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ON A NEW METHOD OF SOLUTION TO VOLTERRA INTEGRAL EQUATION

Abstract

By solving many applied problems we collide with the solution of Volterra type integral equations. Volterra integral equations are investigated long ago, however up to now there hasn't been constructed an effective method to find numerical solution of nonlinear Volterra integral equation. Therefore, different methods are suggested for approximate solution of nonlinear Volterrs type integral equation. One of the popular methods of numerical solution of such equations is replacement of an integral by a quadrature formula. To improve such methods some authors suggested to use quadrature formulae with regard to Runge-Kutta or Adams methods. Unlike these methods here we suggest to use multistep methods for finding numerical solution of nonlinear Volterra type integral equations and give a method to determine the coefficients of the suggested method.

Introduction: Let's introduce the following Volterra integral equation

$$y(x) = f(x) + \int\limits_{x_0}^{x} K(x, s, y(s))\, ds, \qquad x \in [x_0, X]. \tag{1}$$

It is assumed that the function $K(x, s, y)$ continuous in totality of variables has continuous partial derivatives to some order p , inclusively. The derivatives of $p+1$-th order are bounded.

By means of a constant step $h > 0$ we divide the segment $[x_0, X]$ into N equal parts. The partition poinds are taken in the from: $x_i = x_0 + ih$ $(i = 0, 1, 2, ..., N)$ and equation (1) on the segment $[x_0, x_{n+k}]$ is written as follows:

$$y(x) = f(x) + \int\limits_{x_0}^{x_{n+k-1}} K(x, s, y(s))\, ds+$$

$$+ \int\limits_{x_{n+k-1}}^{x} K(x, s, y(s))\, ds, \quad x \in [x_{n+k-1}, x_{n+k}]. \tag{2}$$

Denote

$$\varphi_{n+k-1}(x) = \int\limits_{x_0}^{x_{n+k-1}} K(x, s, y(s))\, ds.$$

Then

$$y\left(x_{n+k}\right) = f(x_{n+k}) + \varphi_{n+k-1}(x_{n+k}) + \int\limits_{x_{n+k-1}}^{x_{n+k}} K(x_{n+k}, s, y(s))\, ds. \tag{3}$$

Depending on the values k in order to calculate the integral in relation (3) we can apply Runge-Kutta or Adams method (for $k = 1$ Runge-Kutta methods, for

$k > 1$ Adams method or their generalizations are used) (see [1]). Obviously, if $\varphi_{n+k-1}(x_{n+k})$ is known, we can determine the value of variable $y(x_{n+k})$. Usually, in all the well known papers in order to determine $\varphi_{n+k-1}(x_{n+k})$ quadrature formulae (see $[1-18]$) are used. It follows from (3) that exactness of quadrature formulae should correspond to exactness of the methods applied to calculation of the integral participating in relation (3), i.e. Runge-Kutta methods or Adams methods. As is known, by increasing the values of n by a unit there arise some difficulties in choosing quadrature formulae coefficiends (see e.g [1]). By refinement the step h, the amount of calculated terms in quadrature formulae very increases and this creates some difficulties in using the indicated methods. In order to be released from these indicated deficiencies here we suggest a new method by using of which the amount of calculated quantities in defining $y(x_{n+k})$ doesn't depend on the values of n. This method is said to be a multistep method with constant coefficients and when applied to equation (1) has the following from:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = \sum_{i=0}^{k}\alpha_i f_{n+i} + h\sum_{j=0}^{k}\sum_{i=0}^{k}\beta_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}), \tag{4}$$

where $\alpha_i, \beta_i^{(j)}$ are some real numbers, k is integer-valued quantity, h is a step of partitioning the segment $[x_0, X]$ into N equal parts, y_m is an approximate value of the solution of Volterra equations at the points $x_m = x_0 + mh$, and $f_m = f(x_m)$ $(m = 0, 1, 2, ...)$.

Notice that, in order to solve equation (1) some authors suggest to use iterative methods (see e.g $[1], [5], [6]$). In some cases there arises a problem on determination of stability domain that was investigated in $[12]$.

§1. Construction of a multistep method with constant coefficients.

