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TORSION OF PRISMATIC PHYSICALLY LINEARLY VISCOELASTIC BODY OF ARBITRARY CROSS-SECTION

Abstract

The problem statement of linear torsion of prismatic viscoelastic body of arbitrary cross-section is given. The solution of the stated problem is submitted by solutions of the corresponding elastic torsion problem. Examples are considered.

Let the forces resulted to torsional pairs, are imposed to the bases of prismatic body. We shall consider, that the lateral surface of the body is free from external forces and body forces are absent. Mechanical properties of prismatic body material are described by ratios of physically linear theory of viscoelasticity [1]:

$$2G_0 e_{ij} = s_{ij} + \int\limits_0^t \Gamma(t-\tau)\, s_{ij}(\tau)\, d\tau; \qquad \theta = \sigma/K; \tag{1}$$

or

$$s_{ij}/2G_0 = e_{ij} - \int\limits_0^t R(t-\tau)\, e_{ij}(\tau)\, d\tau; \quad \sigma = k\theta. \tag{2}$$

Here $i = 1,2,3$; t is time; $s_{ij} = \sigma_{ij} - \sigma\delta_{ij}$ is deviator of stresses tensor; σ_{ij}; $e_{ij} = \varepsilon_{ij} - \varepsilon\delta_{ij}$ is deformation deviator ε_{ij}; $\sigma = \sigma_{ij}\delta_{ij}/\,3$ is average stress; $\varepsilon = \varepsilon_{ij}\delta_{ij}/\,3$ is average deformation; G_0 is the instant modulus of shear; K is modulus of volume elasticity; $\theta = 3\varepsilon$ is volumetric deformation; $\Gamma(t)$ and $R(t)$ are interreciprocal kernels of heredity. There is a ratio between the functions $\Gamma(t)$ and $R(t)$:

$$\Gamma(t) = R(t) + \int\limits_0^t \Gamma(\tau)\, R(t-\tau)\, d\tau \tag{3}$$

The equations (1) and (2) are equivalent ratios.

We shall accept Cartesian rectangular coordinates x_1, x_2, x_3. We shall direct an axis x_3 parallel to the axis of prismatic body. At the constrained torsion of viscoelastic prismatic body of arbitrary cross-section we count, that during the fixed time interval: 1) the cross-sections situated at equal distances from each other, twist for equal angles; 2) the cross-sections are distorted and moreover all sections are identical; the deplanation appears as proportional to time-dependent twist angle that is allowable at linear torsion. Mathematically we shall write the marked assumptions as:

$$u_1 = -\chi(t)\, x_2 x_3, \quad u_2 = -\chi(t)\, x_1 x_3, \quad u_3 = -\chi(t)\, \varphi(x_1, x_2). \tag{4}$$

Here $\varphi(x_1, x_2)$ is deplanation function, $\chi = \chi(t)$ is relative twist angle at the moment t. At $\chi(t) \equiv const$ relation (4) coincides with corresponding ratios of Saint-Venant. It indicates that formulas (4) were written on the basis of Saint-Venant relations.

Proceeding from properties of heredity which are inherent to viscoelastic bodies we shall present function $\chi(t)$ as

$$\chi(t) = \vartheta(t) + \int_0^t \Gamma(t-\tau)\,\vartheta(\tau)\,d\tau \tag{5}$$

where $\vartheta(t)$ is some time function to be defined.

The components of small deformation tensor are represented through components of displacement vector u_i by Cauchy relations

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{6}$$

Allowing (4) in (6) we shall have:

$$\varepsilon_{11} = 0; \quad \varepsilon_{22} = 0; \quad \varepsilon_{33} = 0; \quad \varepsilon_{12} = 0 \tag{7}$$

$$\varepsilon_{13} = \frac{\chi(t)}{2}\left(\frac{\partial\varphi}{\partial x_1} - x_2\right); \quad \varepsilon_{23} = \frac{\chi(t)}{2}\left(\frac{\partial\varphi}{\partial x_2} + x_1\right) \tag{8}$$

