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THE LAPLACE TRANSFORMATION OF THE DISTRIBUTION OF THE UPPER BOUNDARY FUNCTIONAL

Abstract

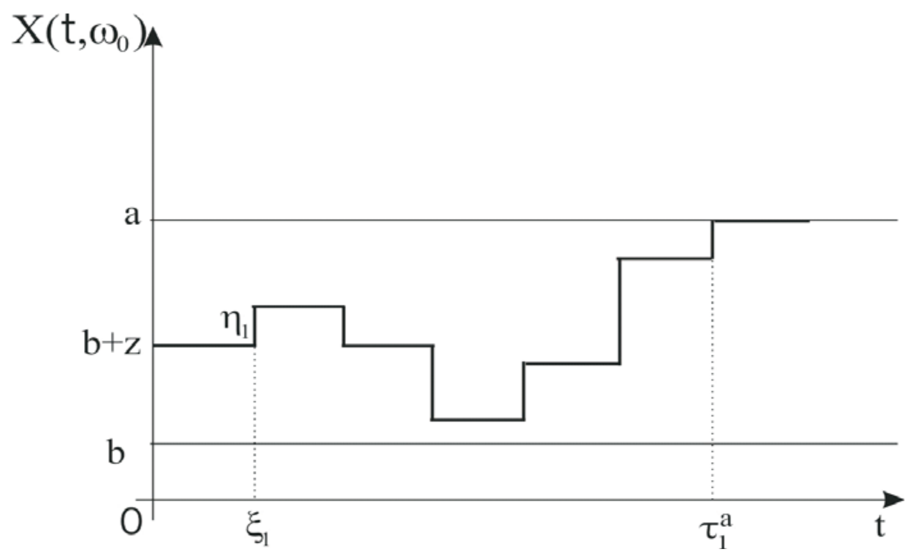
On the given two dimensional sequence of independent identically distributed random variables the semi-markov random walk with delaying screens in the "b" and "a" $(b>0)$ are constructed . Then the Laplace transformation of the first moment reaching the delaying screen in the "a" and its expectation and variance is obtained.

1.Problem statement. Let on a probability space $(\Omega, F, P(\cdot))$ be given a sequence of independent equally distributed random variables $\{\xi_i(\omega), \eta_i(\omega)\}$, $i=\overline{1,\infty}.$ where $\xi_i(\omega)>0$, $E\eta_1(\omega)>0$ and the numbers a, b, and z, be given so that $b>0$, $a>b$ and $0<z\leq a-b$.

By these random variables we construct the process

$$X_1(t,\omega)=b+z+\sum_{i=1}^{k}\eta_i(\omega), \text{ if } \sum_{i=1}^{k}\xi_i(\omega)\leq t<\sum_{i=1}^{k+1}\xi_i(\omega), \ k=\overline{0,\infty}, \ \sum_{1}^{0}=0$$

and call it a step-wise process of semi-markov walk.



The process constructed in such a way is called a semi-markov walk process with delaying screens "b" and "a"

$$X(t,\omega)=\zeta_k(\omega), \text{ if } \sum_{i=1}^{k}\xi_i(\omega)\leq t<\sum_{i=1}^{k+1}\xi_i(\omega), \ k\geq 0, \ \left(\sum_{1}^{0}=0\right)$$

$$\zeta_k(\omega) = \min\{a, \max\{b,\ \zeta_{k-1}(\omega) + \eta_k(\omega)\}\},\ k \geq 1,$$
$$\zeta_0(\omega) = b + z.$$

One of realizations of the semi-markov process $X(t,\omega)$ looks like

Assume

$$\tau_1^a(\omega) = \inf\{t : X(t,\omega) = a\}$$

and $\tau_1^a(\omega) = \infty$, if for all $t > 0$ $X(t,\omega) < a$.

We call the random variable $\tau_1^a(\omega)$ the first moment reaching the screen "a".

The goal of the paper is to find the Laplace transformation of distribution of the random variable $\tau_1^a(\omega)$ and its number characteristics .

A number of papers $[1,4]$ were devodet to probability distributions and number characteristics stady of boundary functionals of the semi-markov walk. Explicit relation for distributions of the first moment reaching of the zero level by a random walk generated by symmetric continuonsly distributed variables was obtained in $[1, p.362]$. When the walk occurs by any distribution, analytic form of distribution of the first moment reaching of the zero level that is not convenient for application, was obtained in [4] .

