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NON-AXIALLY-SYMMETRIC PROBLEMS OF ELASTICITY THEORY FOR TRANSVERSALLY ISOTROPIC HOLLOW SPHERE

Abstract

The non-axially-symmetrical problem of elasticity theory for transversally isotropic hollow sphere is considered. Due to spherical symmetry the general boundary value problem is broken up into two problems, one of which exactly coincides with axially symmetric boundary value problem of hollow sphere, the second one with boundary value problem of pure torsion of hollow sphere.

Let's consider the transversally isotropic spherical layer in spherical coordinate system

$$r_1 \le r \le r_2, \quad \theta_1(\varphi) \le \theta \le \theta_2(\varphi), \quad 0 \le \varphi \le 2\pi.$$

The shell is made of transversally isotropic material.

The spherical parts of layer's boundary we call the face, and the remaining boundary part we call the lateral surface.

Let's give here the complete system of equations describing the spacial stress-strain state of spherical layer. The equilibrium equations in stresses in the absence of body forces, in spherical coordinate system, have the form:

$$\begin{array}{l}
\dfrac{\partial \sigma_r}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{r\theta}}{\partial \theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial \tau_{r\varphi}}{\partial \varphi} + \dfrac{2\sigma_r - \sigma_\varphi - \sigma_\theta + \tau_{r\theta}\cot\theta}{r} = 0 \\
\dfrac{\partial \tau_{r\varphi}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta\varphi}}{\partial \theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial \sigma_\varphi}{\partial \varphi} + \dfrac{3\tau_{r\varphi} + 2\tau_{\theta\varphi}\cot\theta}{r} = 0 \\
\dfrac{\partial \tau_{r\theta}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_\theta}{\partial \theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial \tau_{\theta\varphi}}{\partial \varphi} + \dfrac{(\sigma_\theta - \sigma_\varphi)\cot\theta + 3\tau_{r\theta}}{r} = 0,
\end{array} \tag{1.1}$$

where

$$\begin{array}{l}
\sigma_r = A_{11}\varepsilon_r + A_{12}(\varepsilon_\theta + \varepsilon_\varphi), \quad \sigma_\theta = A_{12}\varepsilon_r + A_{22}\varepsilon_\theta + A_{23}\varepsilon_\varphi \\
\sigma_\varphi = A_{12}\varepsilon_r + A_{23}\varepsilon_\theta + A_{22}\varepsilon_\varphi, \quad \tau_{r\varphi} = G_1\varepsilon_{r\theta} \\
\tau_{r\varphi} = G_1\varepsilon_{r\varphi}, \quad \tau_{r\theta} = G\varepsilon_{\theta\varphi}
\end{array} \tag{1.2}$$

$$\begin{array}{l}
\varepsilon_r = \dfrac{\partial u_r}{\partial r}, \quad \varepsilon_\theta = \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}, \quad \varepsilon_\varphi = \dfrac{u_r}{r} + \dfrac{u_\theta}{r}\cot\theta + \dfrac{1}{r\sin\theta}\dfrac{\partial u_\varphi}{\partial \varphi} \\
\varepsilon_{r\theta} = \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} - \dfrac{u_\theta}{r} + \dfrac{\partial u_\theta}{\partial r}, \quad \varepsilon_{r\varphi} = \dfrac{\partial u_\varphi}{\partial r} - \dfrac{u_\varphi}{r} + \dfrac{1}{r\sin\theta}\dfrac{\partial u_r}{\partial \varphi} \\
\varepsilon_{\theta\varphi} = \dfrac{1}{r}\dfrac{\partial u_\varphi}{\partial \theta} - \dfrac{u_\varphi}{r}\cot\theta + \dfrac{1}{r\sin\theta}\dfrac{\partial u_\theta}{\partial \varphi}
\end{array} \tag{1.3}$$

A_{ij}, G, G_1 are material constants, u_r, u_θ, u_φ are components of displacement vector.

Substituting (1.3), (1.2) in (1.1), after simple computations we get:

$$b_{11}\frac{\partial^2 u_r}{\partial r^2} + \frac{2b_{11}}{r}\frac{\partial u_r}{\partial r} + \frac{2}{r}(b_{11} - b_{22} - b_{23})u_r +$$

$$+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial \varphi^2} +$$

$$+\left[\frac{b_{12}+1}{r}\frac{\partial}{\partial r} + \frac{b_{12} - b_{22} - b_{23} - 1}{r^2}\right]\left(\frac{\partial u_\theta}{\partial \theta} + \cot\theta \cdot u_\theta\right) +$$

