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A VERSION OF THE HOFFMAN-WERMER THEOREM

Abstract

It is obtained a generalization of the Hoffman-Wermer theorem for complex continuous functions on a compact space.

1. Introduction

In the middle of 20-th century M. H. Stone [11] obtained a famous generalization of Weierstrass Theorem that any closed algebra of real continuous functions on a compact space X (or a bicompact space in the terminology of [1, 10]) which separates points of X is equal to the algebra $C_{\mathbb{R}}(X)$ of all real continuous functions on X or coincides with its ideal $\mathrm{id}_{\{x_0\}}(C_{\mathbb{R}}(X))$ of all functions vanishing at some point $x_0 \in X$. The situation for subalgebras of the algebra $C_{\mathbb{C}}(X)$ of all complex continuous functions on X is more intricate, in particular there exist algebras of continuous functions that separate points of X and contain constants, but are not equal to $C_{\mathbb{C}}(X)$ (for instance see [8]). Stone proved that a closed symmetric subalgebra of $C_{\mathbb{C}}(X)$ that separates points of X is equal to $C_{\mathbb{C}}(X)$ or coincides with its ideal $\mathrm{id}_{\{x_0\}}(C_{\mathbb{C}}(X))$ of all functions vanishing at some point $x_0 \in X$. In general, for non-symmetric subalgebras, there is a remarkable theorem of Hoffman and Wermer [9] which states that any closed subalgebra $A \subset C_{\mathbb{C}}(X)$ that separates points of X and contains constants is equal to $C_{\mathbb{C}}(X)$ if and only if the space $\mathrm{Re}\, A$ of all real parts of functions in A is closed in $C_{\mathbb{R}}(X)$. A short proof of the Hoffman-Wermer Theorem was given by Browder [6]. A generalization of the Hoffman-Wermer Theorem was obtained by Bernard [2] for complete (in an other norm) subalgebras $A \subset C_{\mathbb{C}}(X)$ having the uniformly closed space $\mathrm{Re}\, A$. In what follows by $C(X)$ we denote $C_{\mathbb{C}}(X)$ or $C_{\mathbb{R}}(X)$ in the dependence on the case of scalars considered.

B. Bilalov [3] obtained a modification of the Stone-Weierstrass Theorem for an arbitrary subalgebra of $C_{\mathbb{R}}(X)$, more precisely, he described the closure of an arbitrary subalgebra $A \subset C_{\mathbb{R}}(X)$ in terms of subsets of X that are equalized by functions in A or are common zeros of functions in A. For a point $x \in X$, let $K_A(x)$ be the set of all $y \in X$ such that $f(y) = f(x)$ for all functions $f \in A$, $\mathrm{ref}_{C(X)}(A)$ the set of all functions g in $C(X)$ such that $g(y) = g(x)$ for all $x \in X$ and $y \in K_A(x)$, and let $\mathrm{id}_{N_A}(C(X))$ be the set of all functions in $C(X)$ that vanish on the set N_A of common zeros of functions in A. The Bilalov's Theorem states that the closure of a subalgebra $A \subset C_{\mathbb{R}}(X)$ is equal to $\mathrm{id}_{N_A}(C_{\mathbb{R}}(X)) \cap \mathrm{ref}_{C_{\mathbb{R}}(X)}(A)$. In particular if all sets $K_A(x)$ are one-point (this means that A separates points of X), then the algebra $\mathrm{ref}_{C_{\mathbb{R}}(X)}(A)$ is equal to $C_{\mathbb{R}}(X)$ and the ideal $\mathrm{id}_{N_A}(C_{\mathbb{R}}(X))$ coincides with $C(X)$ if $N_A = \varnothing$ or coincides with its ideal $\mathrm{id}_{\{x_0\}}(C_{\mathbb{R}}(X))$ of all functions vanishing at the point $x_0 \in X$, the common zero of functions in A. Thus, Bilalov's Theorem implies the Stone-Weierstrass Theorem.

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Here a generalization of Bilalov's Theorem to algebras of complex continuous functions on a compact space satisfying the Hoffman-Wermer condition is obtained.

2. Preliminaries

We use the terminology of [5]. Let X be a compact space (with Hausdorff topology) and let $C_{\mathbb{C}}(X)$ (respectively, $C_{\mathbb{R}}(X)$) be the algebra of all complex (respectively, real) continuous functions on X, with sup-norm

$$\|f\| = \sup_{x\in X} |f(x)|\,.$$

As usual, by $C(X)$ we denote $C_{\mathbb{C}}(X)$ or $C_{\mathbb{R}}(X)$ in the dependence on scalars considered. So, if scalars are not indicate, our talk relates for both of cases of real and complex scalars. For a subset $Y \subset X$, let $\mathrm{id}_Y(C(X))$ be the ideal of $C(X)$ consisting of all functions vanishing on Y.

