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ON EIGENVALUES AND EIGENFUNCTIONS OF DIRAC OPERATOR WITH DISCONTINUITY CONDITIONS INTERIOR TO INTERVAL

Abstract

In the paper we study properties of eigenvalues and eigenfunctions for the Dirac operator with discontinuity conditions interior to interval.

Denote by L a boundary-value problem, generated by the canonic system of Dirac differential equations

$$By' + \Omega(x)y = \lambda y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{1}$$

with boundary conditions

$$y_1(0) = y_1(\pi) = 0 \tag{2}$$

and with discontinuity conditions at the interior point $x = a$ of the interval $(0, \pi)$

$$y(a+0) = My(a-0) \tag{3}$$

here

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix},$$

$p(x)$, $q(x)$ are real valued functions from the space $L_2(0,\pi)$, M is matrix of second order, $\det M \neq 0$, $M \neq I$ (I is a unit matrix). We can show that, if the matrix M satisfies the condition

$$M^* BM = B \tag{4}$$

then the boundary-value problem L is selfadjoint.

In the sequel we will suppose that, the matrix M has the form

$$M = \begin{pmatrix} \alpha & 0 \\ \beta & \alpha^{-1} \end{pmatrix},$$

where $\alpha > 0$, $\beta \in R$ and condition (4) holds. In the given paper we find asymptotics of eigenvalues and normalizing numbers.

In the case of classical Dirac operator the similar problem was sufficiently studied (see e.g. [1-3]).

Asymptotics of eigenvalues and normalizing numbers of the operator L

At first, we consider the case $\Omega(x) \equiv 0$ and in this case we denote problem L by L_0. It is easy to show, that the solution of the equation $Bs_0' = \lambda s_0$ with initial condition $s_0(0,\lambda) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and conditions (3) has the form

$$s_0(x,\lambda) = \begin{cases} \begin{pmatrix} \sin \lambda x \\ -\cos \lambda x \end{pmatrix}, & 0 < x < a \\ M^- \begin{pmatrix} \sin \lambda x \\ -\cos \lambda x \end{pmatrix} + M^+ \begin{pmatrix} \sin \lambda (2a - x) \\ -\cos \lambda (2a - x) \end{pmatrix}, & a < x < \pi \end{cases} \tag{6}$$

where

$$M^{\pm}=\begin{pmatrix}\alpha^{\mp} & \pm\frac{\beta}{2}\\ \frac{\beta}{2} & \mp\ \alpha^{\mp}\end{pmatrix}.$$

Then the characteristic function of the problem L_0 will have the form:

$$\triangle_0(\lambda)=s_{01}(\pi,\lambda)=\alpha^{+}\sin\lambda\pi+\frac{\beta}{2}\cos\lambda\pi+$$

$$+\alpha^{-}\sin\lambda(2a-\pi)-\frac{\beta}{2}\cos\lambda(2a-\pi). \tag{7}$$

The roots of this function λ_n^0 are the eigenvalues of the problem L_0 and let

$$...<\lambda_{-2}^0<\lambda_{-1}^0<\lambda_0^0=0<\lambda_1^0<\lambda_2^0<...$$

Lemma 1. *Eigenvalues $\left\{\lambda_n^0\right\}$ of the problem L_0 are isolated, i.e.*

$$\inf_{n\neq m}\left|\lambda_n^0-\lambda_m^0\right|=\gamma>0.$$

Proof. We suppose the contrary i.e. let $\gamma=0$. Then we could select also the sequence of the zeros $\widehat{\lambda}_{n_k}^0$, $\lambda_{n_k}^0$ of the function $\triangle_0(\lambda)$, that $\lim\limits_{k\to\infty}\left|\widehat{\lambda}_{n_k}^0-\lambda_{n_k}^0\right|=0$, $\widehat{\lambda}_{n_k}^0\neq\lambda_{n_k}^0$, $\lambda_{n_k}^0\to\infty$, $\widehat{\lambda}_{n_k}^0\to\infty$ as $k\to\infty$. It follows from the form of solution $s_0(x,\lambda)$ that