$$+\frac{1}{r\sin\theta}\left[(b_{12}+1)\frac{\partial}{\partial r}+\frac{b_{11}-b_{22}-b_{23}-1}{r}\right]\frac{\partial u_\varphi}{\partial\varphi}=0$$

$$\frac{b_{12}+1}{r\sin\theta}\frac{\partial^2 u_r}{\partial r\partial\varphi}+\frac{b_{22}+b_{23}+2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi}+\frac{b_{23}+G_0}{r^2\sin\theta}\frac{\partial^2 u_\theta}{\partial\theta\partial\varphi}+$$

$$+\frac{b_{22}+G_0}{r^2\sin\theta}\cot\theta\frac{\partial u_\theta}{\partial\varphi}+\frac{2}{r}\frac{\partial u_\varphi}{\partial r}+\frac{G_0}{r^2}\left(\frac{\partial^2 u_\varphi}{\partial\theta^2}+\cot\theta\frac{\partial u_\varphi}{\partial\theta}-\frac{u_\varphi}{\sin^2\theta}\right)+$$

$$+\frac{\partial^2 u_\varphi}{\partial r^2}+\frac{b_{22}}{r^2\sin^2\theta}\frac{\partial^2 u_\varphi}{\partial\varphi^2}+\frac{2(G_0-1)}{r^2}u_\varphi=0 \quad (1.4)$$

$$\frac{b_{12}+1}{r}\frac{\partial^2 u_r}{\partial r\partial\theta}+\frac{b_{22}+b_{23}+2}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{\partial^2 u_\theta}{\partial r^2}+\frac{2}{r}\frac{\partial u_\theta}{\partial r}+$$

$$+\frac{b_{22}}{r^2}\left(\frac{\partial^2 u_\theta}{\partial\theta^2}+\cot\theta\frac{\partial u_\theta}{\partial\theta}-\frac{u_\theta}{\sin^2\theta}\right)+\frac{2(G_0-1)}{r^2}u_\theta+$$

$$+\frac{G_0}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial\varphi^2}+\frac{b_{23}+G_0}{r^2\sin\theta}\frac{\partial^2 u_r}{\partial\theta\partial\varphi}-\frac{b_{22}+G_0}{r^2\sin\theta}\cot\theta\frac{\partial u_\varphi}{\partial\varphi}=0$$

$$b_{ij}=\frac{A_{ij}}{G_1},\qquad G_0=\frac{G}{G_1}.$$

Let's suppose, that from the face side on the layer there acts the load

$$\sigma_r=q_r^{(k)}(\theta,\varphi),\quad \tau_{r\theta}=q_r^{(k)}(\theta,\varphi),\quad \tau_{r\varphi}=q_r^{(k)}(\theta,\varphi)\quad \text{at}\quad r=r_k\,(k=1,2). \quad (1.5)$$

We'll not revise now the boundary conditions character on lateral face, however, we'll consider them so that the layer is in equilibrium.

Following [1,2,3], let's break up the two-dimensional vector field $\overline{v}=(u_\theta,u_\varphi)$ in potential and vortex part. Supposing

$$u_\theta=r\frac{\partial F}{\partial\theta}+\frac{r}{\sin\theta}\frac{\partial\psi}{\partial\varphi},\quad u_\varphi=\frac{r}{\sin\theta}\frac{\partial F}{\partial\varphi}-r\frac{\partial\psi}{\partial\theta}. \quad (1.6)$$

Substituting (1.6) in equation (1.4) and boundary condition (1.5), respectively, we obtain

$$L_1(u_r,F)=0 \quad (1.7)$$

$$\frac{\partial}{\partial\theta}L_2(u_r,F)+\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}L_3(\psi)=0 \quad (1.8)$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}L_2(u_r,F)+\frac{\partial}{\partial\theta}L_3(\psi)=0 \quad (1.9)$$

$$M_1(u_r,F)|_{r=r_k}=q_r^{(k)}(\theta,\varphi) \quad (1.10)$$

$$\left[\frac{\partial}{\partial\theta}M_2(u_r,F)+\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}M_3(\psi)\right]_{r=r_k}=q_{r\theta}^{(k)}(\theta,\varphi) \quad (1.11)$$