2. Formulation and proof of the main results. Introduce the following denotation:

$$L(\theta/b+z) = E\left(e^{-\theta\tau_1^a}/X(\theta) = b+z\right) =$$
$$= \int\limits_{z=0}^{\infty} e^{-\theta t} dP\{\tau_1^a(\omega) < t/X(0,\omega) = b+z\},$$
$$L(\theta) = Ee^{-\theta\tau_1^a(\omega)} = \int\limits_{z=0}^{\infty} L(\theta, b+z)\, P\{X(0,\omega) \in dz\}, \quad \theta > 0,$$
$$\varphi(\theta) = Ee^{-\theta\xi_1(\omega)} = \int\limits_{i=0}^{\infty} e^{-\theta t} dP\{\xi_1(\omega) < t\}, \quad \theta > 0.$$

Theorem. For the process $X(t,\omega)$ it holds

$$L(\theta/b+z) = \varphi(\theta)\, P\{\eta_1(\omega) > a-b-z\} + \varphi(\theta)\, L(\theta/b)\, P\{\eta_1(\omega) < -z\} +$$
$$+\varphi(\theta) \int\limits_{y=b}^{a} L(\theta/y)\, d_y P\{\eta_1(\omega) < y-b-z\}$$

Proof. First we notice that by the theorem from [5] the first moment reaching of $\tau_1^a(\omega)$ is the eigen random value i.e. $P(\tau_1^a(\omega) < \infty) = 1$.

By total probability relation, we have

$$P\{\tau_1^a(\omega) > t/X(0,\omega) = b+z\} = P\{\xi_1(\omega) > t\} +$$
$$+ \int\limits_{s=0}^{t} \int\limits_{y=b}^{a} P\{\tau_1^a(\omega) > t-s/X(0,\omega) = y\} \times$$

$$\times P\{\xi_1(\omega) \in ds; b+z+\eta_1(\omega) < a; \max[b, b+z+\eta_1(\omega)] \in dy\}.$$

By means of this equality it is easy to get the following integral equation with respect to the conditional Laplace transformation $L(\theta/b+z)$:

$$L(\theta/b+z) = \varphi(\theta) P\{\eta_1(\omega) > a-b-z\} + \varphi(\theta) L(\theta/b) P\{\eta_1(\omega) < -z\} +$$
$$+\varphi(\theta) \int_{y=b}^{a} L(\theta/y)\, d_y P\{\eta_1(\omega) < y-b-z\} \qquad (1)$$

We can solve equation (1) for arbitrarily distributed random variable $\eta_1(\omega)$ by the sequential approximations method, but such a solution is not suitable for applications. We'll show that equation (1) admits exact solution.

Corollary1. Let $\eta_1(\omega)$ have Laplace distribution with parameters λ and μ. Then the Laplace conditional transformation will be of the form:

$$L(\theta/b+z) = \frac{\mu\varphi(\theta)[\lambda - k_2(\theta)]\, k_2(\theta)\, e^{k_1(\theta)(b+z)}}{[\lambda - k_2(\theta)]^2 k_2(\theta)\, e^{k_1(\theta)a} - [\lambda - k_1(\theta)]^2 k_1(\theta)\, e^{-[k_2(\theta)-k_1(\theta)]b + k_2(\theta)a}} -$$
$$- \frac{\mu\varphi(\theta)[\lambda - k_1(\theta)]\, k_1(\theta)\, e^{k_2(\theta)(b+z)}}{[\lambda - k_2(\theta)]^2 k_2(\theta)\, e^{[k_2(\theta)-k_1(\theta)]b + k_1(\theta)a} - [\lambda - k_1(\theta)]^2 k_1(\theta)\, e^{k_2(\theta)a}}.$$

Proof. Let η_1 have the Laplace distribution

$$P\{\eta_1 < x\} = \begin{cases} \dfrac{\lambda}{\lambda+\mu} e^{\mu x}, x < 0, \mu > 0, \\ 1 - \dfrac{\mu}{\lambda+\mu} e^{-\lambda x}, x > 0, \lambda > 0, \end{cases}$$

$$P_\eta(x) = \begin{cases} \dfrac{\lambda\mu}{\lambda+\mu} e^{\mu x}, x < 0, \mu > 0, \\ \dfrac{\lambda\mu}{\lambda+\mu} e^{-\lambda x}, x > 0, \lambda > 0. \end{cases}$$

