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PROPAGATION OF NON-STATIONARY SH WAVES IN ELASTIC STRIPS

Abstract

In the work the propagation of the non-stationary SH (shear horizontal) waves is investigated in the homogeneous isotropic elastic strip. The waves are created by horizontal concentrated tangent force in one surface of the strip. The problem is solved by using the Laplace and Fourier transformations. The inverse transformations are computed by Cagniard-de Hoop method.

Introduction. Non-homogeneous problems on propagation of non stationary waves in deformable bodies with boundary are complicated problems of continuum mechanics $[1-7]$. The problems on elastic half-space excited by concentrated loads belong to these problems. The problem when the surface of an elastic half-space is excited by normal cocentrated load was considered by Lamb. Later on, the Lamb problem was generalizet for anisotropic, non-homogeneous and linear viscous-elastic half-spaces for simple models $[3,6]$.

The recent years horizontally polarized (SH) shift harmonic elastic waves are intensively studied. Non-stationary waves in an elastic half-space were investigated in [7].

In the paper we study a two-dimensional non-stationary problem on propagation of horizontally polarized shift wave in an elastic strip, excited by concentrated shift loads. The problem is solved by joint integral Laplace and Fourier transformations whose inverse transformations are found by Cagniard-de Hoop method [8, 9] that was developed in the paper $[7]$.

Problem statement: Let a permutations field in an elastic strip $0 \le y \le h$ be given in the form

$$\overline{u} = \{0,\ 0,\ w(x,y,t)\} \tag{1}$$

Then, taking into account $u=0,\ v=0$ the deformations will get the form:

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

$$e_{xx} = \frac{\partial u}{\partial x} = 0, \qquad e_{yx} = 0, \qquad e_{zx} = \frac{1}{2}\frac{\partial w}{\partial x},$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0, \quad e_{yy} = \frac{\partial v}{\partial y} = 0, \quad e_{zy} = \frac{1}{2}\frac{\partial w}{\partial x},$$

$$e_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\frac{\partial w}{\partial x}, \quad e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\frac{\partial w}{\partial y}, \quad e_{zz} = \frac{\partial w}{\partial z} = 0.$$

The stress tensor components are expressed by the relations

$$\sigma_{11} = \sigma_{12} = \sigma_{22} = \sigma_{33} = 0, \quad \sigma_{13} = \mu\frac{\partial w}{\partial x}, \quad \sigma_{23} = \mu\frac{\partial w}{\partial y}, \tag{2}$$

where μ is a shift modulus.

Considering (1) – (2) in the motion equation

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2} \qquad (u_1 = u, \quad u_2 = \upsilon, \quad u_3 = w, \quad x_1 = x, \; x_2 = y, \quad x_3 = z)$$

we get a partial differential equation of hyperbolic type

$$\mu \Delta w = \rho \frac{\partial^2 w}{\partial t^2} \qquad (-\infty < x < \infty, \quad 0 \leq y \leq h, \quad t > 0)\,, \tag{3}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a two-dimensional Laplace operator.

We'll solve equation (3) under the following initial and boundary conditions:

$$w = 0, \; \frac{\partial w}{\partial t} = 0 \quad \text{for} \quad t = 0 \quad (-\infty < x < \infty, \; 0 < y < h)\,, \tag{4}$$

$$\sigma_{23} = -\delta(x) f(t) \quad \text{for} \quad y = 0, \quad (-\infty < x < \infty, \quad t > 0)\,, \tag{5}$$

$$\sigma_{23} = 0 \quad \text{for} \quad y = h \quad (-\infty < x < \infty, \quad t > 0)\,, \tag{6}$$

$$w \to 0 \qquad r = \sqrt{x^2 + y^2} \to \infty, \tag{7}$$

where $f(t)$ is a one-valued piece-wise smooth function vanishing for $t < 0$ and alternating no faster than exponential function as $t \to \infty$, $\delta(x)$ is Dirac's delta-function.

in all strips that are parallel to the plane xOy, the wave pattern is identical (cylindrical wave propagates from the source). Therefore we can represent the problem as for a plane strip subjected to the action of momentary concentrated load perpendicular to the strip. On the axis Ox permutations differ from zero, they are created by the stresses σ_{13}, that also differ from zero. However, $\sigma_{23} = 0$ at all the points of the axis Ox $(y = 0)$, except the origin $x = 0$ where it is given in the form of Dirac's δ-function.

