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ON THE CONVERGENCE OF NUMERICAL METHOD OF THE SOLUTION OF NONLINEAR VOLTERRA EQUATION OF THE SECOND KIND

Abstract

As is known, the quadrature method is very urgent up to now in finding the solutions of nonlinear Volterra integral equation of the second kind. Some authors suggest the quadrature method jointly with Runge-Kutta or Adams methods. However in all these cases it is necessary to calculate integral sum wherein the amount of calculations of integrand function (nucleus) increses in passing from a point to a point where the value of the solution of integral equation should be determined. In order to preserve constant amount of calculations of the integral nucleus a multi-step method is suggested in [2]*. Here, in §1 sufficient conditions for the convergence of the indicated method are found. Some concrete methods applied to the solution of typical equations are in §2. Comparison of the obtained results with the known ones is also given.*

Introduction.

We consider a numerical solution of the following Volterra integral equation:

$$y(x) = f(x) + \int_{x_0}^{x} K(x, s, y(s))\, ds, \qquad x \in [x_0, X]. \tag{1}$$

We assume that continuous in totality of variables function $K(x, s, y)$ has continuous partial derivatives up to some order p, inclusively. The derivatives of $p+1$-th order are bounded.

In order to construct the method we divide the segment $[x_0, X]$ into N equal parts by means of the constant step $h > 0$. We take the partitioning points in the form: $x_i = x_0 + ih \quad (i = 0, 1, 2, ..., N)$.

To find numerical solution of equation (1) we suggest the following method:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{i=0}^{k} \alpha_i f_{n+i} + h \sum_{j=0}^{k} \sum_{i=0}^{k} \beta_i^{(j)} K(x_{n+j},\ x_{n+i},\ y_{n+i}), \tag{2}$$

where α_i, $\beta_i^{(j)}$ are some real numbers, k is an integer quantity, h is a partitioning step of the segment $[x_0, X]$ into N equal parts, y_m is an approximate value of the solution of Volterra integral equations at the points $x_m = x_0 + mh$, and $f_m = f(x_m) \quad (m = 0, 1, 2, ..)$ (see [2]).

By solving many applied problems there arises necessity to use the high accuracy methods. To this end, some authors suggest to raise the accurary of the used method. In [3] Richardson's extrapolation method and in [4] Runge method are suggested.

Numerical solution of ordinary differential equations has been well studied and there are many numerical methods for its solutions. Therefore there exist a tendency of substitution of the solution of integral equations by its corresponding differential equation. In [6] by degenerating, the nucleus of equation (1) is substituted by a system of differential equations thereto multi-step method is applied. It is also shown that the solution of the obtained system of difference equations is simple.

Notice that in [2] by means of concrete methods it is shown that if the method (2) is stable and implicit, then its degree of accuracy is higher than the accuracy degree of the explicit stable method. Therefore, here we suggest to use implicit methods in the structure of predictor-corrector method. The construction method of the predictor-corrector method was taken from the paper [5] .

§1. Convergence of multi-step method with constant coefficients.

In [2] it is shown that the use of some methods of type (2) do not always give acceptable results. Therefore, there arises necessity to study the convergence of the method (2) . To this end, in the method (2) we replace $y_{n+i}\,(i = 0, 1, ..., k)$ by their exact values $y\,(x + ih)$, $(x = x_0 + nh)$. Then we get:

$$\sum_{i=0}^{k} \alpha_i y\,(x + ih) = \sum_{i=0}^{k} \alpha_i f\,(x + ih) +$$

$$+h \sum_{j=0}^{k} \sum_{i=0}^{k} \beta_i^{(j)} K\,(x_{n+j},\ x_{n+i},\ y\,(x_{n+i})) + r_n. \tag{1.1}$$

Here r_n is the error of the method (2) . Notice that by using the method (2) it is assumed that the initial values $y_0,\ y_1,\ y_2, ...,\ y_{k-1}$. are known. We can define the convergence of the method (2) by means of the following theorem.