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}M_2(u_r,F)-\frac{\partial}{\partial\varphi}M_3(\psi)\right]_{r=r_k}=q_{r\varphi}^{(k)}(\theta,\varphi), \quad (1.12)$$

where

$$L_1(u,F)=b_{11}\frac{\partial^2 u_r}{\partial r^2}+\frac{2b_{11}}{r}\frac{\partial u_r}{\partial r}+\frac{2}{r^2}(b_{12}-b_{22}-b_{23})u_r+$$

$$+\frac{1}{r^2}\Delta_0 u_r + \left[(b_{12}+1)\frac{\partial}{\partial r} + \frac{2b_{12}-b_{22}-b_{23}}{r}\right]\Delta_0 F$$

$$L_2(u_r,F) = \frac{b_{12}+1}{r}\frac{\partial u_r}{\partial r} + \frac{b_{22}+b_{23}+2}{r^2}u_r + r\frac{\partial^2 F}{\partial r^2} + 4\frac{\partial F}{\partial r} + + \frac{2G_0}{r}F + \frac{b_{22}}{r}\Delta_0 F$$

$$L_3(\psi) = r\frac{\partial^2\psi}{\partial r^2} + 4\frac{\partial\psi}{\partial r} + \frac{2G_0}{r}\psi + \frac{G_0}{r}\Delta_0\psi$$

$$M_1(u_r,F) = b_{11}\frac{\partial u_r}{\partial r} + \frac{2b_{11}}{r}u_r + b_{12}\Delta_0 F$$

$$M_2(u_r,F) = \frac{u_r}{r} + r\frac{\partial F}{\partial r}$$

$$M_3(\psi) = r\frac{\partial\psi}{\partial r},\quad \Delta_0 = \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

Relations (1.8), (1.9) are identically satisfied, if we put

$$L_2(u_r,F) = -\frac{\partial\chi(r,\theta,\varphi)}{\partial\varphi},\quad L_3(F) = \sin\theta\frac{\partial\chi(r,\theta,\varphi)}{\partial\theta}, \tag{1.13}$$

where the function $\chi(r,\theta,\varphi)$ satisfies the equation

$$\Delta_0\chi(r,\theta,\varphi) = 0. \tag{1.14}$$

Now $\left[q_{r\theta}^{(k)}, q_{r\varphi}^{(k)}\right]$ is represented in the form

$$q_{r\theta}^{(k)} = \frac{\partial q_2^{(k)}}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial q_3^{(k)}}{\partial\varphi},\quad q_{r\varphi}^{(k)} = \frac{1}{\sin\theta}\frac{\partial q_2^{(k)}}{\partial\varphi} - \frac{\partial q_3^{(k)}}{\partial\theta}. \tag{1.15}$$

Then initial boundary value problem (1.4), (1.5) is decomposed into two problems

$$L_1(u_r,F) = 0,\quad L_2(u_r,F) = -\frac{\partial\chi}{\partial\varphi} \tag{1.16}$$

$$[M_1(u_r,F)]_{r=r_k} = q_r^{(k)},\quad [M_2(u_r,F)]_{r=r_k} = q_2^{(s)} - \frac{\partial e^{(k)}}{\partial\varphi} \tag{1.17}$$

$$L_3(F) = \sin\theta\frac{\partial\chi}{\partial\theta} \tag{1.18}$$

$$[M_3(u_r,F)]_{r=r_k} = q_3^{(k)} + \sin\theta\frac{\partial e^{(k)}}{\partial\theta}, \tag{1.19}$$

where $e^{(k)}(\theta,\varphi)$ are arbitrary functions satisfying the equation $\Delta_0 e^{(k)}(\theta,\varphi) = 0$.

Validity of representation (1.6) is explicitly discussed in the paper [3] and there is no need to discuss it here. In the same place it is shown, that not loosing generality, we can always put $\chi = 0$, $e^{(s)} = 0$.

2. The nonhomogeneous solutions we call particular solutions of equilibrium equation (1.4) satisfying boundary conditions (1.5) on the face layer. For construction of nonhomogeneous solutions we can use various methods. One of the known methods is the following: the domain $V = [r_1, r_2] \cdot [\theta_1(\varphi), \theta_2(\varphi)] \cdot [0, 2\pi]$ continues up to closed spherical layer $V_0 = [r_1, r_2] \cdot [0, \pi] \cdot [0, 2\pi]$, and the load $q^{(k)} = \left\{ q_r^{(k)}, q_{r\theta}^{(k)}, q_{r\varphi}^{(k)} \right\}$ given on the fase $S^{(k)}$, in a sufficiently arbitrary way, continues on the closed spherical surfaces $S_0^{(k)}$ $(r = r_k)$. We denote by $P^{(k)} = \left\{ P_r^{(k)}, P_{r\theta}^{(k)}, P_{r\varphi}^{(k)} \right\}$ the external forces given on $S_0^{(k)}$. Here for $(\theta, \varphi) \in S^{(k)}$, $P_r^{(k)} = q_r^{(k)}$, $P_{r\theta}^{(k)} = q_{r\theta}^{(k)}$, $P_{r\varphi}^{(k)} = q_{r\varphi}^{(k)}$ and moreover, it is necessary that the external forces $P_r^{(k)}$, $P_{r\theta}^{(k)}$, $P_{r\varphi}^{(k)}$ satisfy equilibrium conditions. Let's express the two-dimensional field $\left\{ P_{r\theta}^{(k)}, P_{r\varphi}^{(k)} \right\}$ of external stress in the form of