Here we suggest several methods to construct a multistep method with constant coefficients. One of them is of the form. Let's consider calculation of quantities $\varphi_{n+k-1}(x_{n+k})$. At first we use expansion $K(x_{n+k}, s, y(s))$ by the first argument around the point x_{n+k-1}. In this case we have :

$$\varphi_{n+k-1}(x_{n+k}) = \int_{x_0}^{x_{n+k-1}} K(x_{n+k}, s, y(s))\,ds = \int_{x_0}^{x_{n+k-1}} \Big(K(x_{n+k-1}, s, y(s)) +$$

$$+hK'_x(x_{n+k-1}, s, y(s)) + \frac{h^2}{2!}K''_{x^2}(x_{n+k-1}, s, y(s)) + ...\Big)\,ds. \tag{1.1}$$

If in relation (1.1) we replace the derivatives by their appropriate difference relations and considering

$$\int_{x_0}^{x_{n+i}} K(x_{n+i}, s, y(s))\,ds = y(x_{n+i}) - f(x_{n+i})$$

we can write

$$\varphi_{n+k-1}(x_{n+k}) = \sum_{i=0}^{m} l_i y(x_{n+i}) + h\sum_{j=0}^{m}\sum_{i=0}^{k}\beta_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}) +$$

$$+\sum_{i=0}^{m} l_i f(x_{n+i}) + R_n. \qquad (1.2)$$

Taking into account (1.2) in (3) and using the Adams method for calculating the integral participating in (3) we get method (4) . Thus, we constructed several methods. However they are turned to be unstable. One of them is of the form:

$$y_{n+2} = 2y_{n+1} - y_n + f_{n+2} - 2f_{n+1} + f_n - h\left(K\left(x_n, x_{n+1}, y_{n+1}\right) +\right.$$

$$\left.+K\left(x_n, x_n, y_n\right)\right)/2 + h\left(3K\left(x_{n+2}, x_{n+1}, y_{n+1}\right) - K\left(x_{n+2}, x_n, y_n\right)\right)/2. \qquad (1.3)$$

The method is of the second order of exactness and is explicit.

Determination 1. Method (4) is said to be stable, if the roots of its characteristic polynomial

$$\rho(\lambda) = \sum_{i=0}^{k} \alpha_i \lambda^i$$

lie interior to a unit circle whose boundaries have no multiple roots.

As it follows from definition, stability of method (4) is determined exactly in the same way as the stability of k -step method with constant coefficients applied to numerical solution of ordinary differential equations.

In order to construct stable methods of quantities $K(x_{n+k}, s, y(s))$ we make substitution in the form:

$$\int_{x_0}^{x_{n+k-1}} K(x_{n+k}, s, y(s))\, ds = \sum_{i=0}^{k-1} \alpha_i y_{n+i} + \sum_{i=0}^{k-1} \alpha_i f_{n+i} +$$

$$+\sum_{j=0}^{k-1} \beta_j \int_{x_{n+j}}^{x_{n+k-1}} K(x_{n+j}, s, y(s))\, ds + R_n. \qquad (1.4)$$

Taking into account (1.4) in (3) and applying Adams method for calculating integrals we get method (4) . Here, in order to find the coefficients $\alpha_i, \beta_i^{(j)}$ $(i, j = 0, 1, ..., k+4)$ we use the method of undetermined coefficients. At first we determine the exactuess of method (4).

Determination 2. Integer-valued p is said to be a order of method (4) if it holds:

$$\sum_{i=0}^{k} \left(\alpha_i \left(y(x+ih) - f(x+ih) -\right.\right.$$

$$\left.\left.-h\sum_{j=0}^{k} \beta_i^{(j)} K(x+jh, x+ih, y(x+ih))\right)\right) = O\left(h^{p+1}\right),\ h \to 0. \qquad (1.5)$$

In some cases it is convenient to determine the quantity $\beta_i^{(j)}$ in the form:

$$\beta_i^{(j)} = \alpha_i \gamma_i^{(j)} \quad (j, i = 0, 1, 2, ..., k).$$

We look for the solution of equation (1) in a class of continuous functions having continuous partial derivatives up to some $p+1$. Naturally, $y(x) = \exp(x)$ is contained in the indicated class of functions. If method (4) has a definite order it should remain this power for the following equation

$$y(x) = 1 + \int_0^x y(s)\,ds \tag{1.6}$$

whose exact solution is an exponential function, i.e. $y(x) = \exp(x)$.

