MECHANICS

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ON NATURAL TRANSVERSE VIBRATIONS OF ORTHOTROPIC PLATE ALLOWING FOR FOUNDATION RESISTANCE INFLUENCE

Abstract

In the paper we solve a problem on natural vibration of continuously in homogeneous rectangular plate. It is assumed that elasticity and shear modules, specific density are exponential functions of length coordinate of a plate. The problem is solved by approximate methods.

It is known that in engineering complexes constructions, in particular in constructions of highways, the structural elements of plate-strip type made of inhomogeneous material are widely used.

As these structural elements are assigned for operations at dynamical conditions and also at conditions of soils inhomogeneous in physical and mechanical properties, as a rule, projection of such structural elements are based on calculation formulae of dynamical stability of inhomogeneous (orthotropic) bodies [1,2].

The problem under investigation is solved under the following assumptions:

-elasticity modules (E_1, E_2), shear module (G) and specific density (ρ) are exponential functions of the length coordinate x. And a coordinate system is chosen in the following way: the axes X and Y are on the mean plane, the axis Z is directed perpendicular to them;

-mechanical characteristics of an inhomogeneous material are accepted in the form [3]:

$$E_1 = E_1^0 e^{-f(x)},\ E_2 = E_2^0 e^{-f(x)},\ G = G_0 e^{-f(x)},\ \rho = \rho_0 e^{-f(x)} \tag{1}$$

here $E_1^0, E_2^0, G_0, \rho_0$ correspond to homogeneous case, $f(x)$ with their derivatives are continuous functions; it is assumed that the plate lies on anisotropic foundation [3]:

$$F = k_1 w - k_2 \frac{h^2}{4}\frac{\partial^2 w}{\partial x^2} - k_3 \frac{h^2}{4}\frac{\partial^2 w}{\partial y^2}. \tag{2}$$

Here F is foundation reaction; k_1, k_2, k_3 are resistance characteristics in principal directions and depend on the properties of foundation (these characteristics are experimentally determined); w is flexure.

Under these suppositions relation between stresses $\sigma_1, \sigma_2, \sigma_{12}$ and strains $\varepsilon_1, \varepsilon_2, \varepsilon_{12}$

is of the form [1]:

$$\sigma_1 = \frac{E_1^0 e^{-f(x)}}{1-\nu_1\nu_2}\left(\varepsilon_1 + \nu_2\varepsilon_2\right),$$
$$\sigma_2 = \frac{E_2^0 e^{-f(x)}}{1-\nu_1\nu_2}\left(\varepsilon_2 + \nu_1\varepsilon_1\right),$$
$$\sigma_{12} = G_0 e^{-f(x)}\varepsilon_{12},$$

where $\nu_1\nu_2$ are Poisson ratios and are accepted to be constant.

We write stresses of bending in arbitrary layer of the plate and also expressions for moments in the form [2]:

$$\begin{gathered}\sigma_{1,n} = -\frac{E_1^0 e^{-f(x)}}{1-\nu_1\nu_2} z\left(\frac{\partial^2 w}{\partial x^2} + \nu_2\frac{\partial^2 w}{\partial y^2}\right),\\ \sigma_{2,n} = -\frac{E_2^0 e^{-f(x)}}{1-\nu_1\nu_2} z\left(\frac{\partial^2 w}{\partial y^2} + \nu_1\frac{\partial^2 w}{\partial x^2}\right),\\ \sigma_{12,n} = -2G_0 e^{-f(x)} z\frac{\partial^2 w}{\partial x\partial y}.\end{gathered} \tag{3}$$

$$M_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{1,n} z dz, \quad M_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{2,n} z dz, \quad M_{12} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{12,n} z dz. \tag{4}$$