$$P_{r\theta}^{(k)} = \frac{\partial P_2^{(k)}}{\partial \theta} + \frac{1}{\sin\theta} \frac{\partial P_3^{(k)}}{\partial \varphi}, \qquad P_{r\varphi}^{(k)} = \frac{1}{\sin\theta} \frac{\partial P_2^{(k)}}{\partial \varphi} - \frac{\partial P_3^{(k)}}{\partial \theta}.$$

Let's expand the functions $P_i^{(k)}(\theta, \varphi)$ $(i = 1, 2, 3)$, $\left[P_1^{(k)} = P_r^{(k)} \right]$ in series on spherical functions in the following form

$$P_i^{(k)}(\theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_{inm}^{(k)} Y_{n(\theta,\varphi)}^{(m)},$$

here $P_{inm}^{(k)}$ are the known constants

$$P_{inm}^{(k)} = \frac{1}{\left\| Y_n^{(m)} \right\|^2} \int_0^{2\pi} \int_0^{\pi} P_i^{(k)}(\theta, \varphi) P_n^m(\cos\theta) \cos m\varphi \sin\theta d\theta d\varphi, \quad m \le 0$$

$$P_{inm}^{(k)} = \frac{1}{\left\| Y_n^{(m)} \right\|^2} \int_0^{2\pi} \int_0^{\pi} P_i^{(k)}(\theta, \varphi) P_n^m(\cos\theta) \sin m\varphi \sin\theta d\theta d\varphi, \quad m > 0$$

$$\left\| Y_n^{(m)} \right\|^2 = \frac{2\pi\varepsilon_m}{(n-m)!} \frac{(n+m)!}{2n+1}; \quad \varepsilon_m = \begin{cases} 2 & \text{at } m = 0 \\ 1 & \text{at } m < 0 \end{cases}$$

Then the functions u, F, ψ, by which displacement vector components

$$u_r = rU,\ u_\theta = r\left(\frac{\partial F}{\partial \theta} + \frac{1}{\sin} \frac{\partial \psi}{\partial \varphi} \right), \quad u_\varphi = r\left(\frac{1}{\sin\theta} \frac{\partial F}{\partial \varphi} - \frac{\partial \psi}{\partial \theta} \right)$$

are defined, can be found in the series form

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} u_{mn}(r) Y_n^{(m)}(\theta, \varphi)$$

$$F(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} F_{mn}(r) Y_n^{(m)}(\theta, \varphi)$$

$$u(r,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Psi_{mn}(r)\,Y_n^{(m)}(\theta,\varphi),$$

here $Y_n^{(m)}(\theta,\varphi)=P_n^m(\cos\theta)\cos m\varphi$ if $m\le 0$, and $Y_n^{(m)}(\theta,\varphi)=P_n^m(\cos\theta)\sin m\varphi$ if $m>0$. The function P_n^m are adjoined Legendre functions of order m. By orthogonality of $Y_n^{(m)}(\theta,\varphi)$ the initial boundary value problem is divided into two sequences of independent one dimensional boundary value problems with respect to the functions u_{nm}, F_{nm}, Ψ_{nm}.

I.

$$\left.\begin{array}{l} b_{11}ru''_{nm}+4b_{11}u''_{nm}+\frac{1}{r}\left[2\left(b_{11}+b_{12}-b_{22}-b_{23}\right)-n(n+1)\right]u_m- \\ -\dfrac{n(n+1)}{r}\left[\left(b_{12}+1\right)F'_{nm}+\dfrac{2b_{12}-b_{22}-b_{23}-1}{r}F_{nm}\right]=0 \\ \left(b_{12}+1\right)u'_{nm}+\dfrac{b_{12}+b_{22}+b_{23}+3}{r}u_{nm}+rF''_{nm}+\dfrac{\left[2G_0-n(n+1)\right]}{r}F_{nm}=0 \end{array}\right\}$$

Boundary conditions

$$\left.\begin{array}{l} \left[b_{11}ru'_{nm}+3b_{11}u_{nm}-b_{12}n(n+1)F_{nm}\right]_{r=r_k}=P_{1nm}^{(k)} \\ \left[u_{nm}+rF'_{nm}\right]_{r=r_k}=P_{2nm}^{(k)} \end{array}\right\}.$$

II.

$$\left.\begin{array}{l} r\Psi''_{nm}+4\Psi'_{nm}+\frac{G_0}{r}\left[2-n(n+1)\right]\Phi_{nm}=0 \\ \left[r\Psi'_{nm}\right]_{r=r_k}=P_{3nm}^{(k)} \end{array}\right\}.$$

For solving obtained problems we can use various methods including numerical, for example, sweep method.

