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THE PROBLEMS OF OPTIMAL CONTROL BY PARABOLIC EQUATIONS COEFFICIENTS

Abstract

For the problems of optimal control by parabolic equations coefficients the questions of correctness of their statement have been investigated, sufficient conditions for aim functional differentiability have been found, formula for its gradient has been obtained, and necessary condition of optimality has been determined.

The problems of optimal control by coefficients of mathematical physics have great theoretical and applied significance [1]-[3]. By investigating the correctness of these problems statement and receiving necessary conditions of optimality there arises a series of difficulties connected with theirs nonlinearity and nonconvexity [1], [4]-[6].

In the given paper a problem of optimal control by parabolic equations coefficients provided, that these coefficents are found in the spaces W_p^1 and L_s ,where p and s are some finite numbers, has been investigated.Earlier, such problems has been studied in the papers [1], [4]-[6] and in others, in cases, when coefficients of parabolic equations are found in the spaces W_∞^1 and L_∞.

For the below considered optimal control problem the questions of correctness of its statement have been investigated, sufficient conditions for aim functional differentiability have been found, formula for its gradient has been obtained, and necessary condition of optimality has been determined.

1. Problem statement.

Let Ω be a bounded domain of n-dimensional Euclidean space R^n, S be a boundary of the domain Ω, which is assumed to be continuous by Lipschitz, $x=(x_1,...,x_n)$ be an arbitrary point of domain Ω, $T>0$ be a given number, $0\le t\le T$, $Q_T=\Omega\times(0,T)$, $S_T=S\times[0,T]$. The functional spaces $C\left(\overline{Q}_T\right)$, $L_S\left(Q_T\right)$, $\overset{\circ}{W}{}_2^1(\Omega)$, $\overset{\circ}{W}{}_2^{1,0}(Q_T)$, $\overset{\circ}{V}{}_2^{1,1/2}(Q_T)$, $W_p^{1,1}(Q_T)$, $\overset{\circ}{W}{}_2^{1,1}(Q_T)$, $W_2^{2,1}(Q_T)$ used below are introduced, for example in [7, ch.I, §1, pp.12-17]. Moreover, everywhere below by M we denote positive constants that don't depend on admissible controls and on evaluated quantities.

Let's consider controlled process described by the equation of parabolic type

$$\frac{\partial u}{\partial t}-\sum_{\alpha=1}^{n}\frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\right)+q(x,t)\,u=f(x,t)\,,\qquad (x,t)\in Q_T, \tag{1.1}$$

where $f\in L_2(Q_T)$ is the given function, $k_\alpha(x,t)$ $\left(\alpha=\overline{1,n}\right)$, $q(x,t)$ are control functions.

Let for equation (1.1) the following boundary condidions be given:

$$u|_{S_T} = 0, \qquad u|_{t=0} = \varphi(x), \quad x \in \Omega, \tag{1.2}$$

where $\varphi \in W_2^1(\Omega)$ is the known function.

Suppose, that control $\nu = (k_1(x,t), ..., k_n(x,t), q(x,t))$ is found on the following set of admissible controls:

$$V = \left\{\nu = (k_1, ...k_n, q) \in B \equiv \left[W_p^{1,1}(Q_T)\right]^n \times L_S(Q_T) : \right.$$

$$0 < \nu_\alpha \le k_\alpha(x,t) \le \mu_\alpha, \ \overset{\circ}{\forall}(x,t) \in Q_T, \quad \left\| \frac{\partial k_a}{\partial x_i} \right\|_{L_p(Q_T)} \le d_i^{(\alpha)},$$

$$\left. \left\| \frac{\partial k_a}{\partial t} \right\|_{L_p(Q_T)} \le d_\alpha, \quad \left(\alpha, i = \overline{1,n}\right), \ \|q\|_{L_S(Q_T)} \le \rho \right\}, \tag{1.3}$$

where $\nu_a, \ \mu_\alpha, d_\alpha, d_i^{(\alpha)}, p > 0 \ \left(\alpha, i = \overline{1,n}\right), \ s, p$ are given numbers, with $s > 2$ at $n \le 2$, $s \ge \dfrac{n+2}{2}$ at $n \ge 3$, $p \ge n+2$ at $n \ge 1$, symbol $\overset{\circ}{\forall}$ denotes "almost for all ".

Let's consider the problem on minimization of functional

$$J(\nu) = \beta_0 \left\| u|_{t=T} - u_0 \right\|^2_{L_2(\Omega)} + \beta_1 \|u - u_1\|^2_{L_2(Q_T)} \tag{1.4}$$

on the solutions $u = u(x,t) = u(x,t;\nu)$ of boundary value problem (1.1), (1.2), which correspond to all admissible controls $\nu \in V$, where $\beta_0, \beta_1 \ge 0$, $\beta_0 + \beta_1 > 0$ are given numbers, $u_0 \in L_2(\Omega)$, $u_1 \in L_2(Q_T)$ are the known functions.