Allowing for (3) from (4) we get:

$$\begin{gathered}M_1 = -D_1^0 e^{-f(x)}\left(\frac{\partial^2 w}{\partial x^2} + \nu_2\frac{\partial^2 w}{\partial y^2}\right),\\ M_2 = -D_2^0 e^{-f(x)}\left(\frac{\partial^2 w}{\partial y^2} + \nu_1\frac{\partial^2 w}{\partial x^2}\right),\\ M_{12} = -2D_k^0 e^{-f(x)}\frac{\partial^2 w}{\partial x\partial y}.\end{gathered} \tag{5}$$

Here D_1^0 and D_2^0 are flexural rigidities in principal directions for a homogeneous orthotropic material, D_k^0 is a torsional rigidity.

$$D_1^0 = \frac{E_1^0 h^3}{12\left(1-\nu_1\nu_2\right)}; \quad D_2^0 = \frac{E_2^0 h^3}{12\left(1-\nu_1\nu_2\right)}; \quad D_k^0 = \frac{G_0 h^3}{12}.$$

We write an equation of motion of transverse natural vibration of the strip in the following form:

$$\begin{gathered}\frac{\partial^2 M_1}{\partial x^2} + 2\frac{\partial^2 M_{12}}{\partial x\partial y} + \frac{\partial^2 M_2}{\partial y^2} + k_1 w - k_2\tfrac{h^2}{4}\frac{\partial^2 w}{\partial x^2} -\\ -k_3\tfrac{h^2}{4}\frac{\partial^2 w}{\partial y^2} + \rho_0 e^{-f(x)}\frac{\partial^2 w}{\partial t^2} = 0\end{gathered} \tag{6}$$

Substituting (5) into (6) we get:

$$\begin{gathered}D_1^0\frac{\partial^4 w}{\partial x^4} + \left(D_1^0\nu_2 + D_2^0\nu_1 + 4D_k^0\right)\frac{\partial^4 w}{\partial x^2\partial y^2} + D_2^0\frac{\partial^4 w}{\partial y^4} - D_1^0\left[\frac{d^2 f(x)}{dx^2}\times\right.\\ \left.\times\left(\frac{\partial^2 w}{\partial x^2} + \nu_2\frac{\partial^2 w}{\partial y^2}\right) + 2\frac{df(x)}{dx}\frac{\partial^3 w}{\partial x^3}\right] - 2\left(D_1^0\nu_2 + 2D_k^0\right)\frac{df}{dx}\frac{\partial^3 w}{\partial x\partial y^2} -\\ -e^{f(x)}\left(k_1 w - k_2\frac{h^2}{4}\frac{\partial^2 w}{\partial x^2} - k_3\frac{h^2}{4}\frac{\partial^2 w}{\partial y^2}\right) + \rho_0\frac{\partial^2 w}{\partial t^2} = 0.\end{gathered} \tag{7}$$

By virtue of mathematical complexity of equation (7) we construct approximate analytic solution of the stated problem. We'll solve the problem by a combined method; i.e. at first we'll use the method of separation of variables and then the Bubnov-Galerkin method. Therefore, we'll look for the solution of equation (7) in the form:

$$w = \upsilon(x,y)e^{i\omega t}, \tag{8}$$

where ω is circular frequency, t is time, the function $\upsilon(x,y)$ should satisfy the appropriate boundary conditions.

Substituting (8) into (7) we get:

$$\begin{gathered} D_1^0\frac{\partial^4\upsilon}{\partial x^4} + \left(D_1^0\nu_2 + D_2^0\nu_1 + 4D_k^0\right)\frac{\partial^4\upsilon}{\partial x^2\partial y^2} + D_2^0\frac{\partial^4\upsilon}{\partial y^4} - D_1^0\left[\frac{d^2f(x)}{dx^2}\times\right. \\ \left.\times\left(\frac{\partial^2\upsilon}{\partial x^2} + \nu_2\frac{\partial^2\upsilon}{\partial y^2}\right) + 2\frac{df(x)}{dx}\frac{\partial^3\upsilon}{\partial x^3}\right] - 2\left(D_1^0\nu_2 + 2D_k^0\right)\frac{df}{dx}\frac{\partial^3\upsilon}{\partial x\partial y^2} - \\ -e^{f(x)}\left(k_1\upsilon - k_2\frac{h^2}{4}\frac{\partial^2\upsilon}{\partial x^2} - k_3\frac{h^2}{4}\frac{\partial^2\upsilon}{\partial y^2}\right) - \rho_0\omega^2\upsilon = 0. \end{gathered} \tag{9}$$