Problem solution. We'll solve problem (3) – (7) by using Fourier integral transformations with respect to the coordinate x and Laplace transformations with respect to time t, determined by the relations

$$w^F = \int_{-\infty}^{\infty} w(x, y, t)\, e^{iqx} dx \qquad (\operatorname{Im} q = 0)\,,$$

$$\overline{w} \equiv w^L = \int_{0}^{\infty} e^{-pt} w(x, y, t)\, dt \qquad (\operatorname{Re}\ p > 0)\,.$$

Applying these transformations and considering initial values (4) we get

$$\mu \left(\frac{d^2 w^{LF}}{\partial y^2} - q^2 w^{LF} \right) = \rho p^2 w^{LF} \quad \text{or} \quad \frac{d^2 w^{LF}}{\partial y^2} = \left(q^2 + \frac{\rho p}{\mu} \right) w^{LF}.$$

The solution of the last equation is

$$w^{LF} = Ae^{-ny} + Be^{ny} \tag{8}$$

where $n = \sqrt{q^2 + \eta^2}$, $\eta^2 = \rho p^2/\mu$. To distinguish one-valued branch of this radical in the plane q the draw a section from the points $\pm i\eta$ to the infinity along the rays $\arg q = \arg \eta \pm \pi/2$ and assume that for $q = 0$ the equality $\sqrt{q^2 + \eta^2} = \eta$ is fulfilled. Then $\operatorname{Re}\sqrt{q^2 + \eta^2} > 0$ for $\operatorname{Im} q = 0$, $\operatorname{Re} p > 0$.

Satisfying condition (6) we find:

$$B = Ae^{-2nh}. \tag{9}$$

Requiring the fulfilment of boundary condition (5) allowing for (9) we get:

$$-\mu n A\left(1 - e^{-2nh}\right) = -\overline{f}.$$

Hence for A we find:

$$A = \frac{\overline{f}}{\mu n\left(1 - e^{-2nh}\right)}.$$

As a result, we get

$$w^{LF} = \frac{\overline{f}}{\mu n\left(1 - e^{-2nh}\right)}\left(e^{-ny} + e^{-n(2h-y)}\right).$$

Considering that in the right half-plane $\operatorname{Re} s > 0$, $\left|e^{-2nh}\right| < 1$, we have a uniform and absolutely convergent series

$$w^{LF} = \frac{\overline{f}}{\mu n}\sum_{k=0}^{\infty}\left[e^{-n(y+2hk)} + e^{-n(2hk+2h-y)}\right]. \tag{10}$$

To calculate the inverse joint transformations of this function we consider the following expression

$$\xi^{LF} = \frac{\overline{f}(p)}{\mu n}e^{-ny} \tag{11}$$

Applying the inverse Fourier transformation with respect to q to (11), we find

$$\overline{\xi}(x,y,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\overline{f}}{\mu\sqrt{q^2+\eta^2}}e^{-iqx-y\sqrt{q^2+\eta^2}}dq. \tag{12}$$

We write relation (12) in the form

$$\overline{\xi}(x,y,p) = \frac{\overline{f}}{2\pi\mu}\int_{-\infty}^{\infty}\frac{1}{\sqrt{q^2+p^2/c^2}}e^{-iqx-y\sqrt{q^2+p^2/c^2}}dq, \tag{13}$$

where $c = \sqrt{\mu/\rho}$ is propagation velocity of a shift wave.

We'll calculate the Laplace inverse transformation of function (13) by using Cagniard-de Hoop method [8, 9]. The essence of the method is that integral (13) is transformed into the integral of the form of Laplace transformation with respect to t. To this end, at first we make change of variable of integration by the formula $q = -isp$ and reduce expression (13) to the form:

$$\overline{\xi} = \frac{i\overline{f}c}{2\pi\mu} \int\limits_L \frac{1}{\sqrt{1-c^2s^2}} e^{-p\left[sx+y\sqrt{c^{-2}-s^2}\right]} ds, \tag{14}$$

where a straight line L passes through the origin of coordinate in the complex plane s with a slope to real positive semi-axis (fig.1).

