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APPROXIMATE SOLUTION OF GOURSAT LINEAR PROBLEM

Abstract

In the paper a variant of Chaplygin method is used for approximate solution of Goursat problem. Bilateral (upper and lower) approximations are constructed, their convergence monotonically increasing and monotonically decreasing, respectively to exact solution, is proved. Besides, convergence speed of the method is estimated.

We consider the equation:

$$u_{xy} = L[u] + f(x,y) \tag{1}$$

with data on characteristics $x = 0$, $y = 0$:

$$u(x,0) = \varphi(x), \quad u(0,y) = \psi(y), \quad (\phi(0) = \psi(0)), \tag{2}$$

where $L[u] \equiv a(x,y)\, u_x + b(x,y)\, u_y + c(x,y)\, u$.

By $\overline{D}$ we denote a closed bounded domain of variables x, y, u, u_x, u_y whose projection on the plane XOY gives the domain $\overline{R} \subset \overline{D}$:

$$\overline{R} = \{0 \le x \le \alpha,\ 0 \le y \le \beta,\ \alpha > 0,\ \beta > 0\}.$$

Let on the domain $\overline{R}$ the function $f(x,y)$, and also the coefficients a, b, c be continuous functions and in $\overline{R}$ they don't take non-negative values, i.e.

$$a(x,y) \ge 0, \quad b(x,y) \ge 0, \quad c(x,y) \ge 0. \tag{3}$$

Let in (3) strong inequality remain even if for one coefficient. Under these suppositions the right hand side of (1) in the domain $\overline{D}$ will be a continuous function.

Introduce the denotation:

$$\begin{array}{l} m = \min\limits_{\overline{D}} \{L[u] + f(x,y)\} - l, \\ M = \max\limits_{\overline{D}} \{L[u] + f(x,y)\} + l. \end{array} \qquad (l > 0)$$

It holds the

Theorem. *If the functions $V(x,y)$ and $W(x,y)$ are known, such that*

1) they are continuous and have continuous partial derivatives contained in (1);

2) these functions satisfy conditions (2);

3) the results of their substituion in (1) give discrepancies $\overline{\alpha}(x,y) < 0$, $\overline{\beta}(x,y) < 0$, relatively, in the domain $\overline{R}$.

Then, for $(x,y)\in R$

$$\left.\begin{array}{c} V(x,y)<U(x,y)<W(x,y)\,,\\ V_x(x,y)<U_x(x,y)<W_x(x,y)\,,\\ V_y(x,y)<U_y(x,y)<W_y(x,y)\,,\\ V_{xy}(x,y)<U_{xy}(x,y)<W_{xy}(x,y) \end{array}\right\} \qquad (4)$$

are valid.

Proof. If we consider the equations

$$V_{xy}(x,y)=m, \quad W_{xy}(x,y)=M$$

under conditions (2), the existence of the functions $V(x,y)$,$W(x,y)$ is obvious.In this case

$$\begin{array}{l} V_{xy}=L[V]+f(x,y)*+\overline{\alpha}(x,y)=m,\\ \overline{\alpha}(x,y)=m-\{L[V]+f(x,y)\}<0;\\ W_{xy}(x,y)=L[W]+f(x,y)+\overline{\beta}(x,y)=M,\\ \overline{\beta}(x,y)=M-\{L[W]+f(x,y)\}>0. \end{array} \qquad (5)$$

Since

$$V_{xy}(x,y)<U_{xy}(x,y)<W_{xy}(x,y)\,,$$

then, allowing for (5) we have

$$\left.\begin{array}{c} V_y(x,y)=V_y(0,y)+m\cdot x,\\ W_y(x,y)=W_y(0,y)+M\cdot x,\\ V(x,y)=\varphi(x)+\psi(y)-\varphi(0)+m\cdot xy,\\ W(x,y)=\varphi(x)+\psi(y)-\varphi(0)+M\cdot xy \end{array}\right\} \qquad (6)$$

Thus, we get validity of all inequalities of (4).

We construct bilateral (upper and lower) approximations for approximate solution of problem (1)-(2). For zero lower and upper aproximations we take the functions $V(x,y)$ and $W(x,y)$, respectively, from (6), i.e. $V^0(x,y)=V(x,y)$, $W^0(x,y)=W(x,y)$, and determine the next approximations by the equalities:

$$\left.\begin{array}{l} V_{xy}^{p+1}=L[V^p]+f(x,y)\,,\\ V^{p+1}(x,0)=\varphi(x)\,, \quad V^{p+1}(0,y)=\psi(y) \end{array}\right\} \qquad (7)$$

$$\left.\begin{array}{l} W_{xy}^{p+1}=L[W^p]+f(x,y)\,,\\ W^{p+1}(x,0)=\varphi(x)\,, \quad W^{p+1}(0,y)=\psi(y) \end{array}\right\} \qquad (8)$$

It is proved that the sequences of lower approximations $\{V^p(x,y)\}$, $\{V_x^p(x,y)\}$, $\{V_y^p(x,y)\}$ monotonically increase and upper bounded, but sequences of upper approximations $\{W^p(x,y)\}$, $\{W_x^p(x,y)\}$, $\{W_y^p(x,y)\}$ monotonically decrease and lower bounded. Let's prove that the enumerated properties are fulfilled for lower approximations.