Then equation (1) will be of the form:

$$L(\theta/b+z) = \frac{\mu\varphi(\theta)}{\lambda+\mu} e^{-\lambda(a-b-z)} + \frac{\lambda\varphi(\theta)}{\lambda+\mu} e^{-\mu z} L(\theta/b) +$$
$$+ \frac{\lambda\mu\varphi(\theta)}{\lambda+\mu} e^{\lambda(b+z)} \int_{y=b+z}^{a} L(\theta/y)\, e^{-\lambda y} dy + \frac{\lambda\mu\varphi(\theta)}{\lambda+\mu} e^{-\mu(b+z)} \int_{y=b}^{b+z} L(\theta/y)\, e^{\mu y} dy \qquad (2)$$

From this integral equation we get a differential equation with constant coefficients:

$$L_a''(\theta/b+z) - (\lambda-\mu) L_a'(\theta/b+z) - \lambda\mu[1-\varphi(\theta)] L_a(\theta/b+z) = 0$$

that has the solution

$$L_a(\theta/b+z) = c_1(\theta)\, e^{k_1(\theta)(b+z)} + c_2(\theta)\, e^{k_2(\theta)(b+z)}, \qquad (3)$$

$k_i(\theta)$, $i=1,2$ are the roots of the characteristic equation

$$k^2(\theta)-(\lambda-\mu)\,k(\theta)-\lambda\mu\left[1-\varphi(\theta)\right]=0.$$

Now, we are to find $c_i(\theta)$, $i=1,2$. To this end, from integral equation (2) we find the following initial conditions:

$$\left.\begin{array}{l} L(\theta/a)=\dfrac{\mu\varphi(\theta)}{\lambda+\mu}+\dfrac{\lambda\varphi(\theta)}{\lambda+\mu}e^{-\mu(a-b)}L(\theta/b)+ \\ +\dfrac{\lambda\mu\varphi(\theta)}{\lambda+\mu}e^{-\mu a}\displaystyle\int_{y=b}^{a}L(\theta/y)\,e^{\mu y}dy, \\ L'(\theta/a)=\dfrac{\lambda\mu\varphi(\theta)}{\lambda+\mu}+\dfrac{\lambda\mu\varphi(\theta)}{\lambda+\mu}e^{-\mu(a-b)}L(\theta/b)- \\ -\dfrac{\lambda\mu^2\varphi(\theta)}{\lambda+\mu}e^{-\mu a}\displaystyle\int_{y=b}^{a}L(\theta/y)\,e^{\mu y}dy. \end{array}\right\}$$

On the other hand, from the solution of differential equation (3) we find

$$L_a(\theta/a)=c_1(\theta)\,e^{k_1(\theta)a}+c_2(\theta)\,e^{k_2(\theta)a},$$

$$L'_a(\theta/a)=c_1(\theta)\,k_1(\theta)\,e^{k_1(\theta)a}+c_2(\theta)\,k_2(\theta)\,e^{k_2(\theta)a}.$$

Hence we find the following system of algebraic inhomogeneous equations for $c_i(\theta)$, $i=1,2$.

$$\left.\begin{array}{l} \{\mu\left[\lambda-k_2(\theta)\right]e^{k_1(\theta)a}-\left[\lambda-k_1(\theta)\right]k_1(\theta)\,e^{-\mu(a-b)+k_1(\theta)b}\}c_1(\theta)+ \\ +\{\mu\left[\lambda-k_1(\theta)\right]e^{k_2(\theta)a}-\left[\lambda-k_2(\theta)\right]k_2(\theta)\,e^{-\mu(a-b)+k_2(\theta)b}\}c_2(\theta)=\mu^2\varphi(\theta), \\ \{\lambda\left[\lambda-k_2(\theta)\right]e^{k_1(\theta)a}+\left[\lambda-k_1(\theta)\right]k_1(\theta)\,e^{-\mu(a-b)+k_1(\theta)b}\}c_1(\theta)+ \\ +\{\lambda\left[\lambda-k_1(\theta)\right]e^{k_2(\theta)a}+\left[\lambda-k_2(\theta)\right]k_2(\theta)\,e^{-\mu(a-b)+k_2(\theta)b}\}c_2(\theta)=\lambda\mu\varphi(\theta). \end{array}\right\}$$

From this system, by Kramer's relation ,we find

$$c_1(\theta)=\frac{\mu\varphi(\theta)\left[\lambda-k_2(\theta)\right]k_2(\theta)}{\left[\lambda-k_2(\theta)\right]^2k_2(\theta)\,e^{k_1(\theta)a}-\left[\lambda-k_1(\theta)\right]^2k_1(\theta)\,e^{-[k_2(\theta)-k_1(\theta)]b+k_2(\theta)a}},$$

$$c_2(\theta)=-\frac{\mu\varphi(\theta)\left[\lambda-k_1(\theta)\right]k_1(\theta)}{\left[\lambda-k_2(\theta)\right]^2k_2(\theta)\,e^{[k_2(\theta)-k_1(\theta)]b+k_1(\theta)a}-\left[\lambda-k_1(\theta)\right]^2k_1(\theta)\,e^{k_2(\theta)a}}.$$