Let S be a set of bounded functions on a set Y, and let $\overline{S}$ be the closure of S in the algebra of all bounded functions on Y with sup-norm. Let us remind that this closure is called *uniform*.

Now let S be a set of continuous functions in $C(X)$. For any $x \in X$ let $K_S(x)$ be the set of all $y \in X$ such that $f(y) = f(x)$ for all functions $f \in S$. By $\mathrm{ref}_{C(X)}(S)$ we denote the *reflexive hull* of S, i.e., the set of all functions g of algebra $C(X)$ such that $g(y) = g(x)$ for all points $x \in X$ and $y \in K_S(x)$.

Let R_S be an equivalence relation on X given by condition $(x, y) \in R_S$ if and only if $y \in K_S(x)$, and let ϕ_S be the canonic map $X \to X/R_S$. For an arbitrary function $f \in \mathrm{ref}_{C(X)}(S)$ we define the function f/R_S on X/R_S by

$$(f/R_S)(\phi_S(x)) = f(x)$$

for every element $\phi_S(x) \in X/R_S$. It is obvious that the definition of f/R_S is correct, i.e., does not depend on a representative x in the equivalence class $K_S(x)$. Following two lemmas in the other notation were proved in [7].

Lemma 1. *Let S be a set of continuous functions on a compact space X. For an arbitrary function $f \in \mathrm{ref}_{C(X)}(S)$ the function f/R_S is continuous on the quotient X/R_S.*

For a sake of completeness we give a simple proof of the following corollary.

Corollary 2. *Let S be a set of continuous functions on a compact space X. Then the quotient X/R_S is a compact space.*

Proof. We remind that X/R_S is a quasi-compact space (see [5]). So we only need to show that X/R_S is a Hausdorff space. Let x and y in X be arbitrary with $\phi_S(x) \neq \phi_S(y)$. Then there exists a continuous function $f \in S$ such that

$$f(x) \neq f(y).$$

It is obvious that $f \in \mathrm{ref}_{C(X)}(S)$. By Lemma 1, f/R_S is a continuous function on X/R_S and

$$(f/R_S)(\phi_S(x)) = f(x) \neq f(y) = (f/R_S)(\phi_S(y)).$$

Take disjoint open neighbourhoods U_x and U_y of points $f(x)$ and $f(y)$. Then $(f/R_S)^{-1}(U_x)$ and $(f/R_S)^{-1}(U_y)$ are disjoint neighbourhoods of points $\phi_S(x)$ and $\phi_S(y)$. So X/R_S is a Hausdorff space.

For a set M of functions in $\mathrm{ref}_{C(X)}(S)$, let M/R_S be the set of all functions f/R_S on X/R_S, where f runs over M. Is is obvious that if S is an algebra then so is S/R_S. Let $\tau_S : \mathrm{ref}_{C(X)}(S) \to \mathrm{ref}_{C(X)}(S)/R_S$ be the map defined by

$$\tau_S(f) = f/R_S$$

for every function $f \in \mathrm{ref}_{C(X)}(S)$. It is obvious that τ_S is a homomorphism of algebras. We have the following assertion.

Lemma 3. *Let S be a set of continuous functions on compact space X. Then τ_S is an isometric homomorphism of algebras $\mathrm{ref}_{C(X)}(S)$ and $\mathrm{ref}_{C(X)}(S)/R_S$.*

As a result, we state the following simple proposition.

Theorem 4. *Let S be a set of continuous functions on a compact space X, and let A be the uniformly closed algebra generated by functions in S. Then $\mathrm{ref}_{C(X)}(S)/R_S$ and A/R_S are uniformly closed algebras of continuous functions on the compact space X/R_S that separate points of X/R_S. Besides, $\mathrm{ref}_{C(X)}(S)/R_S$ contains constants.*

Proof. Since $\mathrm{ref}_{C(X)}(S)$ is a uniformly closed algebra, so is the algebra $\mathrm{ref}_{C(X)}(S)/R_S$ by Lemma 3. The same is clearly true for A/R_S. The separation of points of X/R_S is realized by functions in S/R_S, so both of the algebras separate points of X/R_S. The existence of constants in $\mathrm{ref}_{C(X)}(S)/R_S$ is obvious. The other assertions follow from Lemma 1 and Corollary 2.