Let's consider a special case and apply method (4) to the solution of equation (1.6). Then we have:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{i=0}^{k} \alpha_i + h \sum_{j=0}^{k} \sum_{i=0}^{k} \beta_i^{(j)} y_{n+i}. \tag{1.7}$$

Using the expansion

$$y(x+ih) = y(x) + ihy'(x) + \frac{(ih)^2}{2} y''(x) + \frac{(ih)^p}{P!} y^{(p)}(x) + O\left(h^{p+1}\right)$$

in relation (1.7) we have

$$\sum_{i=0}^{k} \alpha_i \left(y_n + ihy'_n + \frac{(ih)^2}{2!} y''_n + \ldots + \frac{(ih)^p}{P!} y^{(p)}(x) + O\left(h^{p+1}\right) - 1 \right) =$$

$$= \sum_{j=0}^{k} \sum_{i=0}^{k} \beta_i^{(j)} \left(hy_n + ih^2 y'_n + \frac{i^2 h^3}{2!} y''_n + \ldots + \frac{i^{p-1} h^p}{(i-1)!} y_n^{(p-1)} + O\left(h^{p+1}\right) \right)$$

hence we obtain that in order (1.5) hold, it is necessary and sufficient that the coefficients $\alpha_i, \beta_i^{(j)}$ $(i, j = 0, 1, ..., k)$ satisfy the following system of algebraic equations:

$$\begin{gathered} \sum_{i=0}^{k} \alpha_i = 0, \quad \sum_{i=0}^{k} i\alpha_i = \sum_{j=0}^{k} \sum_{i=0}^{k} \beta_i^{(j)}, \\ \sum_{i=0}^{k} \frac{i^l}{l!} \alpha_i = \sum_{j=0}^{k} \sum_{i=0}^{k} \frac{i^{l-1}}{(l-1)!} \beta_i^{(j)}, \quad (l = 2, 3, ..., p). \end{gathered} \tag{1.8}$$

The obtained relation is a homogeneous system of linear-algebraic equations in which the amount of equations equals $p+1$, and the amount of the unknowns equals $(k+2)(k+1)$. If $p < (k+2)(k+1)$, system (1.8) has a solution differ from zero.

To construct stable methods let's consider the case $k = 2$. Then from (1.8) we have:

$$\begin{gathered} \alpha_0 + \alpha_1 + \alpha_2 = 0, \\ \alpha_1 + 2\alpha_2 = \sum_{j=0}^{2} \sum_{i=0}^{2} \beta_i^{(j)}, \\ 1/2\alpha_1 + 2\alpha_2 = \beta_1^{(0)} + \beta_1^{(1)} + \beta_1^{(2)} + 2\beta_2^{(0)} + 2\beta_2^{(1)} + 2\beta_2^{(2)} \\ 1/6\alpha_1 + 4/3\alpha_2 = \left(\beta_1^{(0)} + \beta_1^{(1)} + \beta_1^{(2)}\right)/2 + 2\left(\beta_2^{(0)} + \beta_2^{(1)} + \beta_2^{(2)}\right) \\ 1/24\alpha_1 + 2/3\alpha_2 = \left(\beta_1^{(0)} + \beta_1^{(1)} + \beta_1^{(2)}\right)/6 + 4\left(\beta_2^{(0)} + \beta_2^{(1)} + \beta_2^{(2)}\right)/3. \end{gathered} \tag{1.9}$$

If in this system we put $\alpha_2 = 1, \alpha_1 = \alpha_0 = -1/2$, the amount of the unknowns will equal 9, but if we use the substitution,

$$a = \beta_1^{(0)} + \beta_1^{(1)} + \beta_1^{(2)}, \quad b = \beta_2^{(0)} + \beta_2^{(1)} + \beta_2^{(2)}, \quad c = \beta_0^{(0)} + \beta_0^{(1)} + \beta_0^{(2)},$$

the amount of the unknowns will equal 3. If in this case system (1.9) has a solution differ from zero, then order of the method $p = 4$. At first we construct a method with order $p = 3$. We diminish the amount of equations by a unit. In this case one of stable methods with order $p = 3$ will be of the form:

$$y_{n+2} = (y_{n+1} + y_n)/2 + f_{n+2} - (f_{n+1} + f_n)/2 + h\,(-2K(x_n, x_{n+2}, y_{n+2}) -$$
$$-2K(x_n, x_n, y_n) + 3K(x_{n+1}, x_{n+2}, y_{n+2}) + 4K(x_{n+1}, x_{n+1}, y_{n+1}) +$$
$$+K(x_{n+1}, x_n, y_n) + 2K(x_{n+2}, x_{n+2}, y_{n+2}) +$$
$$+4K(x_{n+2}, x_{n+1}, y_{n+1}) + 2K(x_{n+2}, x_n, y_n))/8. \qquad (1.10)$$