Described construction technique of nongomogeneous solutions is sufficiently universal and doesn't depend on various parameters of shell including its thickness.

However, if relative thickness of a shell is small, and the load given on faces is sufficiently smooth, then for nonhomogeneous solutions construction it is expediently to use the first iterative process of asymptotic method [4] that is simple and allows faster to achieve ultimate aim.

Let's introduce a new radial variable ξ connected with r by relation

$$\xi=\frac{1}{\varepsilon}\ln\frac{r}{r_0},\quad \varepsilon=\frac{1}{2}\ln\frac{r_2}{r_1},\quad r_0=\sqrt{r_1r_2},\quad \xi\in[-1,1]. \tag{2.1}$$

Later on we suppose the functions $q_i^{\pm}$ $(i=1,2,3)$ with regard to ε have the order $O(1)$.

Representation of the vector field $\overline{\nu}=(u_\theta,u_\varphi)$ in form of (1.6) reduces to partition of stressed state. By index 1 above we denote a stress tensor part corresponding to potential problem, and by index 2 the one's to vortex problem.

$$\sigma_r^{(1)}=G_1\varepsilon^{-1}\left[b_{11}\frac{\partial u}{\partial\xi}+\varepsilon\left(b_{11}+2b_{12}\right)u+\varepsilon b_{12}\Delta_0F\right],\quad \sigma_r^{(2)}\equiv 0$$

$$\sigma_\theta^{(1)}=G_1\varepsilon^{-1}\left[b_{12}\frac{\partial u}{\partial\xi}+\varepsilon\left(b_{12}+b_{22}+b_{23}\right)u+\right.$$
$$\left.+\varepsilon\left(b_{22}\frac{\partial^2F}{\partial\theta^2}+b_{23}\cot\theta\frac{\partial F}{\partial\theta}+\frac{b_{23}}{\sin^2\theta}\frac{\partial^2F}{\partial\varphi^2}\right)\right],$$

$$\sigma_{\theta}^{(2)} = 2G\frac{\partial^2}{\partial\varphi\partial\theta}\left(\frac{\psi}{\sin\theta}\right) = -\sigma_{\varphi}^{(2)}$$

$$\sigma_{\varphi}^{(1)} = G_1\varepsilon^{-1}\left[b_{12}\frac{\partial u}{\partial\xi} + \varepsilon\left(b_{12}+b_{22}+b_{23}\right)u + \right. \tag{2.2}$$

$$\left. +\varepsilon\left(b_{23}\frac{\partial^2 F}{\partial\theta^2} + b_{22}\cot\theta\frac{\partial F}{\partial\theta} + \frac{b_{22}}{\sin^2\theta}\frac{\partial^2 F}{\partial\varphi^2}\right)\right]$$

$$\tau_{r\varphi}^{(1)} = G_1\varepsilon^{-1}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{\partial F}{\partial\xi} + \varepsilon u\right), \quad \tau_{r\varphi}^{(2)} = -G_1\varepsilon^{-1}\frac{\partial^2\psi}{\partial\xi\partial\theta}$$

$$\tau_{r\theta}^{(1)} = G_1\varepsilon^{-1}\frac{\partial}{\partial\theta}\left(\frac{\partial F}{\partial\xi} + \varepsilon u\right), \quad \tau_{r\varphi}^{(2)} = G_1\varepsilon^{-1}\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi\partial\xi}$$

$$\tau_{\theta\varphi}^{(1)} = G\frac{\partial^2}{\partial\varphi\partial\theta}\left(\frac{F}{\sin\theta}\right), \quad \tau_{\theta\varphi}^{(2)} = G\left(\Delta_0\psi - 2\frac{\partial^2\psi}{\partial\theta^2}\right).$$

Potential problem (1.16), (1.17) was studied in detail in [5]. Therefore we investigated more simple vortex problem. In case of vortex problem, boundary value problem (1.18), (1.19) in variables ξ, θ, φ is written in the form

$$N\left(\Delta_0,\ \varepsilon\right) = \psi'' + 3\varepsilon\psi' + \varepsilon^2 G_0\left(\Delta_0 + 2\right)\psi = 0$$

$$\left.\psi'\right|_{\xi=\pm1} = q_3^{\pm}\cdot\varepsilon. \tag{2.3}$$

By primes we denote derivatives with respect to ξ.