Under solution of boundary value problem (1.1), (1.2) at each $\nu \in V$ we understand the function $u = u(x,t;\nu)$ from $V_2^{1,1/2}(Q_T)$ satisfying the identity

$$\int\limits_{Q_T} \left(-u\frac{\partial \eta}{\partial t} + \sum_{\alpha=1}^{n} k_a \frac{\partial u}{\partial x_\alpha}\frac{\partial \eta}{\partial x_\alpha} + qu\eta\right) dxdt = \int\limits_{\Omega} \varphi(x)\,\eta(x,0)\,dx + \int\limits_{Q_T} f\eta dxdt, \tag{1.5}$$

for all $\eta = \eta(x,t)$ from $\overset{\circ}{W}{}_2^{1,1}(Q_T)$,equal zero at $t = T$.

At accepted estimates from the results of the paper [7, ch.III, §4, p.189] it follows that at each $\nu \in V$, boundary value problem (1.1), (1.2) has a unique solution from $\overset{\circ}{V}{}_2^{1,1/2}(Q_T)$. Moreover, solution of problem (1.1), (1.2) from $\overset{\circ}{V}{}_2^{1,1/2}(Q_T)$ belongs also to the space $W_{2,0}^{2,1}(Q_T) = W_2^{2,1}(Q_T) \cap \overset{\circ}{W}{}_2^{1,0}(Q_T)$, satisfies equation (1.1) at $\forall (x,t) \in Q_T$, equals $\varphi(x)$ at $t = 0$ and it holds the estimation [7, ch.III, §6, pp.203-212]

$$\|u\|_{W_2^{2,1}(Q_T)} \le M\left[\|f\|_{L_2(Q_T)} + \|\varphi\|_{W_2^2(Q_T)}\right]. \tag{1.6}$$

2. Correctness of problem statement

Let's show, that problem (1.1)-(1.4) has at least one solution.

Theorem 2.1. *Let the conditions accepted at the statement of problem (1.1)-(1.4), where* $s > \dfrac{n+2}{2}$ *at* $n \geq 3$, *be fulfilled. Then functional (1.4) is wekly continuous on* V, *the set* $V_* = \{\nu_* \in V : J(\nu_*) = J_* = \inf\{J(\nu) : \nu \in V\}\}$ *of optimal controls of problem (1.1)-(1.4) is nonempty, weakly compact in* B *and any minimizing sequence* $\{\nu^{(m)}\}$ *weakly converges in* B *to the set* V.

Proof. Let's show that functional (1.4) is weakly continuous on V. Let $\nu = (k_1, ..., k_n, q) \in V$ be some element, $\left\{\nu^{(m)} = \left(k_1^{(m)}, ..., k_n^{(m)}, q^{(m)}\right)\right\} \subset V$ be an arbitrary sequence, such that

$$\nu^{(m)} \to \nu \text{ is weakly in } B, \tag{2.1}$$

as $m \to \infty$.

Let $u^{(m)} = u^{(m)}(x,t) = u\left(x,t;\nu^{(m)}\right)$ be a solution of problrm (1.1), (1.2) from $\overset{\circ}{W}{}_{2,0}^{2,1}(Q_T)$ at $\nu = \nu^{(m)}$. From (1.6) it follows that

$$\left\|u^{(m)}\right\|_{W_2^{2,1}(Q_T)} \leq const \quad (m = 1, 2, ...). \tag{2.2}$$

Moreover, it is known, that the embedding $W_p^{1,1}(Q_T) \to C\left(\overline{Q}_T\right)$ is compact at $p > n+1$, the embedding $\overset{\circ}{W}{}_{2,0}^{2,1}(Q_T) \to L_2(\Omega)$ is compact at any n [7, ch.II, §2, p.78], and the embedding $\overset{\circ}{W}{}_{2,0}^{2,1}(Q_T) \to L_r(Q_T)$ is compact at $1 \leq r < \dfrac{2(n+2)}{n-2}$, if $n \geq 3$ [8, ch.I, §2, p.39] for any finite $r \geq 1$, if $n = 2$ and at $r = \infty$, if $n = 1$ [8, ch.I, §1, p.33]. Therefore, from (2.1) and (2.2) it follows, that from $\left\{\nu^{(m)}, u^{(m)}\right\}$ we can extract such a subsequence that we again denote by $\left\{\nu^{(m)}, u^{(m)}\right\}$, that

$$k_\alpha^{(m)} \to k_\alpha \left(\alpha = \overline{1,n}\right) \text{ is weakly in } W_p^{1,1}(Q_T) \text{ and strongly in } C\left(\overline{Q}_T\right), \tag{2.3}$$

$$q^{(m)} \to q \text{ is weakly in } L_S(Q_T), \tag{2.4}$$

$$u^{(m)} \to u \text{ is weakly in } \overset{\circ}{W}{}_{2,0}^{2,1}(Q_T) \text{ and strongly in } L_r(Q_T), \tag{2.5}$$

$$\left.u^{(m)}\right|_{t=T} \to \left.u\right|_{t=T} \text{ strongly in } L_2(\Omega), \tag{2.6}$$

where $u = u(x,t)$ is some function from $\overset{\circ}{W}{}_{2,0}^{2,1}(Q_T)$, $r = \infty$, if $n = 1$, $r \geq 1$ is any finite number, if $n = 2$ and $1 \leq r < \dfrac{2(n+2)}{n-2}$, if $n \geq 3$.