Theorem. *Let the following conditions be fulfilled:*

1). Continuous function $K\,(x, s, z)$ is determined in some closed domain D and in the same place has continuous partial derivatives up to some $p + 1$, inclusively.

2). The method (2) has degree p and $\alpha_k \neq 0$.

3). Initial data were calculated with accuracy of p, i.e.

$$y\,(x_i) - y_i = O\,(h^p)\,, \quad (i = 0, 1, ..., k - 1)\,.$$

4). Round-off errors have higher accuracy than initial data. Namely

$$\delta_n = O\left(h^{p+1}\right) \quad (n = 0, 1, 2, ...)\,.$$

Then, it holds the following

$$\max_{k \le m \le N} (y\,(x_m) - y_m) = O\,(h^p)\,, \quad h \to 0.$$

Proof. As it is known, in real calculations the method (2) is in the form:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{i=0}^{k} \alpha_i f_{n+i} + h \sum_{j=0}^{k} \sum_{i=0}^{k} \beta_i^{(j)} K\,(x_{n+j},\ x_{n+i},\ y_{n+i}) + \delta_n, \tag{1.2}$$

where δ_n are round-off errors obtained by finding the quantity y_{n+k} .

Denote

$$\varepsilon_m = y(x_m) - y_m \qquad (m = 0, 1, 2, ...).$$

Subtracting (1.2) from (1.1) we get:

$$\sum_{i=0}^{k} \alpha_i \varepsilon_{n+i} = h \sum_{j=0}^{k} \sum_{i=0}^{k} \beta_i^{(j)} L_{n+i}^{(j)} \varepsilon_{n+i} + r_n - \delta_n, \tag{1.3}$$

here

$$L_{n+i}^{(j)} = K_y' \left(x_{n+j},\ x_{n+i},\ \xi_{n+i} \right), \quad j, i = 0, 1, ..., k,$$

ξ_{n+i} is between $y(x_{n+i})$ and y_{n+i} . We rewrite relation (1.3) in the form

$$\varepsilon_{n+i} = \sum_{i=0}^{k-1} \frac{1}{\overline{\alpha_k}} \left(-\alpha_i + h \sum_{j=0}^{k} \beta_i^{(j)} L_{n+i}^{(j)} \right) \varepsilon_{n+i} + R_n , \tag{1.4}$$

here

$$R_n = (r_n - \delta_n) / \overline{\alpha_k} , \ \overline{\alpha_k} = \alpha_k - h \sum_{j=0}^{k} \beta_k^{(j)} L_{n+k}^{(j)}.$$

Denote

$$b_{n+i} = \sum_{j=0}^{k} \beta_i^{(j)} L_{n+i}^{(j)} \qquad (i = 0, 1, ..., k).$$

Let's consider the following relation

$$\frac{-\alpha_i + h b_{n+i}}{\alpha_k - h b_{n+k}} = -\frac{\alpha_i}{\alpha_k} + h \upsilon_{n+i} \tag{1.5}$$

where $\upsilon_{n+i} = (\alpha_k b_{n+i} + \alpha_i b_{n+k}) / \alpha_k (\alpha_k - b_{n+k} h)$.

Considereing sufficient smallness of h we can assume $h b_{n+k} \le \alpha_k / 2$.

Hence, it follows that υ_{n+i} $(i = 0, 1, ..., k-1)$ are bounded, i.e. $|\upsilon_{n+i}| \le \upsilon$.

If we consider (1.5) in (1.4) we can write

$$\varepsilon_{n+k} = -\sum_{i=0}^{k-1} \frac{\alpha_i}{\alpha_k} \varepsilon_{n+i} + h \sum_{i=0}^{k-1} \upsilon_{n+i} \varepsilon_{n+i} + R_n .$$

