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SOME COMMUTATOR PROPERTIES OF A VARIETY WITH A WEAK DIFFERENCE TERM

(Dedicated to the memory of R.A.Bairamov)

Abstract

We establish some commutator-type equations for varieties with a weak difference term.

1^0. **Introduction.** A.I.Mal'cev published [1] which contains the result that all algebras of a variety **V** have permuting congruences if and only if **V** satisfies the equations

$$m(x,x,y) = y = m(y,x,x)$$

for some ternary term $m(x,y,z)$; now it is called a Mal'cev term. For example, any variety of groups, rings, modules, Boolean algebras or Lie algebras has a Mal'cev term.

Later, in [2] it is shown by J.Hagemann and C.Herrmann that it is possible to extend the theory of the group commutator to any congruence modular variety without loss of the generality of the theory; see also [3], [4]. In [3], a generalization of Mal'cev term, so-called difference term plays an essential role. A *difference term* is a term $d(x,y,z)$ which satisfies for any x, y

$$d(x,x,y) = y \quad \text{and} \quad d(x,y,y)\,[\theta,\theta]\,x$$

where [,] is the commutator and θ is any congruence containing the couple (x,y). Many useful properties of varieties with a difference term are described in [5]. In the book [6], D. Hobby and R.McKenzie show that there is a largest special Mal'cev condition for locally finite (not necessary modular) varieties and this Mal'cev condition is equivalent to the existence of a *weak difference term* $w(x,y,z)$ which satisfies

$$w(x,x,y)\,[\theta,\theta]\,y \quad \text{and} \quad w(x,y,y)\,[\theta,\theta]\,x$$

where θ is any congruence containing (x,y). In [7], p. 199, it is shown that in varieties with a weak difference term any algebra is Abelian iff it is affine.

In the present paper we find conditions leading to the commutativity of the commutator for any variety with a weak difference term. Also, we present some congruence identities for varieties with a weak difference term.

2^0. **Some definitions.** For any algebra A, $ConA$ denotes the lattice of all congruences of A. Others standard notations may be found in [6].

Definition 2.1. *If $\alpha,\beta,\gamma \in ConA$ then $C(\alpha,\beta;\gamma)$ holds if for all n whenever $(a,b)\in\alpha$, $(u_i,v_i)\in\beta$, $0\le i<n$, and $p(x,\bar{y})\in Pol_{n+1}(A)$ one has*

$$p(a,\bar{u}) \equiv_\gamma p(a,\bar{v}) \iff p(b,\bar{u}) \equiv_\gamma p(b,\bar{v})\,.$$

As usual, $[\alpha,\beta]$ denotes the least γ such that $C(\alpha,\beta;\gamma)$ holds. The relation $C(\alpha,\beta;\gamma)$ is called the centralizer relation (α centralizes β modulo γ).

We take following definition from [5], also.

Definition 2.2. *Given congruences α, β, γ on A, let $[\alpha, \beta]_\gamma$ denote the least congruence $\delta \geq \gamma$ such that $C(\alpha, \beta; \delta)$ holds.*

Obviously $[\alpha, \beta]_0 = [\alpha, \beta]$.

Definition 2.3. [4]. *Define: $[\theta]^0_\delta := \theta$ and $[\theta]^{n+1}_\delta := [[\theta]^n_\delta, [\theta]^n_\delta]_\delta$. If $\delta = 0$, we write shortly $[\theta]^n$ instead of $[\theta]^n_0$.*

3^0. **Commutatirity of the commutator.** In a congruence modular variety the centralizer relation is symmetric [4], i.e. $C(\alpha, \beta; \gamma)$ if and only if $C(\beta, \alpha; \gamma)$. This fact implies the commutativity of the commutator : $[\alpha, \beta] = [\beta, \alpha]$. For varieties with a weak difference term we have following

Theorem 3.1. *If $\mathbf{V}$ has a weak difference term, $A \in \mathbf{V}, [\alpha, \alpha] \leq [\beta, \alpha]$, and $[\alpha, \beta] \nleq [\beta, \alpha]$, then $[\alpha, \beta] = [\beta, \alpha]$.*

Proof. In the opposite case for α and β we have that $[\beta, \alpha] < [\alpha, \beta]_{[\beta,\alpha]}$. In the algebra $A_1 := A/[\beta, \alpha]$ for congruences $\alpha_1 := \alpha/[\beta, \alpha]$ and $\beta_1 := \beta/[\beta, \alpha]$ it is true that $[\beta_1, \alpha_1] = 0 < [\alpha_1, \beta_1]$. If follows, by changing notation, we may wittingly assume that $[\beta, \alpha] = 0 < [\alpha, \beta]$ in A.