Let's show. that $u = u(x,t) = u(x,t;\nu)$. It is clear, that the function $u^{(m)}(x,t)$, $m = 1, 2, ...$ satisfies the identity

$$\int\limits_{Q_T} \left(-u^{(m)}\frac{\partial \eta}{\partial t} + \sum_{\alpha=1}^{n} k_\alpha^{(m)} \frac{\partial u^{(m)}}{\partial x_\alpha}\frac{\partial \eta}{\partial x_\alpha} + q^{(m)}u^{(m)}\eta\right) dxdt =$$

$$= \int\limits_{\Omega} \varphi(x)\,\eta(x,0)\,dx + \int\limits_{Q_T} f\eta dxdt, \tag{2.7}$$

for all $\eta=\eta(x,t)$ from $\overset{\circ}{W}{}_2^{1,1}(Q_T)$ equal zero at $t=T$.

Moreover, the equality

$$\int\limits_{Q_T}\sum_{\alpha=1}^{n}k_\alpha^{(m)}\frac{\partial u^{(m)}}{\partial x_\alpha}\frac{\partial\eta}{\partial x_\alpha}dxdt=\int\limits_{Q_T}\sum_{\alpha=1}^{n}\left(k_\alpha^{(m)}-k_\alpha\right)\times$$

$$\times\frac{\partial u^{(m)}}{\partial x_\alpha}\frac{\partial\eta}{\partial x_\alpha}dxdt+\int\limits_{Q_T}\sum_{\alpha=1}^{n}k_\alpha\frac{\partial u^{(m)}}{\partial x_\alpha}\frac{\partial\eta}{\partial x_\alpha}dxdt. \tag{2.8}$$

is true.

From (2.2) and (2.3) it follows, that the first item in the right hand side of equality (2.8) tends to zero as $m\to\infty$. Therefore, passing to the limit in (2.8) as $m\to\infty$ and using (2.5) we have

$$\int\limits_{Q_T}\sum_{\alpha=1}^{n}k_\alpha^{(m)}\frac{\partial u^{(m)}}{\partial x_\alpha}\frac{\partial\eta}{\partial x_\alpha}dxdt\to\int\limits_{Q_T}\sum_{\alpha=1}^{n}k_\alpha\frac{\partial u}{\partial x_\alpha}\frac{\partial\eta}{\partial x_\alpha}dxdt. \tag{2.9}$$

Now, let's show that

$$\int\limits_{Q_T}q^{(m)}u^{(m)}\eta dxdt\to\int\limits_{Q_T}qu\eta dxdt, \tag{2.10}$$

as $m\to\infty$. It is clear, that the equality

$$\int\limits_{Q_T}q^{(m)}u^{(m)}\eta dxdt=\int\limits_{Q_T}q^{(m)}\left(u^{(m)}-u\right)\eta dxdt+\int\limits_{Q_T}q^{(m)}u\eta dxdt, \tag{2.11}$$

is true.

Using inequality (1.8) from [7, ch. II, §I, p.75] and the condition $\left\|q^{(m)}\right\|_{L_S(Q_T)}\le\rho$, we have

$$\left|\int\limits_{Q_T}q^{(m)}\left(u^{(m)}-u\right)\eta dxdt\right|\le\left\|q^{(m)}\right\|_{L_S(Q_T)}\left\|u^{(m)}-u\right\|_{L_{\frac{2s}{s-2}}(Q_T)}\|\eta\|_{L_2(Q_T)}\le$$

$$\le\rho\left\|u^{(m)}-u\right\|_{L_{\frac{2s}{s-2}}(Q_T)}\|\eta\|_{L_2(Q_T)}. \tag{2.12}$$

Since, at $n\ge3$ from the condition $s>\dfrac{n+2}{2}$ it follows, that $\dfrac{2s}{s-2}<\dfrac{2(n+2)}{n-2}$, passing to the limit in (2.12) as $m\to\infty$ and using (2.5) we get, that he first item in the right hand side of equality (2.11) tends to zero as $m\to\infty$. Moreover, from embedding theorem [8, ch.I, §1, p.33, §2, p.39] and from condition $s>2$ at $n\le2$, $s>\dfrac{n+2}{2}$ at $n\ge3$ it follows, that if $u\in W_{2,0}^{2,1}(Q_T)$, $\eta\in\overset{\circ}{W}{}_2^{1,1}(Q_T)$, then $u\eta\in L_{\frac{s}{s-2}}(Q_T)$. Therefore passing on to the limit in (2.11) at $m\to\infty$ and using (2.4) we obtain correctness of relations (2.10).

Now, passing to the limit in (2.7) as $m \to \infty$ and using (2.5), (2.9), (2.10) we get, that the function $u = u(x,t)$ satisfies identity (1.5). From here and since the function $u = u(x,t)$ is in element of the space $\overset{\circ}{W}{}_{2,0}^{2,1}(Q_T)$ it follows that the function $u = u(x,t)$ satisfies equation (1.1) $\forall (x,t) \in Q_T$ and boundary conditions (1.2). It means, that $u = u(x,t) = u(x,t;\nu)$.