Solution of (2.3) we find in the form

$$\psi = \varepsilon^{-1}\left(\psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + ...\right) \tag{2.4}$$

Coefficients of expansion of (2.4) are defined by integration on ξ of recurrence system obtained after substitution (2.4) in (2.3). Let's adduce relations defining three terms of expansion (2.4)

$$\left(\Delta_0 + 2\right)\psi_0 = -\frac{q_3}{G_0}, \qquad \left(\Delta_0 + 2\right)\psi_1 = -\frac{3}{2}\cdot\frac{q_3^{+} + q_3^{-}}{G_0}$$

$$\psi_2 = \frac{1}{2}q_3\xi^2 + \frac{q_3^{+} + q_3^{-}}{2}\xi + E\left(\theta,\varphi\right) \tag{2.5}$$

$$\left(\Delta_0 + 2\right)E\left(\theta,\varphi\right) = -\frac{q_3^{+} + q_3^{-}}{2G_0} - 2q_3, \qquad q_3 = q_3^{+} + q_3^{-}.$$

Formalae (2.5) enable to write in asymptotic expansions the stresses $\sigma_{\theta}^{(2)}$, $\sigma_{\varphi}^{(2)}$, $\tau_{\theta\varphi}^{(2)}$ in three terms of expansion, and $\tau_{r\theta}^{(2)}$, $\tau_{r\varphi}^{(2)}$ in one.

$$\sigma_{\theta}^{(2)} = -\sigma_{\theta}^{(2)} = 2G_1G_0\frac{\partial^2}{\partial\varphi\partial\theta}\left[\frac{\sigma\left(\xi,\theta,\varphi\right)}{\sin\theta}\right]$$

$$\tau_{\theta\varphi}^{(2)} = G\left(\Delta_0 - 2\frac{\partial^2}{\partial\theta^2}\right)\sigma\left(\xi,\theta,\varphi\right) \tag{2.6}$$

$$\tau_{r\varphi}^{(2)} = -G_1\varepsilon^{-1}\frac{\partial^2}{\partial\varphi\partial\xi}\sigma\left(\xi,\theta,\varphi\right)$$

$$\tau_{r\theta}^{(2)} = G_1 \varepsilon^{-1} \frac{1}{\sin\theta} \frac{\partial^2}{\partial\varphi\partial\xi} \sigma(\xi,\theta,\varphi)$$

$$\sigma(\xi,\theta,\varphi) = \psi_0 + \varepsilon\psi_1 + \varepsilon^2 \left[\frac{1}{2} q_3 \xi^2 + \frac{q_3^+ + q_3^-}{2} \xi + E(\theta,\varphi) \right].$$

From formulae (2.6) it follows, that the stresses $\sigma_\theta^{(2)}$, $\sigma_\varphi^{(2)}$, $\tau_{\theta\varphi}^{(2)}$, with respect to ε, have the order $O\left(\varepsilon^{-1}\right)$, and $\tau_{r\theta}^{(2)}$, $\tau_{r\varphi}^{(2)}$ the order $O(1)$.

3. The homogeneous solution we call any solution of equilibrium equation (1.4) satisfying no stresses condition on faces.

As we noted above, systems of homogeneous solutions for potential problem were constructed in [5].

Therefore we pass to asymphtotic analysis of vortex homogeneous problem.

$$N(\Delta_0,\varepsilon)\psi = \psi'' + 3\varepsilon\psi' + \varepsilon^2 G_0 (\Delta_0 + 2)\psi = 0$$

$$\psi'\big|_{\xi=\pm1} = 0. \tag{3.1}$$

After separation of variables using the following representation of solution

$$\psi(\xi,\theta,\varphi) = a(\xi) V(\theta,\varphi) \tag{3.2}$$

with regard to the function $a(\xi)$ we obtain self-adjoint spectral problem of the form

$$N\left(\frac{1}{4} - z^2, \varepsilon\right) a(\xi) = 0 \qquad a'(\pm1) = 0. \tag{3.3}$$