3. Characterization of algebras of continuous functions satisfying the Hoffman-Wermer condition

In this section we obtain a generalization of Bilalov's Theorem to algebras of complex continuous functions on a compact space satisfying the Hoffman-Wermer condition. To include algebras without constants into consideration we begin with preliminary lemmas.

Let X be a set, and let A be a vector space of functions on X. The function e defined by $e(x) = 1$ for all $x \in X$ is called *unit.* We say that A does not contain *constants* (or *scalars*), if A does not contain the unit function. Let B be a set of all functions g on X represented as $g = \lambda e + f$, where $f \in A$ and λ is a scalar. It is obvious that B is a vector space of functions. We called B the *space obtained from A by adjoining the unit function.*

The following lemma belongs to the mathematical folklore and holds for real and complex functions.

Lemma 5. *Let A be a uniformly closed space of bounded functions on a set X, without constants, and let B be the space obtained from A by adjoining the unit function. Then B is uniformly closed.*

Proof. Let a sequence $(g_n = \lambda_n e + f_n)_n$ of elements of B tend by sup-norm to the function g, where all f_n belong to A. If there exists a subsequence of scalars (λ_{n_k}) such that $|\lambda_{n_k}| \to \infty$ then

$$\left\| e + \lambda_{n_k}^{-1} f_{n_k} \right\| = \left\| \lambda_{n_k}^{-1} g_{n_k} \right\| = |\lambda_{n_k}|^{-1} \left\| g_{n_k} \right\| \to 0$$

under $k \to \infty$. Since $-\lambda_{n_k}^{-1} f_{n_k} \in A$ and A is closed, the unit function belongs to A, a contradiction.

So the sequence (λ_n) is bounded and thus has a convergent subsequence. Taking a subsequence, assume that $\lambda_n e + f_n \to g$ and (λ_n) tends to a scalar λ under $n \to \infty$. Then

$$f_n \to g - \lambda e$$

in the sup-norm topology, under $n \to \infty$. We obtain that

$$g - \lambda e \in A.$$

Thus

$$g = \lambda e + (g - \lambda e) \in B.$$

Lemma 6. *Let A be a vector space of bounded functions on a set X, without constants, and let B be the space obtained from A by adjoining the unit function. Then $\overline{B}$ is the space obtained from $\overline{A}$ by adjoining the unit function.*

Proof. Let $g \in \overline{B}$ be the limit in the sup-norm topology of a sequence $(\lambda_n e + f_n)$, where all f_n belong to A. As it was shown in the proof of lemma 5, it may be assumed that the sequence (λ_n) of scalars tends to a scalar λ and in the same time the sequence (f_n) tends to some function f in the sup-norm topology. Hence $f \in \overline{A}$ and $g = \lambda e + f$. Thus, $\overline{B}$ is the space obtained from $\overline{A}$ by adjoining the unit function.

It is obvious that if A is an algebra of functions on a set X then the space B obtained from A by adjoining the unit function is an algebra itself.

Lemma 7. *Let A be an algebra of functions on a set X, without constants, and let B be the algebra obtained from A by adjoining the unit function. Then A is a maximal ideal of B.*

Proof. The unit function on X is a unit element of algebra B, so, for every functions $f, g \in A$ and a scalar λ, we have that

$$(\lambda e + f)g = \lambda g + fg \in A.$$

Thus, A is an ideal of the algebra B.

Now let I be a proper ideal of B and $A \subset I$. Then I does not contain constants because if $(\lambda e + f) \in I$ for $f \in A$ then $\lambda = 0$. Thus $I = A$. This means that A is a maximal ideal of B.

As a corollary, we have the following proposition.

Corollary 8. *Let A be an algebra of complex continuous functions on a compact space X, without constants, and let B be the algebra obtained from A by adjoining the unit function. If $B = C(X)$ then there exists a point x_0 such that*

$$A = \mathrm{id}_{\{x_0\}}(C(X)).$$

Proof. By Lemma 7, A is a maximal ideal of $C(X)$. At the same time (see [12, Assertion 4.6.10(b)]) every maximal ideal of $C(X)$ coincides with set of all continuous functions on X vanishing at a point of X.

Let us remind that if A is a complex space of functions on a set X then $\operatorname{Re} A$ means the set of real parts of functions in A. It is obvious that $\operatorname{Re} A$ is a space of real functions on X.