Thus, it is established, that by fulfilling (2.1) from the sequence $\{\nu^{(m)}, u^{(m)}\}$ we can extract a subsequence, that we again denote by $\{\nu^{(m)}, u^{(m)}\}$ for which relations (2.3)-(2.6) are true, where $u = u(x,t;\nu)$. Using uniqueness of solution of problem (1.1), (1.2) it is easy to show, that these relations are true for the whole of sequence $\{\nu^{(m)}, u^{(m)}\}$ too.

Now using (2.5), (2.6) from (1.4) we obtain, that $J(\nu^{(m)}) \to J(\nu)$ as $m \to \infty$, i.e. the functional $J(\nu)$ is weakly continuous on V. Moreover, the set V defined by relation (1.3) is a convex closed bounded set in reflexive banach space B. Then from Weirstrass theorem [9, ch. I, §3, pp.49-51] it follows the problem statement. Theorem 2.1 is proved.

Now let's consider the problem on minimization of functional

$$I_\gamma(\nu) = J(\nu) + \gamma \|\nu - \omega\|_B^2 \tag{2.13}$$

on the set V defined by relation (1.3), under condition (1.1), (1.2), where $\gamma \geq 0$ is given number, $\omega \in B$ is given element, the functional $J(\nu)$ is defined by formula (1.4). This problem is called problem (1.1)-(1.3), (2.13).

Theorem 2.2. *Let the conditions of theorem 2.1 be fulfilled, and $\gamma \geq 0$. Then for each $\omega \in B$ problem (1.1)-(1.3) has at least one solution. If $\gamma > 0$, then there exists a dense subset G of the space B, such that for each $\omega \in G$ problem (1.1)-(1.3), (2.13) has a unique solution.*

Proof. The functional $I_\gamma(\nu)$ is a sum of weakly continuous functional $J(\nu)$ on V and the weakly semicontinuous below functional $\gamma \|\nu - \omega\|_B^2$, $(\gamma \geq 0)$. Hence, the functional $I_\gamma(\nu)$ is weakly semicontinuous below on V. Then from Weirstrass theorem [9, ch. I, §3, pp.49-51] it follows that at $\gamma \geq 0$ problem (1.1)-(1.3), (2.13) has at least one solution.

Now, let $\gamma > 0$. The functional $J(\nu)$ is continuous by the norm of the space B and is bounded below on V, the space B is uniformly convex, and the set V is closed and bounded on B. Then by virtue of the known theorem [10], there exists a dense subset G of the space B such that for any $\omega \in G$ at $\gamma > 0$ problem (1.1)-(1.3), (2.13) has the unique solution. Theorem 2.2 is proved.

3. Differentiability of functional and necessary condition of optimality.

For proving differentiability of functional (1.4) we intoduce the following problem on determination of a function $\psi = \psi(x,t) = \psi(x,t;\nu)$ from the conditions

$$\frac{\partial \psi}{\partial t} + \sum_{\alpha=1}^{n} \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial \psi}{\partial x_\alpha}\right) - q(x,t)\psi =$$

$$= 2\beta_1 \left[u(x,t;\nu) - u_1(x,t) \right], \ (x,t) \in Q_T, \tag{3.1}$$

$$\psi|_{S_T} = 0, \quad \psi|_{t=T} = -2\beta_0 \left[u(x,T;\nu) - u_0(x) \right], \ x \in \Omega, \tag{3.2}$$

where $u(x,t;\nu)$ is the solution of problem (1.1), (1.2).

Under solution of boundary value problem (3.1), (3.2), at each $\nu \in V$, we'll understand generalized solution of this problem from $\overset{\circ}{V}{}_2^{1,1/2}(Q_T)$ [7, ch.3, §1, pp. 160-163]. If in relations (3.1), (3.2) instead of the variable t we take a new independent variable $\tau = T - t$, we get boundary value problem of (1.1), (1.2) type. Then from the results of the papers [7, ch. III, §2, p.189] it follows , that problem (3.1), (3.2), at each $\nu \in V$, has a unique solution from $\overset{\circ}{V}{}_2^{1,1/2}(Q_T)$. Moreover, if $u_0 \in \overset{\circ}{W}{}_2^1(\Omega)$, then $u(\cdot\,,T;\nu) - u_0(\cdot) \in \overset{\circ}{W}{}_2^1(\Omega)$ [11, ch.III, §2, p.160] and therefore solution of problem (3.1), (3.2) belongs to the space $W_{2,0}^{2,1}(Q_T)$, satisfies equation (3.1) $\overset{\circ}{\forall}(x,t) \in Q_T$, equals $-2\beta_0 \left[u(x,T;\nu) - u_0(x) \right]$ at $t = T$ and the estimation [7, ch.III, §6, pp.203-212]

$$\|\psi\|_{W_2^{2,1}(Q_T)} \le M \left[\beta_0 \|u(x,T;\nu) - u_0(x)\|_{W_2^1(\Omega)} + \beta_1 \|u(x,t;\nu) - u_1(x,t)\|_{L_2(Q_T)} \right]$$

is true.