By means of the following vector

$$Y_{n+k} = (\varepsilon_{n+i} , \varepsilon_{n+2} , ..., \ \varepsilon_{n+k}) \tag{1.6}$$

we can write the received relation in the following form (to(1.6) we add the identity $\varepsilon_{n+v} \equiv \varepsilon_{n+v}$ $(v = 1, 2, ..., k-1)$):

$$Y_{n+k} = A Y_{n+k-1} + h V_{n+k} Y_{n+k-1} + W_{n,k} , \tag{1.7}$$

where the matrices A, V_{n+k} and the vector are defined in the form:

$$A = \begin{pmatrix} -\dfrac{\alpha_{k-1}}{\alpha_k} & -\dfrac{\alpha_{k-2}}{\alpha_k} & ... & -\dfrac{\alpha_1}{\alpha_k} & -\dfrac{\alpha_0}{\alpha_k} \\ 1 & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 1 & 0 \end{pmatrix},$$

$$W_{n,k} = \begin{pmatrix} R_n \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$V_{n+k} = \begin{pmatrix} v_{n+k-1} & v_{n+k-2} & ... & v_n \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{pmatrix}$$

The matrix A is a Frobenius matrix. Therefore, the set of eigen value of the matrix A concides with the roots of the characteristic polynomial $\rho(\lambda) \equiv \sum\limits_{i=0}^{k} \alpha_i \lambda^i$ of the method (2). From the statility of the method it follows that eigen value of the matrix A are not greater that a unit in modulus, and the ones equal to a unit in modulus are not multiple. Consequently, there exists a non-singular matrix C such that the matrix $D = C^{-1}AC$ satisfies the condition $\|D\| \leq 1$.

In equation (1.7) we use change of variables $Y_{n+k} = CZ_{n+k}$ and multiplying the obtained equation by C^{-1} from the left we get :

$$Z_{n+k} = DZ_{n+k-1} + h\overline{V}_{n+k} Z_{n+k-1} + \overline{W}_{n,k}\ , \tag{1.8}$$

Here $\overline{V}_j = C^{-1}V_jC\ ,\ \overline{W}_j = C^{-1}W_j$.

Obviously, the norm of the matrix V_j is estimated by the elements of the first row. Then we have (without losing generality we assume $\alpha_k = 1$):

$$\|V_j\| \leq \sum_{i=0}^{k-1} \left| \frac{\alpha_k b_{n+i} + \alpha_i b_{n+k}}{\alpha_k (\alpha_k - b_{n+k} h)} \right| \leq \gamma L,$$

Here $L = \max \left| L_{n+i}^{(j)} \right|, \quad \gamma = 2k \sum\limits_{i=0}^{k} \left| \beta_i^{(j)} \right|$.

Then

$$\|\overline{V}_j\| \leq \left\|C^{-1}\right\| \|V_j\| \|C\| \leq \gamma L \left\|C^{-1}\right\| \|C\|,$$
$$\|w_j\| \leq \left\|C^{-1}\right\| \|W_j\| = \left\|C^{-1}\right\| \max_n |R_n|.$$

Allowing for the obtained estimations in equation (1.8) we have:

$$\|Z_j\| \leq \beta |R_j| + (1 + \gamma L h) \|Z_{j-1}\|. \tag{1.9}$$

where $\beta = \left\| C^{-1} \right\|$.

Considering inequality (1.9) as a recurrent relation and expressing $\|Z_j\|$ $(j \geq k)$ by $\|Z_{k-1}\|$ we get the following:

$$\|Z_m\| \leq \beta \sum_{j=k}^{m} (1+\gamma Lh)^{m-j} |R_j| + (1+\gamma Lh)^{m-k+1} \|Z_{k-1}\| \tag{1.10}$$

Taking into account

$$(1+\gamma Lh)^{m-j} \leq \exp(\gamma Lh (m-j)) \leq \exp(\gamma LX),$$

we can rewrite relation (1.10) in the form:

$$\|Z_m\| \leq \exp(\gamma LX) \left(\beta \sum_{j=k}^{m} |R_j| + \|Z_{k-1}\| \right).$$

It is easy to show that

$$\|\varepsilon_n\| \leq \|Y_n\| \leq \|C\| \, \|Z_n\|,$$

$$\|Z_{k-1}\| \leq \left\| C^{-1} \right\| \|Y_{k-1}\| = \left\| C^{-1} \right\| \max_{0K, i \leq k-1} |\varepsilon_i|.$$