Definition 1. *Let X be a topological space. We say that a vector space A of bounded continuous functions on X satisfies the Hoffman-Wermer condition if $\operatorname{Re} \overline{A}$ is a uniformly closed subspace of algebra of all bounded continuous real functions on X.*

In the case of real functions on a set X we have the equality $A = \operatorname{Re} A$. So vector spaces of real bounded continuous functions on a topological space X satisfy the Hoffman-Wermer condition.

We need the following lemma.

Lemma 9. *Let a vector space A of bounded continuous functions on a topological space X satisfy the Hoffman-Wermer condition, and let B be the space obtained from A by adjoining the unit function. Then B satisfies the Hoffman-Wermer condition.*

Proof. By lemma 6, $\overline{B}$ obtained from $\overline{A}$ by adjoining the unit function. Then $\operatorname{Re} \overline{B}$ obtained from $\operatorname{Re} \overline{A}$ by adjoining the unit function. By condition of the lemma, $\operatorname{Re} \overline{A}$ is closed. Then, by Lemma 5, $\operatorname{Re} \overline{B}$ is also closed, i.e., B satisfies the Hoffman-Wermer condition.

Now we can formulate the main result of this section that generalizes Stone-Weierstrass, Bilalov's and Hoffman-Wermer Theorems.

Theorem 10. *Let A be an algebra of continuous functions on a compact space X satisfying the Hoffman-Wermer condition. Then*

$$\overline{A} = \operatorname{id}_{N_A}(C(X)) \cap \operatorname{ref}_{C(X)}(A).$$

Proof. The result for real functions is proved in [3]. So we assume that A consists of complex functions.

Let us assume that $\overline{A}$ contains constants. Then $\overline{A/R_A}$ is an algebra of continuous functions on a compact space X/R_A that separates points of X/R_A by Theorem 4 and contains constants (since $\tau_A(e)$ is the unit function on X/R_A). In addition, $\overline{A/R_A}$ satisfies the Hoffman-Wermer condition because τ_A is isometric by lemma 3 and maps $\operatorname{Re} \overline{A}$ onto $\operatorname{Re} \overline{A/R_A}$. By the Hoffman-Wermer Theorem, we obtain that

$$\overline{A/R_A} = C(X/R_A).$$

Since $\overline{A} \subset \operatorname{ref}_{C(X)}(A)$, we have that

$$\operatorname{ref}_{C(X)}(A)/R_A = C(X/R_A).$$

So

$$\overline{A} = \tau_A^{-1}(C(X/R_A)) = \operatorname{ref}_{C(X)}(A).$$

Now assume that $\overline{A}$ does not contain constants. Let the algebra B be obtained from A by adjoining the unit function. Then B satisfies the Hoffman-Wermer condition by Lemma 9. It is obvious that $R_B = R_A$, so $\tau_B = \tau_A$ and $\phi_B = \phi_A$. As it was shown, we obtain that

$$\overline{B/R_A} = C(X/R_A).$$

Since $\overline{B}$ is obtained from $\overline{A}$ by adjoining the unit function by Lemma 6, $\overline{B/R_A}$ is also obtained from $\overline{A/R_A}$ by adjoining the unit function. By Corollary 8, there exists a point $\phi_A(x_0) \in X/R_A$ such that

$$\overline{A/R_A} = \mathrm{id}_{\{\phi_A(x_0)\}}(C(X/R_A)).$$

Thus we have

$$\overline{A} = \tau_A^{-1}(\tau_A(\overline{A})) = \tau_A^{-1}(\mathrm{id}_{\{\phi(x_0)\}}(C(X/R_A))).$$

We will show that the equality

$$\tau_A^{-1}(\mathrm{id}_{\{\phi(x_0)\}}(C(X/R_A))) = \mathrm{id}_{K_A(x_0)}(C(X)) \cap \mathrm{ref}_{C(X)}(A)), \tag{1}$$

is valid. It is easy to see that $K_A(x_0) = N_A$. Let us remind that, for any function g on X/R_A, the function $\tau_A^{-1}(g)$ is defined by

$$(\tau_A^{-1}(g))(x) = g(\phi_A(x))$$

for every $x \in X$ (i.e., as the composition $g \circ \phi_A$ of g and the continuous function ϕ_A) , so $\tau_A^{-1}(g)$ is continuous if so is g. Thus $\tau_A^{-1}(\mathrm{id}_{\{\phi_A(x_0)\}}(C(X/R_A)))$ consists of continuous functions that are constant on every set

$$\phi_A^{-1}(x) = K_A(x)$$

and thus belong to $\mathrm{ref}_{C(X)}(A)$. Besides, these functions vanish on the set $K_A(x_0)$. So the following inclusion

$$\tau_A^{-1}(\mathrm{id}_{\{\phi_A(x_0)\}}(C(X/R_A))) \subset \mathrm{id}_{K_A(x_0)}(C(X)) \cap \mathrm{ref}_{C(X)}(A))$$

is valid.