Then taking into account here the inequality

$$\|u(x,T;\nu)\|_{W_2^1(\Omega)} \le M \|u\|_{W_2^{2,1}(Q_T)}$$

from [11, ch.III, §2, p.160] and estimations (1.6) we get

$$\|\psi\|_{W_2^{2,1}(Q_T)} \le M \left[\|f\|_{L_2(Q_T)} + \|\varphi\|_{W_2^1(\Omega)} + \beta_0 \|u_0\|_{W_2^1(\Omega)} + \beta_1 \|u_1\|_{L_2(Q_T)} \right]. \tag{3.3}$$

Now we also introduce the following boundary value problems on definition of the functions $\theta_i = \theta_i(x,t) = \theta_i(x,t;\nu)$ $\left(i = \overline{1,n}\right)$ from the conditions

$$-\sum_{\alpha=1}^{n} \frac{\partial^2 \theta_i}{\partial x_\alpha^2} - \frac{\partial^2 \theta_i}{\partial t^2} + \theta_i = \frac{\partial u}{\partial x_i} \frac{\partial \psi}{\partial x_i}, \quad (x,t) \in Q_T, \tag{3.4}$$

$$\left.\frac{\partial \theta_i}{\partial \nu}\right|_{S_T} = 0, \quad \left.\frac{\partial \theta_i}{\partial t}\right|_{t=0} = \left.\frac{\partial \theta_i}{\partial t}\right|_{t=T} = 0, \quad x \in \Omega, \quad \left(i = \overline{1,n}\right), \tag{3.5}$$

where $u = u(x,t;\nu)$, $\psi = \psi(x,t;\nu)$ are the solutions of problem (1.1), (1.2) and (3.1), (3.2) respectively, ν is exterior normal to the boundary S.

Boundary value problem (3.4), (3.5) is Neuman problem for elliptical equation (3.4) in the domain Q_T. Under solution of problem (3.4), (3.5), at each $\nu \in V$, we'll understand the function $\theta_i = \theta_i(x,t)$ from $W_2^{1,1}(Q_T)$; satisfying the identity

$$\int_{Q_T} \left(\sum_{\alpha=1}^{n} \frac{\partial \theta_i}{\partial x_\alpha} \frac{\partial \eta}{\partial x_\alpha} + \frac{\partial \theta_i}{\partial t} \frac{\partial \eta}{\partial t} + \theta_i \eta \right) dxdt = \int_{Q_T} \frac{\partial u}{\partial x_i} \frac{\partial \psi}{\partial x_i} \eta dxdt, \quad \left(i = \overline{1,n}\right) \tag{3.6}$$

for all $\eta = \eta(x,t)$ from $W_2^{1,1}(Q_T)$.

Lemma 3.1. *Let the conditions of theorem 1.1 be fulfilled and* $1 \le n \le 4$, $u_0 \in \overset{\circ}{W}{}_2^1(\Omega)$. *Then boundary value problem (3.4), (3.5), at each* $\nu \in V$, *is uniquely solvable in* $W_2^{1,1}(Q_T)$ *and the estimation*

$$\|\theta_i\|_{W_2^{1,1}(Q_T)} \le M\left[\|f\|_{L_2(Q_T)} + \|\varphi\|_{W_2^1(\Omega)}\right]\left[\|f\|_{L_2(Q_T)} + \|\varphi\|_{W_2^1(\Omega)} + \right.$$

$$\left. + \beta_0\|u_0\|_{W_2^1(\Omega)} + \beta_1\|u_1\|_{L_2(Q_T)}\right] \quad \left(i = \overline{1,n}\right) \tag{3.7}$$

is true.

Proof. As it follows from the results of the paper [12, ch.III, §6, pp.200-202], for proving unique solvability of boundary value problem (3.4), (3.5) it suffices to show, that

$$\frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_i} \in L_{\frac{2\hat{n}}{\hat{n}+2}}(Q_T) \quad \left(i = \overline{1,n}\right), \tag{3.8}$$

where $\hat{n} = n+1$ at $n \ge 2$, $\hat{n} = 2+\varepsilon$, $\varepsilon > 0$ at $n = 1$.

Using inequality (1.7) from [7, ch.II, §1, p.75] and condition $1 \le n \le 4$ we get

$$\left\|\frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_i}\right\|_{L_{\frac{2\hat{n}}{\hat{n}+2}}(Q_T)} \le \left\|\frac{\partial u}{\partial x_i}\right\|_{L_{\frac{2(\hat{n}+2)}{\hat{n}}}(Q_T)} \left\|\frac{\partial \psi}{\partial x_i}\right\|_{L_{\frac{\hat{n}(\hat{n}+2)}{2(\hat{n}+1)}}(Q_T)} \le$$

$$M\left\|\frac{\partial u}{\partial x_i}\right\|_{L_{\frac{2(n+2)}{n}}(Q_T)} \left\|\frac{\partial \psi}{\partial x_i}\right\|_{L_{\frac{2(n+2)}{n}}(Q_T)} \quad \left(i = \overline{1,n}\right). \tag{3.9}$$

According to lemma 3.3 from [7, ch.II, §3, p.95] for any function $u \in W_2^{1,1}(Q_T)$ the inequality

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{L_{\frac{2(n+2)}{n}}(Q_T)} \le M\|u\|_{W_2^{2,1}(Q_T)} \quad \left(i = \overline{1,n}\right) \tag{3.10}$$

is true.