Considering that

$$V_{xy}^0(x,y)=L\left[V^0\right]+f(x,y)+\overline{\alpha}(x,y)\,, \quad (\alpha(xy)<0)\,,$$

and also (7) for $p = 0$, we get

$$V'_{xy}(x,y) - V^0_{xy}(x,y) = -\overline{\alpha}(x,y) > 0. \tag{9}$$

Multiplying the last one by $dx > 0$, we have

$$\left(V'_{xy}(x,y) - V^0_{xy}(x,y)\right)dx = \frac{\partial}{\partial x}\left[V'_y - V^0_y\right]dx = d_x\left[V'_y - V^0_y\right] > 0.$$

From the last one, considering the initial data we get

$$V'_y(x,y) - V^0_y(x,y) > 0, \;\; V'_x(x,y) - V^0_x(x,y) > 0. \tag{10}$$

Proceeding from (9) and (10) we see that for $x > 0$, $y > 0$ the inequalities

$$\begin{array}{ll} V^0(x,y) < V'(x,y), & V^0_x(x,y) < V'_x(x,y), \\ V^0_y(x,y) < V'_y(x,y), & V^0_{xy}(x,y) < V'_{xy}(x,y). \end{array}$$

are valid.

On the other hand, from (7) by mathematical induction method it is proved that for any $p = 1,2,3,...$

$$\begin{array}{ll} V^p(x,y) < V^{p+1}(x,y), & V^p_x(x,y) < V^{p+1}_x(x,y), \\ V^p_y(x,y) < V^{p+1}_y(x,y), & V^p_{xy}(x,y) < V^{p+1}_{xy}(x,y). \end{array}$$

Consequently, we proved that sequence of lower approximations monotonically increases and upper bounded.

Now, we prove that a sequence of functions $\{V^p(x,y)\}$, $\{V^p_x(x,y)\}$, $\{V^p_y(x,y)\}$ uniformly converge.

Really, if $-\overline{\alpha}(x,y) \leq \overline{P}$ $\left(\overline{P} = const > 0\right)$ then from

$$\begin{array}{l} V'_{xy}(x,y) - V^0_{xy}(x,y) = -\overline{\alpha}(x,y) \leq \overline{P}, \\ V'_x(x,y) - V^0_x(x,y) \leq \overline{P}\cdot y \leq \overline{P}(x+y), \\ V'_y(x,y) - V^0_y(x,y) \leq \overline{P}\cdot x \leq \overline{P}(x+y) \\ V'(x,y) - V^0(x,y) \leq \overline{P}\cdot xy \leq \overline{P}(x+y)^2. \end{array}$$

If we introduce the denotation

$$\begin{array}{rcl} K & = & \max\{\max a, \max b, \max c\}, \\ E & = & 2 + \alpha + \beta, \end{array}$$

then allowing for (7) we have

$$V^2_{xy}(x,y) - V'_{xy}(x,y) = a\left(V'_x - V^0_x\right) + b\left(V'_y - V^0_y\right) + c\left(V' - V^0\right) \leq$$

$$\leq k\overline{P}\left[2(x+y) + (x+y)^2\right] \leq k\overline{P}E(x+y);$$

$$V^2_{xy}(x,y) - V'_{xy}(x,y) \leq k\overline{P}E(x+y);$$

$$V^2_x(x,y) - V'_x(x,y) \leq k\overline{P}E\frac{(x+y)^2}{2!};$$

$$V_y^2(x,y) - V_y'(x,y) \le k\overline{P}E\frac{(x+y)^2}{2!};$$

$$V^2(x,y) - V'(x,y) \le k\overline{P}E\frac{(x+y)^3}{3!}.$$

By the mathematical induction method,

$$V_{xy}^{p+1}(x,y) - V_{xy}^p(x,y) \le \overline{P}\frac{[kE(x+y)]^p}{p!};$$

$$V_x^{p+1}(x,y) - V_x^p(x,y) \le \frac{\overline{P}}{kE}\frac{[kE(x+y)]^{p+1}}{(p+1)!};$$

$$V_y^{p+1}(x,y) - V_y^p(x,y) \le \frac{\overline{P}}{kE}\frac{[kE(x+y)]^{p+1}}{(p+1)!};$$

$$V^{p+1}(x,y) - V^p(x,y) \le \frac{\overline{P}}{(kE)^2}\frac{[kE(x+y)]^{p+2}}{(p+2)!}.$$

are easily proved.