$$\int_0^{\pi}\left\langle s_0\left(x,\lambda_{n_k}^0\right),\ s_0\left(x,\lambda_{n_k}^0\right)\right\rangle dx\geq\int_0^{a}\left\langle s_0\left(x,\lambda_{n_k}^0\right),\ s_0\left(x,\lambda_{n_k}^0\right)\right\rangle dx=$$

$$=\int_0^{a}\left\{\sin^2\lambda_{n_k}^0x+\cos^2\lambda_{n_k}^0x\right\}dx=a.$$

Here and in the sequel by $\langle .,.\rangle$ we denote scalar product in the eucledean space R^2, $\|\cdot\|=\sqrt{\langle .,.\rangle}$. On the other hand, by virtue of selfadjointness of the operator L_0 the eigenfunctions $s_0(x,\lambda_{n_k}^0)$ and $s_0(x,\widehat{\lambda}_{n_k}^0)$ are orthogonal in the space $L_2\left(0,\pi;R^2\right)$.

Therefore

$$0=\int_0^{\pi}\left\langle s_0\left(x,\lambda_{n_k}^0\right),\ s_0\left(x,\widehat{\lambda}_{n_k}^0\right)\right\rangle dx=\int_0^{\pi}\left\langle s_0\left(x,\lambda_{n_k}^0\right),\ s_0\left(x,\lambda_{n_k}^0\right)\right\rangle dx+$$

$$+\int_0^{\pi}\left\langle s_0\left(x,\lambda_{n_k}^0\right),\ s_0\left(x,\widehat{\lambda}_{n_k}^0\right)-s_0\left(x,\lambda_{n_k}^0\right)\right\rangle dx\geq \tag{8}$$

$$\geq a+\int_0^{\pi}\left\langle s_0\left(x,\lambda_{n_k}^0\right),\ s_0\left(x,\widehat{\lambda}_{n_k}^0\right)-s_0\left(x,\lambda_{n_k}^0\right)\right\rangle dx.$$

From the form of solution $s_0(x,\lambda)$ it follows that

$$\left|\int_0^{\pi}\left\langle s_0\left(x,\lambda_{n_k}^0\right),\ s_0\left(x,\widehat{\lambda}_{n_k}^0\right)-s_0\left(x,\lambda_{n_k}^0\right)\right\rangle dx\right|\leq$$

$$\leq C \max_{0\leq x\leq\pi} \left\| s_0\left(x, \widehat{\lambda}^0_{n_k}\right) - s_0\left(x, \lambda^0_{n_k}\right)\right\| \to 0, \quad k\to\infty$$

Thus, passing to limit as $k\to\infty$ in the inequality (8) we have $0\geq a$. Therefore, the maid assumption is not true, $\gamma>0$ and the roots of the function $\triangle_0(\lambda)$ are isolated.

The lemma is proved.

Denote by $\triangle(\lambda)$ and $\{\lambda_n\}$ characteristic function and sequence eigenvalues of the problem L, respectively. Let $s(x,\lambda)=\begin{pmatrix} s_1(x,\lambda)\\ s_2(x,\lambda)\end{pmatrix}$ be a solution of equation (1) satisfying initial condition $s(0,\lambda)=\begin{pmatrix}0\\-1\end{pmatrix}$ and conditions (3). As is known ([4]), the fundamental solution $Y(x,\lambda)$ of Dirac matrix equation exists (for all $\lambda\in\mathbb{C}$) and can be represented in the form

$$Y(x,\lambda) = Y_0(x,\lambda) + \int\limits_{-x}^{x} K(x,t)e^{-\lambda Bt}dt, \tag{9}$$

where

$$Y_0(x,\lambda) = \begin{cases} e^{-\lambda Bx}, & 0<x<a,\\ M^-e^{-\lambda Bx} + M^+e^{-\lambda B(2a-x)}, & a<x<\pi.\end{cases}$$