Then taking into consideration (3.10) and analogous inequality for the function ψ in (3.9), we obtain

$$\left\|\frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_i}\right\|_{L_{\frac{2\hat{n}}{\hat{n}+2}}(Q_T)} \le M\|u\|_{W_2^{2,1}(Q_T)}\|\psi\|_{W_2^{2,1}(Q_T)}, \quad \left(i = \overline{1,n}\right). \tag{3.11}$$

Hence, it folllows relation (3.8). Therefore problem (3.4), (3.5) has a unique solution from $W_2^{1,1}(Q_T)$ and the estimation [12, ch.III, §6, pp.200-202]

$$\|\theta_i\|_{W_2^{1,1}(Q_T)} \le \left\|\frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_i}\right\|_{L_{\frac{2\hat{n}}{\hat{n}+2}}(Q_T)}, \quad \left(i = \overline{1,n}\right).$$

is true.

Here taking into account (3.11), and then (1.6) and (3.3) we obtain (3.7). Lemma 3.1 is proved.

Theorem 3.1. *Let the conditions of lemma 3.1 be fulfilled, and $p \geq n+2$. Then functional (1.4) is continuously differentiable by Freshe on V and its gradient has the form*

$$J'(\nu) = (\theta_1, ..., \theta_n, u\psi). \tag{3.12}$$

Proof. Let $\delta\nu = (\delta k_1, ..., \delta k_n, \delta q) \in B$ be an increment of control on element $\nu \in V$, such that $\nu + \delta\nu \in V$. Then solution of problem (1.1), (1.2) gets the increment $\delta u = \delta u(x,t) = u(x,t;\nu+\delta\nu) - u(x,t;\nu)$. From conditions (1.1), (1.2) it follows that the function δu is the solution from $W_{2,0}^{2,1}(Q_T)$ of the following boundary value problem:

$$\frac{\partial \delta u}{\partial t} - \sum_{\alpha=1}^{n} \frac{\partial}{\partial x_\alpha}\left((k_\alpha + \delta k_\alpha)\frac{\partial \delta u}{\partial x_\alpha}\right) + (q+\delta q)\,\delta u =$$

$$= \sum_{\alpha=1}^{n} \frac{\partial}{\partial x_\alpha}\left(\delta k_\alpha \frac{\partial u}{\partial x_\alpha}\right) - \delta q u, \quad (x,t) \in Q_T, \tag{3.13}$$

$$\delta u|_{t=0} = 0, \quad x \in \Omega, \quad \delta u|_{S_T} = 0. \tag{3.14}$$

As it follows from [7, ch.III, §6, pp.203-212] for the solution of problem (3.13), (3.14) the estimation

$$\|\delta u\|_{W_2^{2,1}(Q_T)} \leq M\left[\left(\sum_{\alpha=1}^{n}\left\|\delta k_\alpha \frac{\partial^2 u}{\partial x_\alpha^2}\right\|_{L_2(Q_T)} + \left\|\frac{\partial \delta k_\alpha}{\partial x_\alpha}\frac{\partial u}{\partial x_\alpha}\right\|_{L_2(Q_T)}\right) + \right.$$

$$\left. + \|\delta q u\|_{L_2(Q_T)}\right]. \tag{3.15}$$

Now let's estimate the summands included in the right hand side of estimation (3.15). Using boundedness of the embedding $W_p^{1,1}(Q_T) \to C(\overline{Q}_T)$ at $p > n+1$ [7, ch.II, §2, p.78], inequality (1.7) from [7, ch.II, §3, p.75], estimations (3.10) and condition $p \geq n+2$, we get

$$\left\|\delta k_\alpha \frac{\partial^2 u}{\partial x_\alpha^2}\right\|_{L_2(Q_T)} + \left\|\frac{\partial \delta k_\alpha}{\partial x_\alpha}\frac{\partial u}{\partial x_\alpha}\right\|_{L_2(Q_T)} \leq$$

$$\leq \|\delta k_\alpha\|_{C(\overline{Q}_T)}\left\|\frac{\partial^2 u}{\partial x_\alpha^2}\right\|_{L_2(Q_T)} + \left\|\frac{\partial \delta k_\alpha}{\partial x_\alpha}\frac{\partial u}{\partial x_\alpha}\right\|_{L_{n+2}(Q_T)}\left\|\frac{\partial u}{\partial x_\alpha}\right\|_{L_{\frac{2(n+2)}{n}}(Q_T)} \leq$$

$$\leq M\|\delta k_\alpha\|_{W_p^{1,1}(Q_T)}\|u\|_{W_2^{2,1}(Q_T)} \quad \left(\alpha = \overline{1,n}\right). \tag{3.16}$$