Using formulas (7), (8) in equations (2) we shall define quantities σ_{ij}:

$$\sigma_{11} = 0; \quad \sigma_{22} = 0; \quad \sigma_{33} = 0; \quad \sigma_{12} = 0 \tag{9}$$

$$\sigma_{13} = G_0\left(\frac{\partial\varphi}{\partial x_1} - x_2\right)\left[\chi(t) - \int_0^t R(t-\tau)\,\chi(\tau)\,d\tau\right] \tag{10}$$

$$\sigma_{23} = G_0\left(\frac{\partial\varphi}{\partial x_2} + x_1\right)\left[\chi(t) - \int_0^t R(t-\tau)\,\chi(\tau)\,d\tau\right] \tag{11}$$

Allowing for (5) and (3) relations (10) and (11) are transformed to the form

$$\sigma_{13} = G_0\left(\frac{\partial\varphi}{\partial x_1} - x_2\right)\vartheta(t), \quad \sigma_{23} = G_0\left(\frac{\partial\varphi}{\partial x_2} + x_1\right)\vartheta(t) \tag{12}$$

From the equations of equilibrium there remain only the followings:

$$\frac{\partial\sigma_{13}}{\partial x_3} = 0; \quad \frac{\partial\sigma_{23}}{\partial x_3} = 0; \quad \frac{\partial\sigma_{13}}{\partial x_1} + \frac{\partial\sigma_{23}}{\partial x_2} = 0 \tag{13}$$

The first two equations of (13) are satisfied identically, and the third equation with regard to (12) gives

$$\frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} \equiv \nabla\varphi = 0. \tag{14}$$

Formula (14) shows, that deplanation function, which else is called Saint-Venant torsion function, should be harmonic function of variables x_1 and x_2 in the area occupied by cross-section of body. From here it follows, that deplanation itself also is harmonic function.

In the considered case as in the theory of elastic torsion, we can show, that deplanation function φ on a contour L of cross-section satisfies the condition:

$$\frac{\partial\varphi}{dn} = \left[x_2 \cos\left(n,\ x_1\right) - x_1 \cos\left(n,\ x_2\right)\right]_L$$

or

$$\frac{\partial\varphi}{dn} = \frac{d}{ds}\left(\frac{x^2+y^2}{2}\right)\Bigg|_L \tag{15}$$

where $\dfrac{d}{dn}$, $\dfrac{d}{ds}$ are derivatives in normal n and on arch s of contour L.

Hence, the problem of viscoelastic prismatic bodies torsion, similarly to the torsion problem in the case of elastic body, is reduced to Neumann's problem (14), (15) for Laplace equation. Thus we can show, that the condition of existence of the Neumann's problem solution $\oint\limits_L \dfrac{\partial\Phi}{\partial n} ds = 0$ is satisfied.

For resultant stresses on a face surface we have

$$\int\limits_{\omega} \sigma_{13} d\omega = 0, \qquad \int\limits_{\omega} \sigma_{23} d\omega = 0, \tag{16}$$

where ω is the area of cross-section of prismatic body.

Taking in the account (16), we conclude, that the shearing stresses applied to cross-section, are reduced to force couple, whose moment equals

$$M\left(t\right) = \int\limits_{\omega} \left(x_1\sigma_{23} - x_2\sigma_{13}\right) d\omega. \tag{17}$$

The equilibrium condition at end faces gives $M\left(t\right) = M_T\left(t\right)$, where $M_T\left(t\right)$ is the assigned torque. Taking this into account, and also formulas (12) in relation (17), we shall receive:

$$\theta\left(t\right) = \frac{M_T\left(t\right)}{D}, \tag{18}$$

where $D = G_0 \int\limits_{\omega} \left(x_1^2 + x_2^2 + x_1\dfrac{\partial\varphi}{\partial x_2} - x_2\dfrac{\partial\varphi}{\partial x_1}\right) d\omega$ is torsion rigidity. We can show, that always $D > 0$.

Hence, the problem of physically linear torsion of viscoelastic prismatic body of arbitrary cross-section is completely solved, if the deplanation function $\varphi\left(x_1,\ x_2\right)$ will be found.