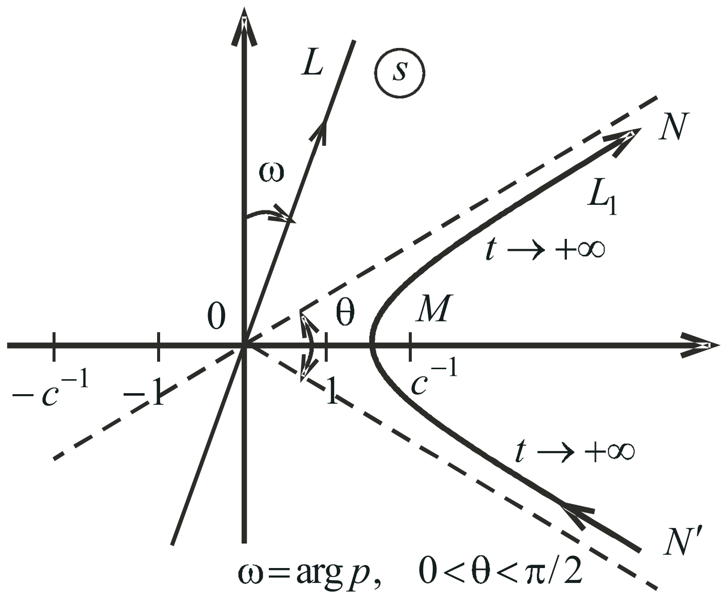

Fig. 1. Contours L and L_1

The branches of the radical $\sqrt{c^{-2}-s^2}$ are determined in such a way that $\sqrt{c^{-2}-s^2} = c^{-1}$ for $s = 0$, and the sections determining this one-valued branch pass along real semi-axes $\left(-\infty, -c^{-1}\right)$ and $\left(c^{-1}, \infty\right)$.

In order integral (14) to look like Laplace integral we deform the contour L to such a path L_1 along which expression $sx + y\sqrt{c^{-2}-s^2}$ is real and assume

$$sx + y\sqrt{c^{-2}-s^2} = t \tag{15}$$

where a real quantity t should be considered as a parameter alternating along the integration path L_1 in the complex domain s.

Solving equation (15) with respect to s, we find

$$s = \frac{xt \pm y\sqrt{r^2c^{-2}-t^2}}{r^2}, \quad |t| < rc^{-1}, \quad r = \sqrt{x^2+y^2}, \quad x \geq 0, \tag{16}$$

$$s = \frac{xt \pm iy\sqrt{t^2-r^2c^{-2}}}{r^2}, \quad |t| > rc^{-1}, \quad r = \sqrt{x^2+y^2}, \quad x \geq 0, \tag{17}$$

where the radicals are assumed to be arithmetical.

For $s = 0$ it follows from expression (15) that $t = y/c$, and therefore, it is necessary to take the sign "minus" in (16) and leave both signs in (17) in order to get a contour described by expression (15) for $t > 0$ and $x \geq 0$ (since it suffices to consider the case $x \geq 0$, considering evenness of w with respect to x). In figure 1, in the domain $\mathrm{Re}\, s > 0$ the contour L_1 (for $x \geq 0$) is described by a bold-face curve. This contour consists of the section OM and the curve $N'MN$ and is described by the expressions

$$s = \frac{xt - y\sqrt{r^2c^{-2} - t^2}}{r^2}, \quad \left(\frac{y}{c} < t < \frac{r}{c}, \quad x \geq 0\right), \tag{18}$$

$$s = \frac{xt \pm iy\sqrt{t^2 - r^2c^{-2}}}{r^2}, \quad \left(t > \frac{r}{c}, \quad x \geq 0\right), \tag{19}$$

The signs "plus" and "minus" in (19) belong to the curves in the upper and lower half-planes s, respectively.