Elements $K_{ij}(x,\cdot)$ of the matrix function $K(x,\cdot)$ belong to the space $L_2(-x,x)$. Since $s(x,\lambda) = Y(x,\lambda)\begin{pmatrix}0\\-1\end{pmatrix}$, then for the solution $s(x,\lambda)$ we have the following formula

$$s(x,\lambda) = s_0(x,\lambda) + \int\limits_{-x}^{x} K(x,t)\begin{pmatrix}\sin\lambda t\\ -\cos\lambda t\end{pmatrix}dt.$$

Consequently, for the characteristic function $\triangle(\lambda) = s_1(\pi,\lambda)$ we obtain the following representation

$$\triangle(\lambda) = \triangle_0(\lambda) + \int\limits_{-\pi}^{\pi} K_{11}(\pi,t)\sin\lambda t dt - \int\limits_{-\pi}^{\pi} K_{12}(\pi,t)\cos\lambda t dt. \tag{10}$$

Lemma 2. *The eigenvalues of the problem L are simple, i.e.*

$$\dot{\triangle}(\lambda_n) = \frac{d}{d\lambda}\triangle(\lambda)\Big|_{\lambda=\lambda_n} \neq 0.$$

Proof. Since

$$Bs'(x,\lambda) + \Omega(x)s(x,\lambda) = \lambda s(x,\lambda),$$
$$Bs'(x,\lambda_n) + \Omega(x)s(x,\lambda_n) = \lambda_n s(x,\lambda_n),$$

then

$$\frac{d}{dx}\left\langle Bs(x,\lambda), s(x,\lambda_n)\right\rangle = (\lambda-\lambda_n)\left\langle s(x,\lambda), s(x,\lambda_n)\right\rangle,$$

and consequently allowing for (2), (3), (4) we have

$$(\lambda-\lambda_n)\int\limits_0^{\pi}\left\langle s(x,\lambda), s(x,\lambda_n)\right\rangle dx = \left\langle Bs(a-0,\lambda), s(a-0,\lambda_n)\right\rangle +$$

$$+\langle Bs(\pi,\lambda),s(\pi,\lambda_n)\rangle-\langle Bs(a+0,\lambda),s(a+0,\lambda_n)\rangle=$$
$$=-s_1(\pi,\lambda)s_2(\pi,\lambda_n)=-\triangle(\lambda)s_2(\pi,\lambda_n).$$

As $\lambda\to\lambda_n$ this gives

$$\int_0^\pi \langle s(x,\lambda_n),s(x,\lambda_n)\rangle\,dx=-\dot{\triangle}(\lambda_n)s_2(\pi,\lambda_n)$$

or

$$\alpha_n=-\dot{\triangle}(\lambda_n)s_2(\pi,\lambda_n). \tag{11}$$

Hence, it follows that, $\dot{\triangle}(\lambda_n)\neq 0$.

We to find asymptotics of eigenvalues of the problem L, i.e. asymptotics of the roots of the function $\triangle(\lambda)$. Denote by Γ_n counter of a rectangle formed by the segments of the lines

$$\operatorname{Re} z=\lambda_n^0+\frac{\gamma}{2},\ \ \operatorname{Re} z=\lambda_{-n}^0-\frac{\gamma}{2},\ \ \operatorname{Im} z\ =\lambda_n^0,\operatorname{Im} z=-\lambda_n^0,$$

where n is sufficiently large natural number, γ is the number from Lemma 1 and $G_\delta=\left\{\lambda:\left|\lambda-\lambda_n^0\right|\geq\delta,\ n=0,\pm1,\pm2,...\right\}$, where δ is a sufficiently small positive number. From representation $\triangle(\lambda)$ and Lemma 1 it follows that, $\triangle_0(\lambda)$ is a function of type “sinus”. Therefore, for each $\lambda\in G_\delta$ is hold the inequality (see [5], pp.118-119)

$$|\triangle_0(\lambda)|>C_\delta e^{|\operatorname{Im}\lambda|\pi},\quad \left|\dot{\triangle}_0(\lambda_n^0)\right|\geq\gamma_0>0$$

is fulfilled.