Let's consider the solution of system (1.9) for $\alpha_2 = 1$. In this case the amount of the unknowns and equations coincide and equal 5 and system (1.9) has a unique solution of the form:

$$\alpha_2 = 1, \alpha_1 = 0, \alpha_0 = -1, a = 4/3, b = 1/3, c = 1/3.$$

Notice that the method having power $p = 4$ is not unique, since we can change the value of the coefficients $\beta_i^{(j)} \quad (i = 0, 1, 2; \quad j = 0, 1, 2)$ is some ranges. One of these methods is of the form:

$$y_{n+2} = y_n + f_{n+2} - f_n + h\,(-K(x_n, x_n, y_n) -$$
$$-K(x_n, x_{n+1}, y_{n+1}) - K(x_n, x_{n+2}, y_{n+2}) + K(x_{n+1}, x_n, y_n) +$$
$$+4K(x_{n+1}, x_{n+1}, y_{n+1}) + K(x_{n+1}, x_{n+2}, y_{n+2}) + K(x_{n+2}, x_n, y_n) +$$
$$+K(x_{n+2}, x_{n+1}, y_{n+1}) + K(x_{n+2}, x_{n+2}, y_{n+2}))/3. \qquad (1.11)$$

Methods (1.10) and (1.11) are implicit. Therefore we construct explicit stable method.

In this case some coefficients equal zero, and namely $\beta_{j,2} = 0 \quad (\ j = 0, 1, 2)$. Hence it follows $b = 0$. It is easy to show that on this case there are not stable methods with order $p > 2$. Hence it follows that we can construct explicit stable methods with order $p = 2$. One of these methods is of the form:

$$y_{n+2} = (y_{n+1} + y_n)/2 + f_{n+2} - (f_{n+1} + f_n)/2 + h\,(K(x_n, x_{n+1}, y_{n+1}) -$$
$$-3K(x_n, x_n, y_n) + 3K(x_{n+1}, x_{n+1}, y_{n+1}) + K(x_{n+1}, x_n, y_n) + \qquad (1.12)$$
$$+3K(x_{n+2}, x_{n+1}, y_{n+1}) + K(x_{n+2}, x_n, y_n))/4.$$

Now, in order to construct a method of type (4) we use another method. In equation (1) we put $x = x_{n+k}$. Then we have

$$y(x_{n+k}) = f(x_{n+k}) + \int_{x_0}^{x_{n+k}} K(x_{n+k}, s, y(s))\,ds. \qquad (1.13)$$

Using Lagrange and Newton interpolation polynomial we represent $K(x_{n+k}, s, y(s))$ in the form:

$$K(x_{n+k}, s, y(s)) \approx -\sum_{i=0}^{k-1} \alpha_i K(x_{n+i}, s, y(s)).$$

Taking the latter into account in (1.12) we have:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{i=0}^{k} \alpha_i f_{n+i} - \sum_{j=0}^{k-1} \alpha_j \int_{x_{n+j}}^{x_{n+k}} K(x_{n+j}, s, y(s))\, ds.$$

Replacing the integral by the quadrature formula

$$\int_{x_{n+j}}^{x_{n+k}} K(x_{n+j}, s, y(s))\, ds \approx h \sum_{i=0}^{k} \gamma_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i})$$

and taking into account in (1.13) we get the following multi-step method

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{i=0}^{k} \alpha_i f_{n+i} - h \sum_{j=0}^{k-1} \alpha_j \sum_{j=0}^{k} \gamma_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}). \qquad (1.14)$$

It in method (4) we replace

$$\beta_i^{(j)} = \alpha_j \gamma_i^{(j)} \qquad (j = 0, 1, ..., k-1; \quad i = 0, 1, ..., k)$$

from it we get method (1.14) For $k = 2$ one of stable methods with power $p = 3$ will be of the form:

$$\begin{gathered} y_{n+2} = (y_{n+1} + y_n)/2 + f_{n+2} - (f_{n+1} + f_n)/2 + h(K(x_n, x_n, y_n) + \\ +8K(x_n, x_{n+1}, y_{n+1}) + 3K(x_n, x_{n+2}, y_{n+2}) + K(x_{n+1}, x_n, y_n) + \\ +8K(x_{n+1}, x_{n+1}, y_{n+1}) + 3K(x_{n+1}, x_{n+2}, y_{n+2}))/16. \end{gathered} \qquad (1.15)$$

§2. Construction of some specific multistep methods.