Put in formulae (18) , (19) $x = r\cos\theta$, $y = r\sin\theta$ $(0 < \theta < \pi/2)$:

$$s(t, r, \theta) = \frac{1}{r}\begin{cases} t\cos\theta - \sqrt{r^2c^{-2} - t^2}\sin\theta & (t < r/c), \\ t\cos\theta \pm i\sqrt{t^2 - r^2c^{-2}}\sin\theta & (t > r/c). \end{cases}$$

The r and θ will be considered to be fixed when $t \to +\infty$. Then we'll get $s \to e^{\pm i\theta}t/r$ and hence the curve L_1 as $t \to +\infty$ is bounded by asymptotes going out from the origin $s = 0$ at the slope angle $\pm\theta$ to the real axis (fig.1) . As $s \to \infty$ the expression $p\left(sx + y\sqrt{c^{-2} - s^2}\right)$ behaves as follows:

$$p\left(sx + y\sqrt{c^{-2} - s^2}\right) = \begin{cases} |p|\,|s|\,re^{i(\omega+\varphi+\theta)} & (\mathrm{Re}\, s > 0), \\ |p|\,|s|\,re^{i(\omega+\varphi-\theta)} & (\mathrm{Re}\, s < 0), \end{cases}$$

where $p = |p|\,e^{i\omega}$ $(|\omega| < \pi/2)$, $s = |s|\,e^{i\varphi}$. Since

$$\min\left(\theta, -\frac{\pi}{2} - \omega\right) \leq \varphi \leq \max\left(\theta, -\frac{\pi}{2} - \omega\right) \quad (\mathrm{Re}\, s < 0),$$

$$\min\left(-\theta, \frac{\pi}{2} - \omega\right) \leq \varphi \leq \max\left(-\theta, \frac{\pi}{2} - \omega\right) \quad (\mathrm{Re}\, s > 0),$$

we get

$$\begin{aligned} &-\frac{\pi}{2} < \min\left(-\frac{\pi}{2} - \theta, \omega\right) \leq (\varphi - \theta + \omega) \leq \\ &\leq \max\left(-\frac{\pi}{2} - \theta, \omega\right) < \frac{\pi}{2} \quad (\mathrm{Re}\, s < 0), \\ &-\frac{\pi}{2} < \min\left(\frac{\pi}{2} + \theta, \omega\right) \leq (\varphi + \theta + \omega) \leq \\ &\leq \max\left(\frac{\pi}{2} + \theta, \omega\right) < \frac{\pi}{2} \quad (\mathrm{Re}\, s < 0). \end{aligned} \tag{20}$$

Consequently, the index of the exponent equal $-p\left(sx + y\sqrt{c^{-2} - s^2}\right)$, has a negative real part as $|s| \to \infty$ in the domain between the curves L and L_1 in figure 1. Therefore, using Jordan lemma [10] we can deform an integration contour L in L_1 passing in such order: $N'MOMN$, moreover, a section of the real axis OM is passed twice in contrary directions along lower and upper coasts. Then, deforming the contour L in L_1 and making change of variable s in t by formula (15) we get $(x \geq 0)$:

$$\overline{\xi} = \frac{-i\overline{f}c}{2\pi\mu}\int\limits_{L_1} Fds = \frac{-i\overline{f}c}{2\pi\mu}\left[\int\limits_{N'MO} Fds + \int\limits_{OMN} Fds\right] =$$

$$= \frac{-i\overline{f}c}{2\pi\mu}\left[\int\limits_{+\infty}^{y/c} F\frac{ds}{d\tau}d\tau + \int\limits_{y/c}^{+\infty} F\frac{ds}{d\tau}d\tau\right] = \frac{\overline{f}}{\pi\mu}\int\limits_{y/c}^{+\infty} \operatorname{Im}\left[\frac{y}{(sx-t)}\frac{ds}{dt}\right]e^{-pt}dt. \qquad (21)$$

Here, in integrand expression s is expressed by the relations (18) and (19) and in relation (19) the sign "plus" is taken. By obtaining formula (21) we take into account that the curves $N'MO$ and OMN are symmetric with respect to a real axis, and the integrand function is complex conjugate at complex conjugate points s and $\overline{s}$ on OMN and $N'MO$, respectively. And at these points the function $\left(sx + y\sqrt{c^{-2} - s^2}\right)$ takes real values equal t.