On the other hand, by virtue of Lemma 1 [6] and from formula (10) we have

$$\lim_{|\lambda|\to\infty} e^{-|\operatorname{Im}\lambda|\pi}\left(\triangle(\lambda)-\triangle_0(\lambda)\right)=$$
$$=\lim_{|\lambda|\to\infty}\left(e^{-|\operatorname{Im}\lambda|\pi}\int_0^\pi \widetilde{K}_{11}(\pi,t)\sin\lambda t dt+e^{-|\operatorname{Im}\lambda|\pi}\int_0^\pi \widetilde{K}_{12}(\pi,t)\cos\lambda t dt\right)=0.$$

Consequently, for sufficiently large n for $\lambda\in\Gamma_n$ the inequality

$$|\triangle(\lambda)-\triangle_0(\lambda)|<\frac{C_\delta}{2}e^{|\operatorname{Im}\lambda|\pi}$$

is fulfilled.

Hence, for $\lambda\in\Gamma_n$, where n is a sufficiently large natural number, we have

$$|\triangle_0(\lambda)|>C_\delta e^{|\operatorname{Im}\lambda|\pi}>|\triangle(\lambda)-\triangle_0(\lambda)|\,.$$

Now by using Rouche theorem we have that, interior to Γ_n for sufficiently large n the functions $\triangle(\lambda)$ and $\triangle_0(\lambda)$ have the same number of zeros, i.e. $2n+1$ zeros. We denote the zeros of function $\triangle(\lambda)$ interior Γ_n by $\lambda_{-n},...,\lambda_1,\lambda_0,\lambda_1,...,\lambda_n$. Analogously, using the Rouche theorem, we can prove, that for sufficiently large n in every circle $\left|\lambda_n^0-\lambda\right|<\delta$, where δ is an any sufficiently small number, there is only one zero of the function $\triangle(\lambda)$. Therefore, we can write $\lambda_n = \lambda_n^0 + \varepsilon_n$, where $\lim\limits_{n\to\infty} \varepsilon_n = 0$. Consequently,

$$\triangle(\lambda_n) = \triangle_0\left(\lambda_n^0 + \varepsilon_n\right) + \int\limits_{-\pi}^{\pi} K_{11}(\pi,t)\sin\left(\lambda_n^0 + \varepsilon_n\right) tdt -$$

$$-\int\limits_{-\pi}^{\pi} K_{12}(\pi,t)\cos\left(\left(\lambda_n^0 + \varepsilon_n\right)\right) tdt = 0. \tag{12}$$

From the results of the papers [7], [8] it follows that $\lambda_n^0 = n + h_n$, where $\sup\limits_n |h_n| < \infty$. Therefore (see [6], pp.67)

$$\{k_n\} \stackrel{def}{=} \left\{ \int\limits_{-\pi}^{\pi} K_{11}(\pi,t)\sin\left(\lambda_n^0 + \varepsilon_n\right) tdt - \int\limits_{-\pi}^{\pi} K_{12}(\pi,t)\cos\left(\left(\lambda_n^0 + \varepsilon_n\right)\right) tdt \right\} \in l_2.$$

Further, it is obvious that

$$\triangle_0\left(\lambda_n^0 + \varepsilon_n\right) = \dot{\triangle}_0(\lambda_n^0)\left[1 + 0(1)\right]\varepsilon_n.$$

Consequently, equality (12) can be written in the form

$$\varepsilon_n \dot{\triangle}_0(\lambda_n^0)\left[1 + 0(1)\right] + k_n = 0,$$

and hence we obtain that, $\{\varepsilon_n\} \in l_2$. Thus, it holds

Lemma 3. *Let $p(\cdot), q(\cdot) \in L_2(0,\pi)$. Then for the eigenvalues $\{\lambda_n\}$ and normalizing $\{\alpha_n\}$ numbers of the problem L the asymptotic equalities*

$$\lambda_n = \lambda_n^0 + \varepsilon_n, \tag{13}$$

$$\alpha_n = \alpha_n^0 + \frac{\widetilde{k}_n}{n} \tag{14}$$

are valid, where $\{\varepsilon_n\} \in l_2$, $\left\{\widetilde{k}_n\right\} \in l_2$.