Apply the Bubnov-Galerkin method to equation (9). We'll look for the solution of the function $\upsilon(x,y)$ in the form:

$$\upsilon(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{l} A_{ij}F_i(x)\Phi_j(y). \tag{10}$$

Here, it is assumed that each of functions $F_i(x)$, $\Phi_j(y)$ satisfies the boundary conditions of the stated problem.

Introduce the following denotation:

$$\begin{gathered} H_1 = D_1^0\Phi_j\frac{d^4F_i}{dx^4} + \left(D_1^0\nu_2 + D_2^0\nu_1 + 4D_k^0\right)\frac{d^2F_i}{dx^2}\frac{d^2\Phi_j}{dy^2} + D_2^0F_i\frac{d^4\Phi_j}{dy^4}; \\ H_2 = D_1^0\left[\frac{d^2f(x)}{dx^2}\left(\Phi_j\frac{d^2F_i}{dx^2} + \nu_2F_i\frac{d^2\Phi_j}{dy^2}\right) + 2\frac{df(x)}{dx}\Phi_j\frac{d^3F_i}{dx^3}\right] + \\ +2\left(D_1^0\nu_2 + 2D_k^0\right)\frac{df}{dx}\frac{dF_i}{dx}\frac{d^2\Phi_j}{dy^2}; \\ H_3 = e^{f(x)}\left(k_1F_i\Phi_j - k_2\frac{h^2}{4}\Phi_j\frac{\partial^2F_i}{\partial x^2} - k_3\frac{h^2}{4}F_i\frac{\partial^2\Phi_j}{\partial y^2}\right). \end{gathered} \tag{11}$$

It is known that in solving general problem, circular frequency should be determined from a system of linear homogeneous algebraic equations. Moreover, for the existence of non-trivial solution of the problem the main determinant of this system composed of coefficients A_{ij} should equal zero. However in creating engineering calculations we are restricted with the first approximation. Therefore, for the first approximation we get:

$$\int_0^a\int_0^b\left[H_1\left(F_1,\Phi_1\right) - H_2\left(F_1,\Phi_1\right) - H_3\left(F_1,\Phi_1\right) - \rho_0\omega^2F_1\Phi_1\right]F_1\Phi_1dxdy = 0,$$

whence we determine expression of circular frequency ω in the form:

$$\omega^2 = \frac{\int\limits_0^a \int\limits_0^b \left[H_1(F_1,\Phi_1) - H_2(F_1,\Phi_1) - H_3(F_1,\Phi_1)\right] F_1\Phi_1 dxdy}{\rho_0 \int\limits_0^a \int\limits_0^b F_1^2\Phi_1^2 dxdy} \tag{12}$$

It we ignore the influence of foundation resistance the formulae (12) accepts the following simplified form:

$$\omega_1^2 = \frac{\int\limits_0^a \int\limits_0^b \left[H_1(F_1,\Phi_1) - H_2(F_1,\Phi_1)\right] F_1\Phi_1 dxdy}{\rho_0 \int\limits_0^a \int\limits_0^b F_1^2\Phi_1^2 dxdy} \tag{13}$$