We substitute the found $c_1(\theta),c_2(\theta)$ in (3) and get

$$L(\theta/b+z)=\frac{\mu\varphi(\theta)\left[\lambda-k_2(\theta)\right]k_2(\theta)\,e^{k_1(\theta)(b+z)}}{\left[\lambda-k_2(\theta)\right]^2k_2(\theta)\,e^{k_1(\theta)a}-\left[\lambda-k_1(\theta)\right]^2k_1(\theta)\,e^{-[k_2(\theta)-k_1(\theta)]b+k_2(\theta)a}}-$$

$$-\frac{\mu\varphi(\theta)\left[\lambda-k_1(\theta)\right]k_1(\theta)\,e^{k_2(\theta)(b+z)}}{\left[\lambda-k_2(\theta)\right]^2k_2(\theta)\,e^{[k_2(\theta)-k_1(\theta)]b+k_1(\theta)a}-\left[\lambda-k_1(\theta)\right]^2k_1(\theta)\,e^{k_2(\theta)a}}.$$

Corollary 2. Let $\eta_1(\omega)$ have the Laplace distribution with parameters λ and μ, moreover $\lambda<\mu$. Then

$$E\tau_1^a(\omega)=\left\{\frac{\lambda^3}{\mu(\lambda-\mu)^2}+\frac{\lambda\mu}{\lambda-\mu}(a-b)-\frac{\lambda^2}{\mu(\lambda-\mu)}e^{-\mu(a-b)}+\right.$$

$$+\frac{\mu}{\lambda-\mu}e^{-\lambda(a-b)}-\frac{\lambda^3}{\mu(\lambda-\mu)^2}e^{-(\lambda-\mu)(a-b)}\Big\}E\xi_1(\omega),$$

$$D\tau_1^a(\omega)=\Big\{\frac{\lambda^3}{\mu(\lambda-\mu)^2}+\frac{\lambda\mu}{\lambda-\mu}(a-b)-\frac{\lambda^2}{\mu(\lambda-\mu)}e^{-\mu(a-b)}+$$

$$+\frac{\mu}{\lambda-\mu}e^{-\lambda(a-b)}-\frac{\lambda^3}{\mu(\lambda-\mu)^2}e^{-(\lambda-\mu)(a-b)}\Big\}D\xi_1(\omega)+$$

$$+\Big\{-\lambda^3\frac{\lambda^3+3\lambda^2\mu-2\lambda\mu^2-5\mu^3}{(\lambda-\mu)^4}+\frac{\lambda\mu\left(\lambda^2+\mu^2\right)}{(\lambda-\mu)^2}(a-b)+$$

$$+\frac{2\lambda^2}{(\lambda-\mu)^3}e^{-(\lambda+\mu)(a-b)}+\frac{\lambda^6}{\mu^2(\lambda-\mu)^4}e^{2(\lambda-\mu)(a-b)}-\frac{\lambda^4}{\mu^2(\lambda-\mu)^2}e^{2\mu(a-b)}-$$

$$-\frac{\mu^2}{(\lambda-\mu)^2}e^{-2\lambda(a-b)}+\left[\frac{\mu\left(\lambda^2+\mu^2\right)-2\lambda\left(\lambda^2+2\mu^2\right)}{(\lambda-\mu)^3}-\frac{2\lambda\mu^2}{(\lambda-\mu)^2}(a-b)\right]e^{-\lambda(a-b)}+$$

$$+\left[\frac{2\lambda^2\left(\lambda^3+\mu^3\right)-\lambda^3\mu(5\lambda+4\mu)}{\mu^2(\lambda-\mu)^3}-\frac{2\lambda^3}{(\lambda-\mu)^2}(a-b)\right]e^{-\mu(a-b)}+$$

$$+\left[\frac{2\lambda^3(\lambda+\mu)^2-\left(\lambda^2+\mu^2\right)}{\mu(\lambda-\mu)^4}-\frac{4\lambda^4}{(\lambda-\mu)^3}(a-b)\right]e^{-(\lambda-\mu)(a-b)}\Big\}\left[E\xi_1(\omega)\right]^2$$