As $[\alpha, \beta] > 0$ there are a polynomial $p(x, \bar{y})$ and couples $(a, b) \in \alpha$, $(u_i, v_i) \in \beta$, $0 \leq i < k$, such that

$$p(a, u_0, ..., u_{k-1}) = p(a, v_0, ..., v_{k-1}),$$

but simultaneously

$$c := p(b, \bar{u})\,[\alpha, \beta]\,p(b, \bar{v}) =: e \text{ and } c \neq e. \tag{1}$$

For $\theta := Cg(c, e)$ clearly $\theta \leq [\alpha, \beta] \leq \alpha$, hence $[\theta, \theta] \leq [\alpha, \alpha] \leq [\beta, \alpha] = 0$.

Let $p_1(x, \bar{y}) := w(p(x, \bar{y}), p(x, \bar{u}), p(b, \bar{u}))$ for a weak difference term $w(x, y, z)$. As $(a, b) \in \alpha$ then

$$p_1(a, \bar{u})\,[\alpha, \alpha]\,p(b, \bar{u}) = p_1(b, \bar{u}). \tag{2}$$

Here $[\alpha, \alpha] = 0$ and hence $p_1(a, u) = p_1(b, \bar{u})$. Consequently

$$c = p_1(a, \bar{v})\,[\beta, \alpha]\,p_1(b, \bar{v}) = w(e, c, c)\,[\theta, \theta]\,e.$$

Since $[\beta, \alpha] = [\theta, \theta] = 0$, then $c = e$. But this contradicts to the second part of (1). So, the opposite case is impossible.

In the next proposition we consider some conditions which are equivalent to the centralizer relation.

Theorem 3.2. *Suppose that $\alpha, \beta, \gamma, \delta \in ConA$ and $A \in \mathbf{V}$ for a variety with a weak difference term. If $[\alpha, \alpha] \leq [\beta, \alpha]$ and $[\alpha, \beta] \nleq [\beta, \alpha]$ then the following conditions are equivalent.*

(i) $C(\alpha, \beta; \delta)$

(ii) $[\alpha, \beta]_\delta = \delta$

(iii) $[\alpha, \beta] \vee [\gamma, \gamma] \leq \delta$, where $\gamma := \beta \wedge [\alpha, \beta]_\delta$.

Proof. The implications $(i) \implies (ii) \implies (iii)$ follow immediately from definitions 2.1. and 2.2.

Assume (iii) and simultaneously that (i) fails. This means that there exist a polynomial $p(x,\bar{y}) \in Pol_{n+1}(A)$, and couples $(a,b) \in \alpha$ and $(u_i, v_i) \in \beta$, $0 \le i < n$, such that in the matrix

$$\begin{pmatrix} p(a,\bar{u}) & p(a,\bar{v}) \\ p(b,\bar{u}) & p(b,\bar{v}) \end{pmatrix} \in M(\alpha,\beta)$$

for the first row we have $p(a,\bar{u})\,\delta p(a,\bar{v})$, while for the second row

$$c := p(b,\bar{u})\,[\alpha,\beta]_\delta\, p(b,\bar{v}) =: e \text{ and } (c,e) \notin \delta. \tag{3}$$

Since each couple $(u_i, v_i) \in \beta$, $0 \le i < n$, then $(c,e) \in \beta$. Consequently $(c,e) \in \gamma = \beta \wedge [\alpha,\beta]_\delta$. Let $p_1(x,\bar{y})$ be defined as in the proof of Theorem 3.1. Then for the left column of the matrix

$$\begin{pmatrix} p_1(a,\bar{u}) & p_1(a,\bar{v}) \\ p_1(b,\bar{u}) & p_1(b,\bar{v}) \end{pmatrix} \in M(\alpha,\beta)$$

we have (as in (2)):

$$p_1(a,\bar{u})\,[\alpha,\alpha]\,p_1(b,\bar{u}) = p(b,\bar{u}).$$

Therefore for the right column of the same matrix we have (as $[\alpha,\alpha] \le [\beta,\alpha]$):

$$p_1(a,\bar{v})\,[\beta,\alpha]\,p_1(b,\bar{v})$$

On the other hand

$$p_1(a,\bar{v})\,\delta w\left(p(a,\bar{u}), p(a,\bar{u})\, p(b,\bar{u})\right)[\alpha,\alpha]\,p(b,\bar{n}) =: c$$

and

$$p_1(b,\bar{v}) := w\left(p(b,\bar{v}), p(b,\bar{u})\right), p(b,\bar{u}) = w(e,c,c)\,[\gamma,\gamma]\,e.$$