Then we can write

$$\|Y_m\| \leq \|C\| \exp(\gamma LX) \left(\beta \sum_{j=k}^{m} |R_j| + \left\| C^{-1} \right\| \|Y_{k-1}\| \right).$$

Consequently, for the error of the method (2) we get the following estimates:

$$\|\varepsilon_n\| \leq \exp(\gamma LX) \left(M_1 \sum_{j=k}^{m} |R_j| + M_2 \max_{0 \leq i \leq k-1} |\varepsilon_i| \right), \tag{1.11}$$

Here $M_1 = \beta \|C\|$, $\quad M_2 = \|C\| \left\| C^{-1} \right\|$.

If in (1.11) we take in to account the conditions of the theorem, we get:

$$\varepsilon_n = O(h^p), \ h \to 0.$$

§2 Construction of some concrete multi step methods and their comparison with the known methods.

Here, in constructing concrete methods we use the ways suggested in [2]. In order to find the coefficients α_i , $\beta_i^{(j)}$ $(i, j = 0, 1, ..., k)$ for $k = 2$ we use the following system of algebraic equations:

$$
\begin{aligned}
&\alpha_0+\alpha_1+\alpha_2=0,\\
&\alpha_1+2\alpha_2=\beta_0^{(0)}+\beta_1^{(0)}+\beta_2^{(0)}+\beta_0^{(1)}+\\
&+\beta_1^{(1)}+\beta_2^{(1)}+\beta_0^{(2)}+\beta_1^{(2)}+\beta_2^{(2)},\\
&\frac{1}{2}\alpha_1+2\alpha_2=\beta_1^{(0)}+\beta_2^{(0)}+\beta_1^{(1)}+\\
&+\beta_2^{(1)}+\beta_1^{(2)}+\beta_2^{(2)},\\
&\frac{1}{6}\alpha_1+\frac{4}{3}\alpha_2=\frac{1}{2}\beta_1^{(0)}+2\beta_2^{(0)}+\\
&+\frac{1}{2}\beta_1^{(1)}+2\beta_2^{(1)}+\frac{1}{2}\beta_1^{(2)}+2\beta_2^{(2)},\\
&\frac{1}{24}\alpha_1+\frac{2}{3}\alpha_2=\frac{1}{6}\beta_1^{(0)}+\frac{4}{3}\beta_2^{(0)}+\\
&+\frac{1}{6}\beta_1^{(1)}+\frac{4}{3}\beta_2^{(1)}+\frac{1}{6}\beta_1^{(2)}+\frac{4}{3}\beta_2^{(2)},\\
&\frac{1}{120}\alpha_1+\frac{4}{15}\alpha_2=\frac{1}{24}\beta_1^{(0)}+\frac{2}{3}\beta_2^{(0)}+\\
&+\frac{1}{24}\beta_1^{(1)}+\frac{2}{3}\beta_2^{(1)}+\frac{1}{24}\beta_1^{(2)}+\frac{2}{3}\beta_2^{(2)}.
\end{aligned}
\tag{2.1}
$$

If the obtained system has a solution differ from zero, then we get a method with degree $p=5$. First of all we consider construction of stable explicit method. To this end we write the last equations from (2.1) in the form:

$$
\begin{aligned}
&a+2b=\alpha_1/2+2\alpha_2\ ,\\
&a/2+2b=\alpha_1/6+4\alpha_2\ /3\ ,\\
&a/6+4b/3=\alpha_1/24+2\alpha_2\ /3\ ,\\
&a/24+2b/3=\alpha_1/120+4\alpha_2\ /3\ ,
\end{aligned}
\tag{2.2}
$$

here $a=\beta_1^{(0)}+\beta_1^{(1)}+\beta_1^{(2)},\ b=\beta_2^{(0)}+\beta_2^{(1)}+\beta_2^{(2)}$.