Suppose that, $p(\cdot) \in W_2^1[0,\pi]$, $q(\cdot) \in W_2^1[0,\pi]$. Then we can revise asymptotic formula for eigenvalues.

Lemma 4. *Let $p(\cdot) \in W_2^1[0,\pi]$, $q(\cdot) \in W_2^1[0,\pi]$. Then for eigenvalues λ_n of the problem L it holds the following asymptotic formula*

$$\lambda_n = \lambda_n^0 + \frac{h_1\cos\lambda_n^0\pi + h_2\sin\lambda_n^0\pi + h_3\cos\lambda_n^0(2a-\pi) + h_4\sin\lambda_n^0(2a-\pi)}{\lambda_n^0} + \frac{\widetilde{\varepsilon}_n}{n}, \tag{15}$$

where $\{\widetilde{\varepsilon_n}\} \in l_2$,

$$h_1 = \frac{1}{2}\left(\alpha^+ q(0) + \frac{\beta}{2}p(0)\right) - \frac{1}{2}\alpha^+ q(\pi) + \frac{\beta}{4}p(\pi) - \alpha^- q(a) +$$

$$+\frac{\beta}{2}p(a)-\frac{1}{2}\left(\alpha^{+}-\frac{\beta^{2}}{4}\right)\int\limits_{0}^{a}\left\{p^{2}\left(s\right)+q^{2}\left(s\right)\right\}ds-$$

$$-\frac{1}{2}\alpha^{+}\int\limits_{a}^{a}\left\{p^{2}\left(s\right)+q^{2}\left(s\right)\right\}ds,$$

$$h_{2}=-\frac{1}{2}\left(\frac{\beta}{2}q(0)-\alpha^{-}p(0)\right)+\frac{\beta}{4}q(\pi)+\frac{1}{2}\alpha^{+}p(\pi)+\frac{1}{2}\alpha^{+}\beta\times$$

$$\times\int\limits_{0}^{a}\left\{p^{2}\left(s\right)+q^{2}\left(s\right)\right\}ds+\frac{\beta}{2}\int\limits_{a}^{\pi}\left\{p^{2}\left(s\right)+q^{2}\left(s\right)\right\}ds,$$

$$h_{3}=\frac{1}{2}\alpha^{-}q(\pi)-\frac{\beta}{4}p(\pi)-\frac{\beta}{2}p(a)+\frac{1}{2}\alpha^{-}\int\limits_{a}^{\pi}\left\{p^{2}\left(s\right)+q^{2}\left(s\right)\right\}ds,$$

$$h_{4}=\frac{1}{2}\left(\alpha^{-}p(\pi)-\frac{\beta}{2}q(\pi)\right)-\alpha^{+}p(a)+\frac{\beta^{-}}{4}\int\limits_{a}^{\pi}\left\{p^{2}\left(s\right)+q^{2}\left(s\right)\right\}ds$$

Proof. Provided $p(\cdot)\in W_{2}^{1}\left[0,\pi\right],\ q(\cdot)\in W_{2}^{1}\left[0,\pi\right]$ from integral equation [4] for the kernel of the representation (9) it follows that, the elements of the matrix-function $\frac{\partial}{\partial t}K(x,.)$ belong to $L_{2}(-x,x)$. Therefore, integrating by part, from (10) we have

$$\triangle(\lambda)=\triangle_{0}(\lambda)+h_{1}\frac{\cos\lambda\pi}{\lambda}+h_{2}\frac{\sin\lambda\pi}{\lambda}+h_{3}\frac{\cos\lambda\left(2a-\pi\right)}{\lambda}+$$

$$+h_{4}\frac{\sin\lambda\left(2a-\pi\right)}{\lambda}++\frac{1}{\lambda}\int\limits_{-\pi}^{\pi}\frac{\partial K_{11}(\pi,t)}{\partial t}\cos\lambda tdt+\frac{1}{\lambda}\int\limits_{-\pi}^{\pi}\frac{\partial K_{12}(\pi,t)}{\partial t}\sin\lambda tdt,$$