Solving jointly (12) and (13) we express the value of the circular frequency ω arising allowing for orthotropic foundation influence from circular frequency ω_1, appearing in the strip without influence of foundation resistance in the form:

$$\overline{\omega}^2 = \frac{\omega^2}{\omega_1^2} = \frac{\int\limits_0^a \int\limits_0^b \left[H_1(F_1,\Phi_1) - H_2(F_1,\Phi_1) - H_3(F_1,\Phi_1)\right] F_1\Phi_1 dxdy}{\int\limits_0^a \int\limits_0^b \left[H_1(F_1,\Phi_1) - H_2(F_1,\Phi_1)\right] F_1\Phi_1 dxdy}$$

or

$$\overline{\omega}^2 = 1 - \frac{\int\limits_0^a \int\limits_0^b H_3(F_1,\Phi_1) F_1\Phi_1 dxdy}{\int\limits_0^a \int\limits_0^b \left[H_1(F_1,\Phi_1) - H_2(F_1,\Phi_1)\right] F_1\Phi_1 dxdy} \tag{14}$$

It follows from (14) that

$$C = \frac{\int\limits_0^a \int\limits_0^b H_3(F_1,\Phi_1) F_1\Phi_1 dxdy}{\int\limits_0^a \int\limits_0^b \left[H_1(F_1,\Phi_1) - H_2(F_1,\Phi_1)\right] F_1\Phi_1 dxdy} < 1 \tag{15}$$

For F_1 and Φ_1 we accept the expressions of the form:

$$F_1 = \sin\frac{m\pi}{a}x, \quad \Phi_1 = \sin\frac{n\pi}{b}y \tag{16}$$

substitute into (11), for $f(x) = 1 + \varepsilon\frac{x}{a}$ we get:

$$\begin{gathered} H_1 = \pi^4\left[D_1^0 m^4 + \left(D_1^0\nu_2 + D_2^0\nu_1 + 4D_k^0\right)m^2n^2 + D_2^0 n^4\right]\sin m\pi\overline{x}\sin n\pi\overline{y}; \\ H_2 = -2\varepsilon\pi^3\left[\left(D_1^0\nu_2 + 2D_k^0\right)mn^2 + D_1^0 m^3\right]\cos m\pi\overline{x}\sin n\pi\overline{y}; \\ H_3 = \left[k_1 + \frac{\pi^2h^2}{4}\left(k_2m^2 + k_3n^2\right)\right]e^{1+\varepsilon\overline{x}}\sin m\pi\overline{x}\sin n\pi\overline{y}; \end{gathered} \tag{17}$$

where

$$\overline{x} = \frac{x}{a}, \quad \overline{y} = \frac{y}{b}.$$

Allowing for (17) formulae (15) admits to determine dependence of the constant C on all entering constants, in the form:

$$C = \frac{4k_1 + h^2\pi^2 \left(k_2 m^2 + k_3 n^2\right)}{\pi^4 \left[D_1^0 m^4 + \left(D_1^0 \nu_2 + D_2^0 \nu_1 + 4D_k^0\right) m^2 n^2 + D_2^0 n^4\right]} \cdot \frac{e(e^{\varepsilon} - 1)}{2\varepsilon} \qquad (18).$$

Considering (18) in (14) we determine dependence of circular frequency of orthotropic strip allowing for influence of physically inhomogeneous foundation.

$$\overline{\omega}^2 = 1 - B \cdot \frac{e(e^{\varepsilon} - 1)}{2\varepsilon}, \qquad (19)$$

where

$$B = \frac{4k_1 + h^2\pi^2 \left(k_2 m^2 + k_3 n^2\right)}{\pi^4 \left[D_1^0 m^4 + \left(D_1^0 \nu_2 + D_2^0 \nu_1 + 4D_k^0\right) m^2 n^2 + D_2^0 n^4\right]}. \qquad (20)$$

Graphical dependence of circular frequency $\overline{\omega}^2 = \dfrac{\omega^2}{\omega_1^2}$ on the inhomogeneity degree ε of the material of the strip is in figure 1.

Fig. 1. Dependence of circular frequency on the degree of inhomogeneity of the strip material for different values of the reduced coefficient B.

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Received May 17, 2007; Revised July 27, 2007.