Obviously, for explicit methods $b\equiv 0$. We can show that in this case explicit stable methods with degree $p>2$ do not exist. Assume that there exist explicit stable methods with degree $p=3$. Then, from (2.2) we have:

$$a=\alpha_1/2+2\alpha_2 \quad \text{and} \quad a/2=\alpha_1/6+4\alpha_2\ /3.$$

Without losing generality we can assume $\alpha_2=1$. Hence we get that $\alpha_1=4$ and the existing methods are unstable. Therefore we consider construction of explicit stable method with degree $p=2$.One of these methods is of the form

$$
\begin{gathered}
y_{n+2}=(3y_{n+1}+y_n)/4+f_{n+2}-(3f_{n+1}+f_n)/4+\\
h\left(-K\left(x_n\ ,\ x_n\ ,\ y_n\ \right)-K\left(x_{n+1}\ ,\ x_n\ ,\ y_n\ \right)-\right.\\
-K\left(x_{n+2}\ ,\ x_n\ ,\ y_n\ \right)+4K\left(x_n\ ,\ x_{n+1}\ ,\ y_{n+1}\ \right)+\\
\left.+4K\left(x_{n+1}\ ,\ x_{n+1}\ ,\ y_{n+1}\ \right)+5K\left(x_{n+2}\ ,\ x_{n+1}\ ,\ y_{n+1}\ \right)\right)/8
\end{gathered}
\tag{2.3}
$$

Using system (2.2) we construct stable implicit method with degree $p = 3$. Such methods are more than one. The following methods belong to the of stable methods with degree $p = 3$:

$$y_{n+2} = (y_n + y_{n+1})/2 + f_{n+2} - (f_{n+1} + f_n)/2+$$

$$h\left(4K(x_{n+1}, x_{n+1}, y_{n+1}) + K(x_{n+2}, x_n, y_n) +\right.$$

$$\left.+4K(x_{n+2}, x_{n+1}, y_{n+1}) + 3K(x_{n+2}, x_{n+2}, y_{n+2})\right)/8 \qquad (2.4)$$

$$y_{n+2} = (y_n + y_{n+1})/2 + f_{n+2} - (f_{n+1} + f_n)/2+$$

$$h\left(K(x_n, x_n, y_n) + 4K(x_n, x_{n+1}, y_{n+1}) +\right.$$

$$\left.+3K(x_n, x_{n+2}, y_{n+2}) + 4K(x_{n+1}, x_{n+1}, y_{n+1})\right)/8 \qquad (2.5)$$

We can easily prove that system (2.2) has no solution different from zero. Consequently, for $k = 2$ there are no stable methods with degree $p = 5$. Therefore, we consider construction of stable methods with degree $p = 4$. Such methods are more than one. One of them has the following form:

$$y_{n+2} = y_n + f_{n+2} - f_n + h\left(-K(x_n, x_{n+2}, y_{n+2}) -\right.$$

$$-K(x_n, x_{n+1}, y_{n+1}) - K(x_n, x_n, y_n) +$$

$$K(x_{n+1}, x_{n+2}, y_{n+2}) + 4K(x_{n+1}, x_{n+1}, y_{n+1}) +$$

$$+K(x_{n+1}, x_n, y_n) + K(x_{n+2}, x_{n+2}, y_{n+2}) +$$

$$\left.K(x_{n+2}, x_{n+1}, y_{n+1}) + K(x_{n+2}, x_n, y_n)\right)/3 \qquad (2.6)$$

Now, let's consider construction of explicit method and select the coefficients in the form: $\beta_i^{(2)} = -\overline{\alpha_i}\gamma_i^{(j)}$. In this one of the explicit stable methods with degree $p = 2$ is of the form:

$$y_{n+2} = (3y_{n+1} + y_n)/4 + f_{n+2} - (3f_{n+1} + f_n)/4 + h\left(-3K(x_{n+2}, x_n, y_n) +\right.$$

$$\left.+6K(x_{n+1}, x_{n+1}, y_{n+1}) + 7K(x_{n+2}, x_{n+1}, y_{n+1})\right)/8.$$

Notice that in this case the method with degree $p = 4$ is unique and is of the form: $\left(\beta_i^{(2)} = 0 \quad (i = 0, 1, 2)\right)$:

$$y_{n+2} = y_n + f_{n+2} + h\left(K(x_n, x_n, y_n) +\right.$$

$$\left.+4K(x_n, x_{n+1}, y_{n+1}) + K(x_n, x_{n+2}, y_{n+2})\right)/3. \qquad (2.7)$$

Thus, we considered construction of 6 methods of type (4), whose coefficients are the solution of system (2.1). Two of these methods were constructed in [2].