In the first section we suggested the methods for constructing multistep method with constant coefficients of type (4). Here, using the indicated methods we construct specific methods and give their comparison with the known methods.

Notice that all the methods were constructed for $k = 2$ and $\alpha_2 \neq 0$.

Let's consider the first method and in the equation (1) put $x = x_n$ and write the obtained relation in the form:

$$y(x_n) = f(x_n) + \int_{x_0}^{x_{n-1}} K(x_n, s, y(s))\, ds + \int_{x_{n-1}}^{x_0} K(x_n, s, y(s))\, ds. \qquad (2.1)$$

Calculate the first integral by the following scheme:

$$\int_{x_0}^{x_{n-1}} K(x_n, s, y(s))\, ds + \int_{x_0}^{x_{n-1}} K(x_{n-1}, s, y(s))\, ds +$$

$$+h\int\limits_{x_0}^{x_{n-1}}\frac{K\left(x_{n-1},s,y\left(s\right)\right)ds-K\left(x_{n-2},s,y\left(s\right)\right)}{h}ds+O\left(h^2\right). \qquad (2.2)$$

Hence

$$\int\limits_{x_0}^{x_{n-1}}K\left(x_n,s,y\left(s\right)\right)ds=2y\left(x_{n-1}\right)-2f\left(x_{n-1}\right)+f\left(x_{n-2}\right)-$$

$$-\int\limits_{x_{n-2}}^{x_{n-1}}K\left(x_{n-2},s,y\left(s\right)\right)ds+O\left(h^2\right).$$

To calculate the integral participating in (2.2) we apply the trapezoid method, for calculating the second integral in (2.1) we use the following method:

$$\int\limits_{x_{n-1}}^{x_n}K\left(x_n,s,y\left(s\right)\right)ds=\frac{3h}{2}K\left(x_n,x_{n-1},y\left(x_{n-1}\right)\right)-\frac{h}{2}K\left(x_n,x_{n-2},y\left(x_{n-2}\right)\right)$$

then we get the following method:

$$y_n-2y_{n-1}+y_{n-2}=f_n-2f_{n-1}+f_{n-2}-h\left(K\left(x_{n-2},x_{n-1},y_{n-1}\right)+\right.$$

$$\left.+K\left(x_{n-2},x_{n-2},y_{n-2}\right)\right)/2+h\left(3K\left(x_n,x_{n-1},y_{n-1}\right)-\right.$$

$$\left.-K\left(x_n,x_{n-2},y_{n-2}\right)\right)/2. \qquad (2.3)$$

This explicit method is unstable, since the $\alpha=1$ roots of the characteristic equation are multiple.

Now, using the second method, construct a stable method of type (4). It is easy to show that in this case a stable method with power $p>2k$ doesn't exist and maximal order of the explicid method equals $2\,(p=2)$. Such methods are more than one. One of these methods is of the form:

$$y_{n+2}=\left(y_{n+1}+y_n\right)/2+f_{n+2}-\left(f_{n+1}+f_n\right)/2+h\left(-K\left(x_n,x_n,y_n\right)+\right.$$

$$\left.+7K\left(x_n,x_{n+1},y_{n+1}\right)-K\left(x_{n+1},x_n,y_n\right)+7K\left(x_{n-1},x_{n-1},y_{n-1}\right)\right)/8. \qquad (2.4)$$

Notice that stable methods with power $p=4$ are also not unique. One of stable implicit methods with order $p=4$ is of the form:

$$y_{n+2}=y_n+f_{n+2}-f_n+h\left(K\left(x_n,x_n,y_n\right)+\right.$$

$$\left.+4K\left(x_n,x_{n+1},y_{n+1}\right)+K\left(x_n,x_{n+2},y_{n+2}\right)\right)/3. \qquad (2.5)$$

In comparison with method (1.11) this method is simple and may be easily applied to specific problems.

Thus, we constructed seven methods of different exactness power that were applied to numerical solution of some Volterra integral equations. In many cases the results are better than the ones of the quadrature methods and Runge-Kutta and Adams methods as well.

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