Proof. Obviously

$$L_a(\theta)=\int_{z=0}^{a-b}L_a(\theta/b+z)\,dP\{X\{0,\omega\}<b+z\}=$$

$$=\int_{z=0}^{a-b}L_a(\theta/b+z)\,dP\{\min\left(a,b+\eta_1^+(\omega)\right)<b+z\}=L_a(\theta/a)\,P\{\eta_1^+(\omega)>a-b\}-$$

$$-\int_{z=0}^{a-b}L_a(\theta/b+z)\,dP\{\eta_1^+(\omega)>z\}=-\sum_{i=1}^{2}\frac{k_i(\theta)}{\lambda-k_i(\theta)}c_i(\theta)\,e^{\lambda b-[\lambda-k_1(\theta)]a}+$$

$$+\sum_{i=1}^{2}\frac{\lambda}{\lambda-k_i(\theta)}c_i(\theta)\,e^{k_i(\theta)b}. \qquad (4)$$

Now, let's find $E\tau_1^a(\omega)$, $D\tau_1^a(\omega)$.
By the Laplace transformation property we have

$$E\tau_1^a(\omega)=-L'(0)\ \text{ and }\ D\tau_1^a(\omega)=L''(0)-\left[L'(0)\right]^2$$

Then from (4) we get

$$E\tau_1^a(\omega)=\Big\{\frac{\lambda^3}{\mu(\lambda-\mu)^2}+\frac{\lambda\mu}{\lambda-\mu}(a-b)-\frac{\lambda^2}{\mu(\lambda-\mu)}e^{-\mu(a-b)}+$$

$$+\frac{\mu}{\lambda-\mu}e^{-\lambda(a-b)}-\frac{\lambda^3}{\mu\left(\lambda-\mu\right)^2}e^{-(\lambda-\mu)(a-b)}\Bigg\}E\xi_1\left(\omega\right),$$

$$D\tau_1^a\left(\omega\right)=\left\{\frac{\lambda^3}{\mu\left(\lambda-\mu\right)^2}+\frac{\lambda\mu}{\lambda-\mu}\left(a-b\right)-\frac{\lambda^2}{\mu\left(\lambda-\mu\right)}e^{-\mu(a-b)}+\right.$$

$$\left.+\frac{\mu}{\lambda-\mu}e^{-\lambda(a-b)}-\frac{\lambda^3}{\mu\left(\lambda-\mu\right)^2}e^{-(\lambda-\mu)(a-b)}\right\}D\xi_1\left(\omega\right)+$$

$$+\left\{-\lambda^3\frac{\lambda^3+3\lambda^2\mu-2\lambda\mu^2-5\mu^3}{\left(\lambda-\mu\right)^4}+\frac{\lambda\mu\left(\lambda^2+\mu^2\right)}{\left(\lambda-\mu\right)^2}\left(a-b\right)+\right.$$

$$+\frac{2\lambda^2}{\left(\lambda-\mu\right)^3}e^{-(\lambda+\mu)(a-b)}+\frac{\lambda^6}{\mu^2\left(\lambda-\mu\right)^4}e^{2(\lambda-\mu)(a-b)}-\frac{\lambda^4}{\mu^2\left(\lambda-\mu\right)^2}e^{2\mu(a-b)}-$$

$$-\frac{\mu^2}{\left(\lambda-\mu\right)^2}e^{-2\lambda(a-b)}+\left[\frac{\mu\left(\lambda^2+\mu^2\right)-2\lambda\left(\lambda^2+2\mu^2\right)}{\left(\lambda-\mu\right)^3}-\frac{2\lambda\mu^2}{\left(\lambda-\mu\right)^2}\left(a-b\right)\right]e^{-\lambda(a-b)}+$$

$$+\left[\frac{2\lambda^2\left(\lambda^3+\mu^3\right)-\lambda^3\mu\left(5\lambda+4\mu\right)}{\mu^2\left(\lambda-\mu\right)^3}-\frac{2\lambda^3}{\left(\lambda-\mu\right)^2}\left(a-b\right)\right]e^{-\mu(a-b)}+$$

$$\left.+\left[\frac{2\lambda^3\left(\lambda+\mu\right)^2-\left(\lambda^2+\mu^2\right)}{\mu\left(\lambda-\mu\right)^4}-\frac{4\lambda^4}{\left(\lambda-\mu\right)^3}\left(a-b\right)\right]e^{-(\lambda-\mu)(a-b)}\right\}\left[E\xi_1\left(\omega\right)\right]^2.$$

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Received January 10, 2007; Revised March 16, 2007