To illustrate the obtained results we consider application of the methods constructed here to concrete equations. We also compare the obtained results on the method constructed above with the results obtained by quadrature methods successfully applied in numerical methods for solving integral equations. All examples have been taken from the paper [1].

Example 1. *For numerical solution of the integral equation (exact solution* $y(x) = x + x^3/6$*) :*

$$y(x) = x + \int_0^x \sin(x-s)\, y(s)\, ds, \quad x \in [0,1], \tag{2.8}$$

we use the following method (see [2]*)*

$$y_n - 2y_{n-1} + y_{n-2} = f_n - 2f_{n-1} + f_{n-2} -$$
$$h\left(K\left(x_{n-2}\ ,\ x_{n-1}\ ,\ y_{n-1}\right) + K\left(x_{n-2}\ ,\ x_{n-2}\ ,\ y_{n-2}\right)\right)/2$$
$$+h\left(3K\left(x_n\ ,\ x_{n-1}\ ,\ y_{n-1}\right) - K\left(x_n\ ,\ x_{n-2}\ ,\ y_{n-2}\right)\right)/2$$

but in [1, p. 87] *the trapezoid method is used.*

The error obtained in [1] at the finite point $x = 1$ equals $0,255$, the error of the above-indicated methods at the same point equals $0,036$.

All the methods constructed above has degree $p = 2,3,4$. Consequently, if the solution of equation (1) is a polynomial with degree no more thatn 2, the error of methods with degree $p \geq 3$ will equal zero. In this relation we consider the following example whose solution equals $y(x) = x^2$.

Example 2. *The methods constructed above were applied to numerical solution of the integral equation*

$$y(x) = 1 + \int_1^x \sqrt{y(s)}ds, \quad x \in [1,2], \tag{2.9}$$

The calculations were carried out to within 10^{-12} *and for all methods with degree* $p > 2$ *the error was equal to zero, i.e. approximate and exact values of the solution of equation* (2.9) *coincided. However, in* [1, p. 83], *equation* (2.9) *was solved by iterated methods and the error at final step was equal to* $0,1802$. *Then the next integral equation whose solution equals* $y(x) = x$, *was considered.*

Example 3. *We solve the integral equation:*

$$y(x) = \int_0^x \left(1 + s^2\right) ds, \quad x \in [0,1] \tag{2.10}$$

by the methods constructed above.

The calculations were carried out with accuracy 10^{-12} *and error for all methods at the points* x_i $(i \geq 2)$ *was equal to zero.*

Now, let's consider the numerical solution of integral equations whose solution is not a polynomial. One of these equations is of the form:

Example 4.

$$y(x) = 2 + \int_0^x \left(y(s) / (s+1) - (s + 1/y(s)) \right) ds, \quad x \in [0,6] \tag{2.11}$$

The solution of equation (2.11) *is written in the form:*

$$y(x) = (x+1)\sqrt{4 - 2\ln(x+1)}.$$

We find the numerical solution of the equation by predictor-corrector methods that use methods $(2.4) - (2.6)$ for the step $h = 0,025$. At the step I high accuracy is attained, then the error of the method hesitate (increases, then decreases) and at the point $x = 6$ the error of the method has order 10^{-7}. The obtained result coincides with the result obtained in [1, p. 93] where for numerical solution of equation (2.11) a method with degree $p = 5$ is applied and the principle of finding of solution by the regions that provide attainment of higher accuracy of results whose use is more complicated than the use of the method (2), is used.

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