Moreover, it is known, that the embedding $W_2^{2,1}(Q_T) \to L_r(Q_T)$ is bounded at any finite $r \geq 1$, if $n \leq 2$ and at any $r \leq \dfrac{2(n+2)}{n-2}$, if $n \geq 3$ [8, ch.I, §2, p.39]. Therefore, using inequality (1.7) from [7, ch.II, §1, p.75] and taking into account, that $s > 2$ at $n \leq 2$ and $s > \dfrac{n+2}{2}$ at $n \geq 3$, we get

$$\|\delta q u\|_{L_2(Q_T)} \leq \|\delta q\|_{L_s(Q_T)}\|u\|_{L_{\frac{2s}{s-2}}(Q_T)} \leq M\|\delta q\|_{L_s(Q_T)}\|u\|_{W_2^{2,1}(Q_T)}. \tag{3.17}$$

Then taking into account (3.16), (3.17) in (3.15) and using (1.6) we obtain the estimation

$$\|\delta u\|_{W_2^{2,1}(Q_T)} \leq M\left[\|f\|_{L_2(Q_T)} + \|\varphi\|_{W_2^1(\Omega)}\right]\|\delta\nu\|_B . \quad (3.18)$$

Now, consider increment of the functional $J(\nu)$. Using formula (1.4), we get

$$\delta J(\nu) = J(\nu + \delta\nu) - J(\nu) = 2\beta_0 \int_\Omega [u(x,T;\nu) - u_0(x)]\,\delta u(x,T)\,dx +$$

$$+2\beta_1 \int_\Omega [u(x,T;\nu) - u_1(x,t)]\,\delta u(x,t)\,dxdt + \beta_0\|\delta u(x,T)\|^2_{L_2(\Omega)} + \beta_1\|\delta u\|^2_{L_2(Q_T)} .$$

Using conditions (3.1), (3.2) and (3.13), (3.14) this expression can be represented as

$$\delta J(\nu) = \int_\Omega \left(\sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_i}\delta k_i + u\psi\delta q\right) dxdt + R(\delta\nu), \quad (3.19)$$

where

$$R(\delta\nu) = \int_\Omega \left(\sum_{i=1}^n \frac{\partial \delta u}{\partial x_i}\frac{\partial \psi}{\partial x_i}\delta k_i + \delta u\psi\delta q\right) + \beta_0\|\delta u(x,T)\|^2_{L_2(\Omega)} + \beta_1\|\delta u\|^2_{L_2(\Omega)} . \quad (3.20)$$

Putting $\eta = \delta k_i$ in (3.6) and taking into consideration the obtained equality in (3.19), we get

$$\delta J(\nu) = \int_{Q_T} \left[\sum_{i=1}^n \left(\frac{\partial \theta_i}{\partial x_\alpha}\frac{\partial \delta k_i}{\partial x_\alpha} + \frac{\partial \theta_i}{\partial t}\frac{\partial \delta k_i}{\partial t} + \theta_i\delta k_i\right) + u\psi\delta q\right] dxdt + R(\delta\nu), \quad (3.21)$$

Now, let's estimate the remainder term $R(\delta\nu)$. Using Cauchy-Bunyakovskii inequality, inequality (1.7) from [7, ch.II, §1, p.75], boundedness of the embeddings $W_p^{1,1}(Q_T) \to C(\overline{Q}_T)$, $W_2^{2,1}(Q_T) \to L_{\frac{2s}{s-2}}(Q_T)$ at $p > n+1$, $s > 2$ [7, ch.II, §1, p.78], [8, ch.I, §2, p.39] and estimations (3.18), (3.3) we have

$$\left|\int_{Q_T} \sum_{\alpha=1}^n \left(\frac{\partial \delta u}{\partial x_\alpha}\frac{\partial \psi}{\partial x_\alpha}\delta k_\alpha + \delta u\psi\delta q\right) dxdt\right| \leq$$

$$\leq \sum_{\alpha=1}^n \|\delta k_\alpha\|_{C(\overline{Q}_T)} \left\|\frac{\partial \delta u}{\partial x_\alpha}\right\|_{L_2(Q_T)} \left\|\frac{\partial \psi}{\partial x_\alpha}\right\|_{L_2(Q_T)} +$$

$$+\|\delta q\|_{L_s(Q_T)}\|\delta u\|_{L_{\frac{2s}{s-2}}(Q_T)}\|\psi\|_{L_2(Q_T)} \leq M\|\delta u\|_{W_2^{2,1}(Q_T)}\|\psi\|_{W_2^{2,1}(Q_T)}\|\delta\nu\|_B \leq$$

$$\leq M\left[\|f\|_{L_2(Q_T)} + \|\varphi\|_{W_2^1(\Omega)}\right]\left[\|f\|_{L_2(Q_T)} + \|\varphi\|_{W_2^1(\Omega)} +\right.$$

$$\left.+\beta_0\|u_0\|_{W_2^1(\Omega)} + \beta_1\|u_1\|_{L_2(Q_T)}\right]\|\delta\nu\|_B^2 .$$

Hence, from boundedness of the embedding $W_2^{2,1}(Q_T) \to L_2(Q_T)$ [8, ch.I, §2, p.39] and from (3.18) it follows that for the remainder term $R(\delta\nu)$ defined by equality (3.20) the estimation

$$|R(\delta\nu)| \leq M\left[\|f\|_{L_2(Q_T)} + \|\varphi\|_{W_2^1(\Omega)}\right]\left[\|f\|_{L_2(Q_T)} + \|\varphi\|_{W_2^1(\Omega)} + \right.$$

$$\left. +\beta_0 \|u_0\|_{W_2^1(\Omega)} + \beta_1 \|u_1\|_{L_2(Q_T)}\right] \|\delta\nu\|_B^2 .$$

is true.