Putting $q_3 = 0$ in formulae (2.5) we get homogeneous solutions corresponding the first iterative process of vortex problem. To these solutions there correspond two eigenvalues $z_0 = \pm\frac{3}{2}$. To the lasts there corresponds the same eigenfunction $a_{00}(\xi) = c = const$. Other group is formed by the countable set of eigenvalues of the form

$$z_t = \varepsilon^{-1} i (\gamma_{0t} + \varepsilon\gamma_{1t} + ...), \qquad t = 1, 2, ... \tag{3.4}$$

where γ_{0t} in its turn are the nonzero eigenvalues of the spectral problem

$$T\psi_0 = G_0 \gamma_0^2 \psi, \quad T\psi_0 = \left\{ -\psi_0'', \psi_0'(\pm1) = 0 \right\} \tag{3.5}$$

$$\gamma_{1t} = -\frac{3}{2G_0} \int_{-1}^{1} \psi_{0t}' \cdot \psi_{0t} d\xi.$$

The corresponding eigenfunctions have the form

$$\psi_1 = \psi_{0t} = \varepsilon\psi_{1t} + O\left(\varepsilon^2\right), \quad \psi_{1t} = \sum_{\substack{m=0 \\ m\neq t}}^{\infty} l_{tm} \psi_{0m}$$

$$\int_{-1}^{1} \psi_{0t} \cdot \psi_{0m} d\xi = \delta_{tm}, \quad l_{tm} = \frac{3}{G_0 \left(\gamma_{0m}^2 - \gamma_{0t}^2\right)} \int_{-1}^{1} \psi_{0t}' \cdot \psi_{0m} d\xi. \tag{3.6}$$

To the eigenvalues z_0 there corresponds the following elementary solution of equilibrium equation

$$u_r^{(20)} \equiv 0, \quad u_\theta^{(20)} = c\frac{r_0 e^{\varepsilon\xi}}{\sin\theta}\frac{\partial Y_0}{\partial\varphi}, \quad u_\varphi^{(20)} = -c_1 r_0 e^{\varepsilon\xi}\frac{\partial Y_0}{\partial\theta},$$

where $Y_0(\theta,\varphi)$ is spherical function satisfying the equation $(\Delta_0+2)\,Y_0(\theta,\varphi)=0$.

To the other eigenvalues there corespond the elementary vortex solutions of the form

$$u_{rt}^{(22)} \equiv 0, \quad u_{\theta t}^{(22)} = \left[\psi_{0t} + \varepsilon\psi_{1t} + O\left(\varepsilon^2\right)\right]\frac{r_0 e^{\varepsilon\xi}}{\sin\theta}\frac{\partial Y_t}{\partial\varphi} \tag{3.7}$$

$$u_{\varphi t}^{(22)} = -\left[\psi_{0t} + \varepsilon\psi_{1t} + O\left(\varepsilon^2\right)\right] r_0 e^{\varepsilon\xi}\frac{\partial Y_t}{\partial t}.$$

Let's give the characteristic of deflected mode of vortex problem. In compliance with two groups of eigenvalues of spectral problem (3.1) the stress tensor is transformed to the form

$$\underline{\underline{\sigma}}^{(2)} = \underline{\underline{\sigma}}^{(20)} + \underline{\underline{\sigma}}^{(22)}. \tag{3.8}$$

The stress tenzor $\underline{\underline{\sigma}}^{(20)}$ corresponds to the eigenvalues $z_0 = \pm\dfrac{3}{2}$ and its components are defined by the formulae

$$\sigma_r^{(20)} = \tau_{r\theta}^{(20)} = \tau_{r\varphi}^{(20)} = 0$$

$$\sigma_\theta^{(20)} = -\sigma_\varphi^{(20)} = 2GC\frac{\partial^2}{\partial\theta\partial\varphi}\left[\frac{Y_0(\theta,\varphi)}{\sin\theta}\right] \tag{3.9}$$

$$\tau_{\theta\varphi}^{(20)} = GC\left(\Delta_0 Y_0 - 2\frac{\partial^2 Y_0(\theta,\varphi)}{\partial\theta^2}\right)$$

$$\sigma_\theta^{(22)} = -\sigma_\varphi^{(22)} = 2GG_0\psi_k(\xi)\frac{\partial^2}{\partial\theta\partial\varphi}\left[\frac{Y_k(\theta,\varphi)}{\sin\theta}\right] \tag{3.10}$$

$$\tau_{r\theta}^{(22)} = \varepsilon^{-1}\frac{G_1}{\sin\theta}\psi_k'(\xi)\frac{\partial Y_k(\theta,\varphi)}{\partial\varphi}$$

$$\tau_{r\varphi}^{(22)} = \varepsilon^{-1}G_1\psi_k'(\xi)\frac{\partial Y_k(\theta,\varphi)}{\partial\varphi}$$

$$\tau_{\theta\varphi}^{(22)} = G\psi_k(\xi)\left(\Delta_0 Y_0 - \frac{\partial^2}{\partial\theta^2}Y_0(\theta,\varphi)\right)$$

$$\psi_k(\xi) = \psi_{0k}(\xi) + \varepsilon\psi_{1k}(\xi)$$

$$Y_0(\theta) = A_0 P_1(\cos\theta) + B_0 Q_1(\cos\theta), \quad P_1(\cos\theta) = \cos\theta$$

$$Q_1(\cos\theta) = \frac{1}{2}\cos\theta\ln\frac{1+\cos\theta}{1-\cos\theta} - 1.$$

Let's explain the view of stressed state of vortex problem.