So we have: $c = p(b,\bar{u})\,(\alpha,\alpha)\circ\delta p_1(a,\bar{v})\,[\beta,\alpha]\,p_1(b,\bar{v})\,[\gamma,\gamma]\,e$. As $[\alpha,\alpha] \le [\beta,\alpha] =$ (by Theorem 3.1.) $= [\alpha,\beta] \le \delta$ and $[\gamma,\gamma] \le \delta$, then lastly $(c,e) \in \delta$. But this contradicts to the second condition of (3). So (i) is true.

From here we immediately obtain

Corollary 3.3. *Under the conditions of theorem 3.2, if $[\alpha,\beta] \le \delta$ and $[\beta,\beta] \le \delta$ then $C(\alpha,\beta;\delta)$.*

The following proposition may be proved by argument similar to that of Theorem 3.2.

Proposition 3.4. *If* $\mathbf{V}$ *has a weak difference term,* $A \in \mathbf{V}$, $\alpha,\beta,\gamma \in ConA$, $[\alpha,\alpha] \le [\beta,\alpha]$, $[\alpha,\beta] \not\le [\beta,\alpha]$ *and* $\gamma \le \alpha \wedge \beta$, *then*

$$[\alpha,\beta]_\gamma = [\alpha,\beta] \vee \gamma.$$

Note also that the next proposition was proved for varieties with a difference term in [5], but in fact the same arguments work for a weak difference term also.

Proposition 3.5 ([5], p. 943) *Assume that* $\mathbf{V}$ *has a weak difference term* $w(x,y,z)$, *that* $A \in \mathbf{V}$ *and* $\alpha \in ConA$. *Then* $[\alpha,\alpha] = 0$ *if and only if*

(i) $w(b,b,a) = w(a,b,b) = a$ *for all* $(a,b) \in \alpha$, *and* *(ii)* $w : A \times_\alpha A \times_\alpha A \to A$ *is a homomorphism.*

4^0. **Commutator identities.** In [5], it is found a commutator identity for any variety with a difference term:

$$\mathbf{V} \models_{con} \quad \alpha \circ \beta \subseteq [\alpha, \alpha] \circ \beta \circ \alpha$$

(moreover this identity is equivalent to the presence of a difference term for any variety). Our aim here is to present some commutator identities for varieties with a weak difference term. While the following theorem has a short proof, it has interesting consequences.

Theorem 4.1. *If* $\mathbf{V}$ *has a weak difference term* $w(x, y, z)$*, then*

$$\mathbf{V} \models_{con} \quad \alpha \circ \beta \circ \gamma \subseteq [\alpha, \alpha] \circ \beta \circ \gamma \circ \alpha \circ \beta \circ [\gamma, \gamma] \tag{4}$$

for any algebra $A \in \mathbf{V}$ *and any congruences* $\alpha, \beta, \gamma \in ConA$.

Proof. Let $(x, t) \in \alpha \circ \beta \circ \gamma$. Then there exist $y, z \in A$ such that $x\ \alpha\ y\ \beta\ z\ \gamma\ t$ holds. For the weak difference term w we have $x\,[\alpha, \alpha]\, w(x, y, y)$. Then

$$w(x, y, y)\, \beta w(x, z, z)\, \gamma w(x, z, t)\, \alpha w(y, z, t)\, \beta w(z, z, t)$$

and lastly $w(z, z, t)\,[\gamma, \gamma]\, t$. After all we have $(x, t) \in [\alpha, \alpha] \circ \beta \circ \gamma \circ \alpha \circ \beta \circ [\gamma, \gamma]$.