Then, hence and from (3.21) it follows that functional (1.4) is differentiable by Freshe on V and for its gradient equality (3.12) is true.

It remains to show, that $\nu \to J'(\nu)$ is a continuous mapping from V into B^*, where B^* is a space conjugate to B. Let $\delta\psi = \psi(x,t;\nu+\delta\nu) - \psi(x,t;\nu)$, $\delta\theta_i = \theta_i(x,t;\nu+\delta\nu) - \theta_i(x,t;\nu)$ $\left(i=\overline{1,n}\right)$ be an increment of solutions of problems (3.1), (3.2) and (3.4), (3.5), respectively. Reasoning by analogy as estimation (3.18) for the function δu and estimation (3.7) for the function θ_i had been obtained, it is easy to show that fot the functions $\delta\psi$ and $\delta\theta_i$ $\left(i=\overline{1,n}\right)$ the estimations

$$\|\delta\psi\|_{W_2^{2,1}(Q_T)} \leq M\left[\|f\|_{L_2(Q_T)} + \|\varphi\|_{W_2^1(\Omega)} + \right.$$

$$\left. +\beta_0 \|u_0\|_{W_2^1(\Omega)} + \beta_1 \|u_1\|_{L_2(Q_T)}\right] \|\delta\nu\|_B \tag{3.22}$$

$$\max_{1\leq i\leq n} \|\delta\psi\|_{W_2^{2,1}(Q_T)} \leq M\left[\|f\|_{L_2(Q_T)} + \|\varphi\|_{W_2^1(\Omega)}\right]\left[\|f\|_{L_2(Q_T)} + \|\varphi\|_{W_2^1(\Omega)} + \right.$$

$$\left. +\beta_0 \|u_0\|_{W_2^1(\Omega)} + \beta_1 \|u_1\|_{L_2(Q_T)}\right]\left[\|\delta\nu\|_B + \|\delta\nu\|_B^2\right]. \tag{3.23}$$

are true.

Then using equality (3.12) and estimations (1.6), (3.3), (3.18), (3.22), (3.23) we get the estimations

$$\left\|J'(\nu+\delta\nu) - J'(\nu)\right\|_{B^*} \leq M\left[\|f\|_{L_2(Q_T)} + \|\varphi\|_{W_2^1(\Omega)}\right]\left[\|f\|_{L_2(Q_T)} + \|\varphi\|_{W_2^1(\Omega)} + \right.$$

$$\left. +\beta_1 \|u_1\|_{L_2(Q_T)}\right]\left[\|\delta\nu\|_B + \|\delta\nu\|_B^2\right],$$

from which it continuity of $J'(\nu)$ on V follows. Theorem 3.1 is proved.

Now, let's formulate necessary condition of optimality for solution of problem (1.1)-(1.4).

Theorem 3.2. *Let the conditions of theorem 3.1 be fulfilled, and* $\nu_* = (k_{1^*}, ..., k_{n^*}, q_*) \in V$ *be optimal control for problem (1.1)-(1.4). Ten for any control* $\nu = (k_1, ..., k_n, q) \in V$ *the inequality*

$$\int\limits_{Q_T} \left\{ \left[\sum_{\alpha=1}^{n} \theta_{\alpha^*}(x,t)\left(k_\alpha(x,t) - k_{\alpha^*}(x,t)\right) + \sum_{n=1}^{n} \frac{\partial\theta_{\alpha^*}}{\partial x_i}\left(\frac{\partial k_\alpha}{\partial x_i} - \frac{\partial k_{\alpha^*}}{\partial x_i}\right) + \right.\right.$$

$$+\frac{\partial\theta_{\alpha^*}}{\partial t}\left(\frac{\partial k_\alpha}{\partial t}-\frac{\partial k_{\alpha^*}}{\partial t}\right)\Bigg]+u_*(x,t)\,\psi_*(x,t)\,(q(x,t)-q_*(x,t))\Bigg\}\,dxdt\geq 0, \quad (3.24)$$

is fulfilled, where $u_*(x,t)$, $\psi_*(x,t)$ *and* $\theta_{\alpha^*}(x,t)$ $\left(\alpha=\overline{1,n}\right)$ *are solutions of problems (1.1), (1.2); (3.1), (3.2) and (3.4), (3.5), respectively, at* $\nu=\nu_*$.

Proof. By theorem 3.1 the functional $J(\nu)$ is continuously differentiable by Freshe on V and for its gradient formula (3.12) is true. The set V defined by relation (1.3) is convex. Then by virtue of the known theorem [9, ch.I, §2, p.28], on the element $\nu^*\in V$ providing minimum to the functional $J(\nu)$ the inequality

$$<J'(\nu_*)\,,\nu-\nu_*>_B\geq 0$$

is to be fulfilled for any $\nu\in V$. Hence and from (3.12) it follows the validity of inequality (3.24). Theorem 3.2 is proved.

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