We show now that the opposite inclusion is also valid. For this we have to prove that

$$\tau_A(\mathrm{id}_{K_A(x_0)}(C(X)) \cap \mathrm{ref}_{C(X)}(A)) \subset \mathrm{id}_{\{\phi_A(x_0)\}}(C(X/R_A)).$$

The left part of this inclusion determines continuous functions on X/R_A that vanish at the point $\phi_A(x_0)$ and thus belong to the right part. So we obtain that the equality (1) is valid.

Thus, for all cases, the theorem is true.

Corollary 11. *Let A and B be algebras of (simultaneously complex or real) continuous functions on compact space X satisfying the Hoffman-Wermer condition. Then $\overline{A} \supset \overline{B}$ if and only if $N_A \subset N_B$ and $K_A(x) \subset K_B(x)$ for all $x \in X$.*

Proof. First let us note that all sets $K_A(x)$ are closed (as intersections of corresponding closed sets). Hence $N_A = N_{\overline{A}}$ and $K_A(x) = K_{\overline{A}}(x)$ for all $x \in X$. Thus it is obvious that if $\overline{A} \supset \overline{B}$, then $N_A \subset N_B$ and $K_A(x) \subset K_B(x)$ for all $x \in X$.

Now let $N_A \subset N_B$ and $K_A(x) \subset K_B(x)$ for all $x \in X$. Then

$$\mathrm{id}_{N_A}(C(X)) \cap \mathrm{ref}_{C(X)}(A) \supset \mathrm{id}_{N_B}(C(X)) \cap \mathrm{ref}_{C(X)}(B)).$$

By Theorem 10, we have that

$$\overline{A} \supset \overline{B}.$$

Corollary 12. *Let A and B are algebras of (simultaneously complex or real) continuous functions on compact space X satisfying the Hoffman-Wermer condition. Then $\overline{A} = \overline{B}$ if and only if $N_A = N_B$ and $K_A(x) = K_B(x)$ for all $x \in X$.*

Proof. Follows from Corollary 11.

Lemma 13. *Let R be an equivalence relation on a compact space X such that the quotient X/R is a Hausdorff space (and hence a compact space). Then there exists a closed algebra A of continuous functions on X satisfying the Hoffman-Wermer condition such that $R = R_A$ and N_A is equal to one of the following sets: $\varnothing$, $K_A(x)$ for an arbitrary element $x \in X$.*

Proof. Let $\phi : X \to X/R$ be the canonic map (that is continuous in the quotient topology). We define map $\upsilon : C(X/R) \to C(X)$ by

$$(\upsilon g)(x) = g(\phi(x))$$

for all $x \in X$. The map υ is an isometric homomorphism and in particular maps real subspaces of $C(X/R)$ into real subspaces of $C(X)$. Thus

- The algebra $A = \upsilon(C(X/R))$ is closed and satifies the Hoffman-Wermer condition with $N_A = \varnothing$.
- The algebra $A = \upsilon(C(X/R) \cap \mathrm{id}_{\{\phi(x)\}}(C(X/R)))$ is closed and satisfies the Hoffman-Wermer condition with $N_A = K_A(x)$.

In all cases we have that $R = R_A$.

Theorem 14. *Let X be a compact space and $C(X) = C_{\mathbb{C}}(X)$. Then*

(i) There exists a one-to-one map from the set of all closed subalgebras $A \subset C(X)$ satisfying the Hoffman-Wermer condition with $N_A = \varnothing$ onto the family of all equivalence relations $R = R_A$ such that X/R is Hausdorff.

(ii) There exists a one-to-one map from the set of all closed subalgebras $A \subset C(X)$ satisfying the Hoffman-Wermer condition with a nonempty set N_A of common zeros of functions in A onto the family of all pairs (R, N_A), where $R = R_A$ is an equivalence relation such that X/R is Hausdorff, and $N_A = K_A(x)$ for a point $x \in N_A$.

Proof. Follows from Theorem 10, Corollary 12 and Lemma 13.

Note that the corresponding proposition is also true for algebras of real continuous functions on a compact space.

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