where

$$h_{1}=K_{11}(\pi,-\pi)-K_{11}(\pi,\pi),\quad h_{2}=-K_{12}(\pi,-\pi)-K_{12}(\pi,\pi),$$

$$h_{3}=K_{11}(\pi,2a-\pi+0)-K_{11}(\pi,2a-\pi-0),$$

$$h_{4}=K_{12}(\pi,2a-\pi+0)-K_{12}(\pi,2a-\pi-0).$$

Since $\triangle\left(\lambda_{n}^{0}+\varepsilon_{n}\right)=0$, the acting as in the proof of Lemma 2 and using the formula

$$K_{11}(\pi,\pi)=\frac{1}{2}\alpha^{-}q(\pi)-\frac{\beta}{4}p(\pi)+\alpha^{-}q(a)-\frac{\beta}{2}p(a)+\frac{1}{2}\left(\alpha^{+2}-\frac{\beta^{2}}{4}\right)\times$$

$$\times\int\limits_{0}^{a}\left\{p^{2}\left(s\right)+q^{2}\left(s\right)\right\}ds+\frac{1}{2}\alpha^{+}\int\limits_{a}^{\pi}\left\{p^{2}\left(s\right)+q^{2}\left(s\right)\right\}ds;$$

$$K_{12}(\pi,\pi)=-\frac{\beta}{4}q(\pi)-\frac{1}{2}\alpha^{+}p(\pi)-\frac{1}{2}\alpha^{+}\beta\int\limits_{0}^{a}\left\{p^{2}(s)+q^{2}(s)\right\}ds-$$

$$-\frac{\beta}{4}\int\limits_{a}^{\pi}\left\{p^{2}(s)+q^{2}(s)\right\}ds;$$

$$K_{11}(\pi,-\pi)=\frac{1}{2}\left(\alpha^{+}q(0)+\frac{\beta}{2}p(0)\right),\quad K_{12}(\pi,-\pi)=\frac{1}{2}\left(\frac{\beta}{2}q(0)+\alpha^{+}p(0)\right),$$

$$K_{11}(x,2a-x+0)-K_{11}(x,2a-x-0)=$$

$$=\frac{1}{2}\alpha^{-}q(x)-\frac{\beta}{4}p(x)-\frac{\beta}{2}p(a)+\frac{1}{2}\alpha^{-}\int\limits_{a}^{x}\left\{p^{2}(s)+q^{2}(s)\right\}ds;$$

$$K_{12}(x,2a-x+0)-K_{12}(x,2a-x-0)=$$

$$=\frac{1}{2}\alpha^{-}p(x)+\frac{\beta}{4}q(x)-\alpha^{+}p(a)+\frac{\beta}{4}\int\limits_{a}^{x}\left\{p^{2}(s)+q^{2}(s)\right\}ds.$$

We obtain asymptotic formula (14) for λ_n.

Lemma 5. *Let $p(\cdot)\in W_2^1[0,\pi]$, $q(\cdot)\in W_2^1[0,\pi]$. Then the normalizing numbers have the form*

$$\alpha_n=\alpha_n^0+\frac{\widetilde{d}_n}{\lambda_n^0}+\frac{\widetilde{k}_n}{n},\quad \left\{\widetilde{k}_n\right\}\in l_2 \tag{16}$$

here α_n^0 are the normalizing numbers of the problem L for $\Omega(x)\equiv 0$ and they are of the form

$$\alpha_n^0=-\left(\pi\alpha^{+}\cos\lambda_n^0\pi-\frac{\pi\beta}{2}\sin\lambda_n^0\pi+(2a-\pi)\alpha^{-}\cos\lambda_n^0(2a-\pi)+\right.$$

$$\left.+\frac{\beta}{2}(2a-\pi)\sin\lambda_n^0(2a-\pi)\right)\times$$

$$\times\left(\frac{\beta}{2}\sin\lambda_n^0\pi+\frac{\beta}{2}\sin\lambda_n^0(2a-\pi)-\alpha^{+}\cos\lambda_n^0\pi+\alpha^{-}\cos\lambda_n^0(2a-\pi)\right).$$

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