Consequently,

$$V^{p+1}(x,y) - V^p(x,y) \le \frac{\overline{P}}{(kE)^2}\frac{[kE(x+y)]^{p+2}}{(p+2)!}$$

$$V_x^{p+1}(x,y) - V_x^p(x,y) \le \frac{\overline{P}}{kE}\frac{[kE(x+y)]^{p+1}}{(p+1)!};$$

$$V_y^{p+1}(x,y) - V_y^p(x,y) \le \frac{\overline{P}}{kE}\frac{[kE(x+y)]^{p+1}}{(p+1)!}.$$

Hence, it follows that the sequences:

$$V^p(x,y) = V^0(x,y) + \left[V'(x,y) - V^0(x,y)\right] + ... + \left[V^p(x,y) - V^{p-1}(x,y)\right],$$

$$V_x^p(x,y) = V_x^0(x,y) + \left[V_x'(x,y) - V_x^0(x,y)\right] + ... + \left[V_x^p(x,y) - V_x^{p-1}(x,y)\right],$$

$$V_y^p(x,y) = V_y^0(x,y) + \left[V_y'(x,y) - V_y^0(x,y)\right] + ... + \left[V_y^p(x,y) - V_y^{p-1}(x,y)\right]$$

uniformly converge:

$$V(x,y) = \lim_{p\to\infty} V^p(x,y), \quad \Phi(x,y) = \lim_{p\to\infty} V_x^p(x,y), \quad \overline{\Phi}(x,y) = \lim_{p\to\infty} V_y^p(x,y).$$

Then from (7), passing to the limit we get

$$\Phi(x,y) = \Phi(x,0) + \int_0^y f(x,\eta)\,d\eta + \int_0^y \left\{a\Phi(x,\eta) + b\overline{\Phi}(x,\eta) + cV(x,\eta)\right\}d\eta,$$

$$\overline{\Phi}(x,y) = \overline{\Phi}(0,y) + \int_0^x f(\xi,y)\,d\xi + \int_0^x \left\{a\Phi(\xi,y) + b\overline{\Phi}(\xi,y) + cV(\xi,y)\right\}d\xi,$$

$$V(x,y)=\varphi(x)+\varphi(y)-\varphi(0)+\int_0^x\int_0^y f(\xi,\eta)\,d\xi d\eta+$$

$$+\int_0^x\int_0^y\left\{a\Phi(\xi,\eta)+b\overline{\Phi}(\xi,\eta)+cV(\xi,\eta)\right\}d\xi d\eta$$

It follows from the last ones, that

$$\Phi(x,y)=V_x(x,y),\qquad \overline{\Phi}(x,y)=V_x(x,y),$$

i.e. the function $V(x,y)$ is the solution of problem (1)-(2).

It is easy to prove that the sequence $\{V^p(x,y)\}$ converges to the solution of the problem $u(x,y)$ monotonically increasing, i.e. $u(x,y)-V^p(x,y)>0$.

Therefore, the sequence of functions determined by equalities (7) is said to be lower approximations.

Remark. In a similar way as for lower approximations, we can prove that the sequence of functions $\{W^p(x,y)\}$, $\{W^p_x(x,y)\}$, $\{W^p_y(x,y)\}$ determined by equalities (8) monotonically decrease, and are lower bounded, i.e. they converge, and the limit of sequence$\{W^p(x,y)\}$ is the solution of the given problem. Moreover, $\{W^p(x,y)\}$ converges monotonically decreasing, i.e. $W^p(x,y)-u(x,y)>0$.

The functions defined by equalities (8) will be said to be upper approximations.

Finally, let's estimate the convergence speed of the constructed approximations to the exact solution.

From (7), (8), assuming $z^p(x,y)=W^p(x,y)-V^p(x,y)$, we have

$$z^{p+1}_{xy}=L[z^p]=az^p_x(x,y)+bz^p_y(x,y)+cz^p(x,y). \tag{11}$$

Considering $z^0_{xy}(x,y)=M-m=R_0$.

$$z^0_x(x,y)=R_0y\le R_0(x+y),$$

$$z^0_y(x,y)=R_0x\le R_0(x+y),$$

$$z^0(x,y)=R_0xy\le R_0(x+y)^2.$$

Then from (11)

$$z'_{xy}(x,y)\le kR_0(2+x+y)(x+y)\le KR_0E(x+y).$$

$$z'_x(x,y)\le kR_0E\frac{(x+y)^2}{2!},$$

$$z'_y(x,y)\le kR_0E\frac{(x+y)^2}{2!},$$

$$z'(x,y)\le kR_0E\frac{(x+y)^3}{3!}.$$

By the mathematical induction method we prove:

$$z^p(x,y)\le\frac{R_0}{(kE)^2}\frac{[kE(x+y)]^{p+2}}{(p+2)!}\le\frac{R_0}{(kE)^2}\frac{[kE(\alpha+\beta)]^{p+2}}{(p+2)!},$$

$$z_x^p(x,y) \le \frac{R_0}{kE}\frac{[kE(x+y)]^{p+1}}{(p+2)!} \le \frac{R_0}{kE}\frac{[kE(\alpha+\beta)]^{p+1}}{(p+1)!},$$

$$z_y^p(x,y) \le \frac{R_0}{kE}\frac{[kE(x+y)]^{p+1}}{(p+2)!} \le \frac{R_0}{kE}\frac{[kE(\alpha+\beta)]^{p+1}}{(p+1)!}$$

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