For $x \geq 0,\ y \geq 0$, we can rewrite formula (21) in the form:

$$\overline{\xi}(x,y,p) = \frac{\overline{f}}{\pi\mu}\int\limits_{0}^{\infty} \operatorname{Im}\left[\frac{y}{(sx-t)}\frac{ds}{dt}\right] H(t - r/c)\, e^{-pt}dt, \qquad (22)$$

where $H(t - r/c)$ is a Heaviside unique function.Since the integral in formula (22) is a Laplace integral, we have

$$\overline{\xi}(x,y,p) = \overline{f}\left\{\frac{1}{\pi\mu}\operatorname{Im}\left[\frac{y}{(sx-t)}\frac{ds}{dt}\right] H(t - r/c)\right\}^{L}.$$

Then, the original $\xi(x,y,p)$ is found by using convolution of the functions

$$\xi(x,y,t) = \frac{1}{\pi\mu} f(\tau) * \operatorname{Im}\left[\frac{y}{(sx-t)}\frac{ds}{dt}\right] H(t - r/c), \qquad (23)$$

where asterisks between the functions mean their convolution

$$f(t) * g(t) = \int\limits_{0}^{t} f(t-\tau)\, g(\tau)\, d\tau.$$

Considering the expression of the function $s(t)$ in (23) we finally get:

$$\xi(x,y,t) = \frac{1}{\pi\mu}\int\limits_{r/c}^{t} \frac{f(t-\tau)\, d\tau}{\sqrt{\tau^2 - r^2/c^2}}. \qquad (24)$$

Now, using (23) we find the original of the function (10) :

$$w(x,y,t) = \frac{f(t)}{\pi\mu} * \sum_{k=0}^{ki} \operatorname{Im}\left[\left(\frac{y+2hk}{s_1 x - t}\frac{ds_1}{dt}\right) H\left(t - \frac{r_1}{c}\right) + \right.$$

$$+\left(\frac{2hk+2h-y}{s_2x-t}\frac{ds_2}{dt}\right)H\left(t-\frac{r_2}{t}\right)\right],$$

where

$$s_1(t)=\frac{xt-(y+2hk)\sqrt{r_1^2c^{-2}-t^2}}{r_1^2}\qquad \text{for}\quad \frac{y+2hk}{c}<t<\frac{r_1}{c},\quad x\geq 0,$$

$$s_1(t)=\frac{xt\pm i(y+2hk)\sqrt{t^2-r_1^2c^{-2}}}{r_1^2}\qquad \text{for}\quad t>\frac{r_1}{c},\quad x\geq 0,$$

$$s_2(t)=\frac{xt-(2hk+2h-y)\sqrt{r_1^2c^{-2}-t^2}}{r_2^2}\qquad \text{for}\quad \frac{(2hk+2h-y)}{c}<t<\frac{r_2}{c},\quad x\geq 0,$$

$$s_2(t)=\frac{xt\pm i(2hk+2h-y)\sqrt{t^2-r_1^2c^{-2}}}{r_2^2}\qquad \text{for}\quad t>\frac{r_2}{c},\quad x\geq 0,$$

$$r_1^2=x^2+(y+2hk)^2,\qquad r_2^2=x^2+(2hk+2h-y)^2,\qquad k=0,1,2,...$$

Now we use formula (24) and get:

$$w(x,y,t)=\frac{1}{\mu\pi}\sum_{k=0}^{\infty}\left[\int\limits_{r_1/c}^{t}\frac{f(t-\tau)\,d\tau}{\sqrt{\tau^2-r_1^2c^2}}+\int\limits_{r_2/c}^{t}\frac{f(t-\tau)\,d\tau}{\sqrt{\tau^2-r_2^2c^2}}\right].\qquad (25)$$

It is easy to prove the theorem.

Theorem. If $f(t)$ is a one-valued piecewise-smooth function vanishing for $t<0$ and alternating no faster than exponential function as $t\to\infty$, the function $w(x,y,t)$ expressed by relation (25) is an exact solution of $(3)-(7)$.

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