The equilibruim conditions of layer have the form

$$2\pi\int_{r_1}^{r_2}\tau_{\theta\varphi}(r,\theta_1)\,r^2\sin^2\theta_1 dr = 2\pi\int_{r_1}^{r_2}\tau_{\theta\varphi}(r,\theta_2)\,r^2\sin^2\theta_2 dr. \tag{3.11}$$

Consider the relation of homogeneous solutions of vortex problem with the torque M_{kp} of stresses acting in section $\theta = const$.

We have

$$M_{kp} = 2\pi \sin^2 \theta r_0^3 \int_{r_1}^{r_2} \tau_{\theta\varphi} (\xi, \theta) e^{3\varepsilon\xi} d\xi. \tag{3.12}$$

Let the stresses $\tau_{\theta\varphi}$ be represented in the form

$$\tau_{\theta\varphi} = \tau_{\theta\varphi}^{(20)} + G_1 \sum_{k=1}^{\infty} G_0 \psi_k (\xi) \left[\left(z_k^2 - \frac{1}{4} \right) Y_k (\theta) - 2 \cot \theta \frac{dY_k (\theta)}{d\theta} \right]. \tag{3.13}$$

The summand $\tau_{\theta\varphi}^{(20)}$ corresponds to the eigenvalues $z_0 = \pm \frac{3}{2}$ and has the form

$$\tau_{\theta\varphi}^{(20)} = \frac{G}{r} \left[A_0 P_1 (\cos \theta) + B_0 Q_1 (\cos \theta) \right]. \tag{3.14}$$

Other part of the stresses $\tau_{\theta\varphi}$ is defined by the eigenfunctions ψ_k and by the eigenvalues z_k of spectral problem (3.5).

Let's transform expressions for M_{kp} with taking into account (3.13), (3.14).

We have

$$M_{kp} = 2\pi r_0^3 B_0 M_0 + 2\pi r_0^3 \sin^2 \theta G_i \left[\left(z_k^2 - \frac{1}{4} \right) Y_k (\theta) - 2 \cot \theta \frac{dY_k (\theta)}{d\theta} \right] \int_{-1}^{1} \psi_k (\xi) e^{3\varepsilon\xi} d\xi$$

$$M_0 = G e^{\varepsilon\xi} d\xi = \frac{2sh\varepsilon}{\varepsilon} G_0. \tag{3.15}$$

Starting directly from spectral problem (3.1), we show that in each summand of sum in formula (3.15) the multiplier

$$\int_{-1}^{1} \psi_k (\xi) e^{3\varepsilon\xi} d\xi = 0 \tag{3.16}$$

vanishes; multiplying the both parts of the equation

$$\psi_k'' + 3\varepsilon\varphi_k' - G_0 \left(z_k^2 - 9/4 \right) \psi_k = 0$$

on $\varepsilon^{3\varepsilon\xi}$, integrating in the range $[-1, 1]$, we get

$$\int_{-1}^{1} \psi_k'' \varepsilon^{3\varepsilon\xi} d\xi + 3\varepsilon \int_{-1}^{1} \varphi_k' \varepsilon^{3\varepsilon\xi} d\xi = \varepsilon^2 \left(z_k^2 - 9/4 \right) \int_{-1}^{1} \psi_k (\xi) e^{3\varepsilon\xi} d\xi.$$

The left hand side of the last equality vanishes. We can easily see it with the help of integration by parts and using the boundary condition $\psi' (\pm 1) = 0$, whence it follows equality (3.16).

Thus for M_{kp} we obtain

$$M_{kp} = 2\pi r_0^3 B_0 M_0$$

The stressed state corresponding to zeros of the second group of vortex problem, is self-balanced in each section $\theta = const$.

Thus, the general problem of elasticity theory for spherical layer is dismembered in two ones. However, solutions of these problems are connected through the boundary conditions on lateral face. Therefore, by satisfying boundary condition on lateral face there arise the problems, chiefly connected with nonorthogonality of homogeneous solutions. As it shown in [5], solutions of potential problem posses the property of generalized orthogonality, the solutions of vortex problem are orthogonal. But slutions of different groups don't posses these properties. Therefore, in general case the boundary value problem is reduced to solution of infinite sytems of linear algebraic equations as in axially-symmetric case.

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