Corollary 4.2. (See also [8], p. 166) *If* $\mathbf{V}$ *has weak difference term then*

$$\mathbf{V} \models_{con} \quad \alpha \circ \beta \subseteq [\alpha, \alpha] \circ \beta \circ \alpha \circ [\beta, \beta].$$

Proof. In (4), take the middle congruence equals to the diagonal congruence 0. From here it follows that

$$\mathbf{V} \models_{con} \quad \alpha \circ [\beta]^k \subseteq [\alpha]^1 \circ [\beta]^k \circ \alpha \circ [\beta]^{k+1}$$

and

$$\mathbf{V} \models_{con} \quad [\alpha]^k \circ \beta \subseteq [\alpha]^{k+1} \circ \beta \circ [\alpha]^k \circ [\beta]^1.$$

Thus we have: every variety with a weak difference term congruence-satisfies the inclusions

$$\alpha \circ \beta \subseteq [\alpha]^1 \circ \beta \circ \left(\alpha \circ [\beta]^1\right) \subseteq [\alpha]^1 \circ \beta \circ [\alpha]^1 \circ [\beta]^1 \circ \alpha \circ [\beta]^2 \subseteq ...$$

$$\subseteq \left([\alpha]^1 \circ \beta\right) \circ \left([\alpha]^1 \circ [\beta]^1\right) \circ \left([\alpha]^1 \circ [\beta]^2\right) \circ ... \circ \left([\alpha]^1 \circ [\beta]^k\right) \circ \left(\alpha \circ [\beta]^{k+1}\right).$$

Of course, here the length is bounded for solvable varieties.

In another direction Theorem 4.1. implies following corollaries.

Corollary 4.3. *If* $\mathbf{V}$ *has a weak difference term then*

$$\mathbf{V} \models_{con} \quad \alpha \circ \beta \circ \alpha \subseteq [\alpha, \alpha] \circ \beta \circ \alpha \circ \beta \circ [\alpha, \alpha]$$

Proof. Indeed, take $\gamma = \alpha$ is (4).

Corollary 4.4. *If* $\mathbf{V}$ *has a weak difference term then*

$$\mathbf{V} \models_{con} \quad \alpha \circ \beta \circ [\alpha, \alpha] \subseteq [\alpha, \alpha] \circ \beta \circ \alpha \circ \beta \circ [\alpha]^2$$

and

$$\mathbf{V} \models_{con} \quad [\alpha,\alpha] \circ \beta \circ \alpha \subseteq [\alpha]^2 \circ \beta \circ \alpha \circ \beta \circ [\alpha,\alpha] \,.$$

Proof. For the first equation it is enough take $\gamma = [\alpha,\alpha]$ in (4):

$$\alpha \circ \beta \circ [\alpha,\alpha] \subseteq [\alpha,\alpha] \circ \beta \circ ([\alpha,\alpha] \circ \alpha) \circ \beta \circ [\alpha]^2 =$$

$$= [\alpha]^1 \circ \beta \circ \alpha \circ \beta \circ [\alpha]^2 \,.$$

For the second take $\gamma = \alpha$ and $[\alpha,\alpha]$ instead of α in (4):

$$[\alpha,\alpha] \circ \beta \circ \alpha \subseteq [\alpha]^2 \circ \beta \circ (\alpha \circ [\alpha,\alpha]) \circ \beta \circ [\alpha]^1 =$$

$$= [\alpha]^2 \circ \beta \circ \alpha \circ \beta \circ [\alpha]^1 \,.\square$$

Now combining corollaries 4.3 and 4.4 it is easy to see that

$$\alpha \circ \beta \circ \alpha \subseteq ([\alpha,\alpha] \circ \beta \circ \alpha) \circ (\alpha \circ \beta \circ [\alpha,\alpha]) \subseteq$$

$$\subseteq [\alpha]^2 \circ \beta \circ \alpha \circ \beta \circ \left([\alpha]^1 \circ [\alpha]^1\right) \circ \beta \circ \alpha \circ \beta \circ [\alpha]^2 =$$

$$= [\alpha]^2 \circ \beta \circ \alpha \circ \beta \circ [\alpha]^1 \circ \beta \circ \alpha \circ \beta \circ [\alpha]^2 \,.$$

Of course, the repeatition of Corollary 4.4 gives us the following equations:

$$\mathbf{V} \models_{con} \alpha \circ \beta \circ [\alpha]^k \subseteq [\alpha,\alpha] \circ \beta \circ \alpha \circ \beta \circ [\alpha]^{k+1}$$

and

$$\mathbf{V} \models_{con} [\alpha]^k \circ \beta \circ \alpha \subseteq [\alpha]^{k+1} \circ \beta \circ \alpha \circ \beta \circ [\alpha,\alpha] \,.$$

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