.

Let

$$\begin{aligned} b_{r\varepsilon}&\equiv\left\langle\Pi_{\varepsilon}^{(r)}\psi,\pi_{\varepsilon}\right\rangle, \qquad r=\overline{1,k}\\ b_{k+1,\varepsilon}(z)&=\left\langle\Pi_{\varepsilon}^{(k+1)}(z)\,\psi,\pi_{\varepsilon}\right\rangle. \end{aligned} \tag{60}$$

For z indicated in theorem 1 the existence of these functionals are provided by estimations (49), (50).

Corollary 3. *In fulfilling the conditions of theorem 1, for the indicated z it holds the expansion*

$$b_{\varepsilon z}=1+\sum_{r=1}^{k}b_{r\varepsilon}z^r+b_{k+1,\varepsilon}(z), \tag{61}$$

moreover, uniformly with respect to ε

$$|b_{r\varepsilon}|\leq\frac{26}{69}M(\rho,h)\,[D(L_k)]^r, \qquad r=\overline{1,k} \tag{62}$$

$$|b_{k+1,\varepsilon}(z)|\leq 6M(\rho,h)\,[D(L_k)]^{k+1}|z|^{k+1}. \tag{63}$$

In lemma 2 it is confirmed that the eigen function of the operator $P_{\varepsilon z}$ is analytic for all complex z for which

$$|z| < \min\left\{\frac{1}{4M^2(\rho,h)\,L_k^{1/(k+1)}}, b\right\}.$$

It follows from (51) and (59) that

$$\Lambda_\varepsilon(z) = a_{\varepsilon z}\cdot b_{\varepsilon z}^{-1}. \tag{64}$$

The goal of the next step is to obtain the expansion for $\Lambda_\varepsilon(z)$.

At first we expand $b_{\varepsilon z}^{-1}$ in powers of z.

Introduce the denotation:

$$d_{r\varepsilon} = \sum_{m=1}^{r}(-1)^m \sum_{\substack{m_1+\ldots+m_k\ =m\\ m_1+\ldots+km_k=r}} \frac{m!}{m_1!\ldots m_k!} b_{1\varepsilon}^{m_1}\ldots b_{k\varepsilon}^{m_k}, \quad r=\overline{1,k} \tag{65}$$

$$d_{(k+1)\varepsilon}(z) =$$

$$= \sum_{r=k+1}^{\infty}\sum_{m=1}^{r}(-1)^m \sum_{\substack{m_1+\ldots+m_k=m\\ m_1+\ldots+(k+1)m_{k+1}=r}} \frac{m!}{m_1!\ldots m_k!} b_{1\varepsilon}^{m_1}\ldots b_{k\varepsilon}^{m_k}\left(b_{k+1,\varepsilon}(z)\,z^{-k-1}\right)^{m_{k+1}} z^r.$$

It holds

Lemma 3. *Let the condition $A_{\rho,h}$ and for some $k\geq 2$ the condition B_k be fulfilled. Then for all z for which*

$$|z| < \min\left\{\frac{1}{14M^2(\rho,h)\,[D(L_k)]^{k+1}}, b\right\},$$

$b_{\varepsilon z}\neq 0$ *and it holds the expansion*

$$b_{\varepsilon z}^{-1} = 1 + \sum_{r=1}^{k} d_{r\varepsilon}z^r + d_{k+1,\varepsilon}(z), \tag{66}$$

moreover, uniformly with respect to ε

$$|d_{r\varepsilon}| \leq \frac{101}{100}\left\{\frac{2}{5}M(\rho,h)\,[D(L_k)]^k\right\}^r, \qquad r=\overline{1,k} \tag{67}$$

$$|d_{k+1,\varepsilon}(z)| \leq 3\left\{7M(\rho,h)\,[D(L_k)]^{k+1}\right\}^{k+1}|z|^{k+1}. \tag{68}$$

Because of analyticity we omit the proof of this lemma.

Thus, by means of (66) and (53) we can get expansions in powers of z for the eigen function $\Lambda_\varepsilon(z)$.

Introduce the denotation:

$$\vartheta_{r\varepsilon} = \sum_{s=1}^{k-1} a_{r-s,\varepsilon}d_{s\varepsilon} + a_{r\varepsilon} + d_{s\varepsilon}$$

$$r = 1, 2, ..., k$$

$$\sum_1^0 = 0 \tag{69}$$

$$\vartheta_{k+1,\varepsilon}(z) = \sum_{s=k+1}^{2k} \sum_{r=s-k}^{k} a_{r-s,\varepsilon} d_{s\varepsilon} z^s + \left(1 + \sum_{r=1}^{1} a_{r\varepsilon} z^r\right) d_{k+1,\varepsilon}(z) +$$

$$+ a_{k+1,\varepsilon}(z) \left(1 + \sum_{r=1}^{1} d_{r\varepsilon} z^r\right) + a_{k+1,\varepsilon}(z)\, d_{k+1,\varepsilon}(z).$$

The following theorem is valid.

Theorem 2. *Let the conditions $A_{\rho,h}$ and for some $k \geq 2$ the conditions B_k be fulfilled when for all complex z for which*

$$|z| < \min\left\{\frac{1}{14M(\rho, h)\,[D(L_k)]^{k+1}}, b\right\}$$

it holds the expansion

$$\Lambda_\varepsilon(z) = 1 + \sum_{r=1}^{k} \vartheta_{r\varepsilon} z^r + \vartheta_{k+1,\varepsilon}(z), \tag{70}$$

moreover, uniformly with respect to ε

$$|\vartheta_{r\varepsilon}| \leq 2\left\{\frac{2}{5} M(\rho, h)\,[D(L_k)]^k\right\}^r \tag{71}$$

$$|\vartheta_{k+1,\varepsilon}(z)| \leq 4\left\{7M(\rho, h)\,[D(L_k)]^{k+1}\right\}^{k+1} |z|^{k+1}. \tag{72}$$

Proof. By fulfilling the conditions of the present theorem the conditions of theorem 1 and lemma 3 are also fulfilled, and from $(53), (66)$ grouping in powers of z, allowing for (69) we get (70).

For obtaining the estimations $(71), (72)$ we use the estimation $(54), (55), (67)$ and (68).

It follows from the proved theorem that by fulfilling the conditions of the theorem we can choose z so that the eigen function $\Lambda_\varepsilon(z)$ were isolated from zero. For such z the definition $\ln \Lambda_\varepsilon(z)$ will be correct, and its expansion in powers of z will have very great value later on.

This will be considered in the second part of the paper.

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