Now, following [3], we'll present the solution of the problem of linear torsion of viscoelastic prismatic body as:

$$u_i = u_i' + \int\limits_0^t \Gamma\left(t-\tau\right) u_i' d\tau; \quad \sigma_{ij} = \sigma_{ij}' \tag{19}$$

Using relations (6) and the first formula of (19), we shall receive:

$$\varepsilon_{ij} = \varepsilon_{ij}' + \int\limits_0^t \Gamma\left(t-\tau\right) \varepsilon_{ij}' d\tau \tag{20}$$

where

$$\varepsilon'_{ij} = \frac{1}{2}\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right) \tag{21}$$

The equilibrium equations (13) with taking into account the second formula (19) are kept:

$$\frac{\partial \sigma'_{13}}{\partial x_3} = 0; \quad \frac{\partial \sigma'_{23}}{\partial x_3} = 0; \quad \frac{\partial \sigma'_{13}}{\partial x_1} + \frac{\partial \sigma'_{23}}{\partial x_2} = 0 \tag{22}$$

Besides with taking into account (9) and (12) we have:

$$\sigma'_{11} = 0; \quad \sigma'_{22} = 0; \quad \sigma'_{33} = 0; \quad \sigma'_{12} = 0$$

$$\sigma'_{13} = G_0\left(\frac{\partial \varphi}{\partial x_1} - x_2\right)\vartheta(t), \quad \sigma'_{23} = G_0\left(\frac{\partial \varphi}{\partial x_2} + x_1\right)\vartheta(t). \tag{23}$$

Formulas (7) and (8) using (20), (5) will be transformed to the form:

$$\varepsilon'_{11} = 0; \quad \varepsilon'_{22} = 0; \quad \varepsilon'_{33} = 0; \quad \varepsilon'_{12} = 0$$

$$\varepsilon'_{13} = \frac{\vartheta(t)}{2}\left(\frac{\partial \varphi}{\partial x_1} - x_2\right); \quad \varepsilon'_{23} = \frac{\vartheta(t)}{2}\left(\frac{\partial \varphi}{\partial x_2} + x_1\right) \tag{24}$$

From formulas (4) for components of displacement vector subject to the first formula of (19), and also the formula (5), it follows:

$$u'_1 = -\vartheta(t)\, x_2 x_3; \quad u'_2 = \vartheta(t)\, x_1 x_3, \quad u'_3 = \vartheta(t)\, \varphi(x_1, x_2). \tag{25}$$

At using the second formula of (19) the relation (17) will be rewritten as:

$$M_T(t) = M(t) = \int\limits_{\omega} \left(x_1 \sigma'_{23} - x_2 \sigma'_{13}\right) d\omega. \tag{26}$$

Relation (18) keeps its form.

As we see, relations (21) - (26) with addition (18) are relations of the elastic quasi-static torsion theory. It means, that the quantities u'_i, σ'_{ij}, ε'_{ij} included in formulas (19), (20) are components of displacement vector, stress and deformation tensors which arise in the considered prismatic body at its quasi-static elastic torsion by the torsion torque $M_T(t)$. In this case time t plays only a role of a parameter.

Hence, if any existing method solves the problem of elastic torsion of prismatic body with the given cross-section at the known shear modulus G and torque M, i.e. elastic displacements u^e_i, deformations ε^e_{ij}, stresses σ^e_{ij} have been found, then having replaced in expressions the last G by G_0, M by $M_T(t)$, we find quantities u'_i, σ'_{ij}, ε'_{ij}. After that, according to formulas (19) and (20) we determine the required solution of the corresponding viscoelastic problem. Here we notice, that at the solution of the considered elastic and viscoelastic torsion problems instead the deplanation function φ we can use, according [2], the harmonic function ψ adjoint to function φ, or the torsion function of Prandtl Φ. connected with function ψ by the relation:$\Phi(x_1, x_2) = \psi(x_1, x_2) - \left(x_1^2 + x_2^2\right)/2$. Thus, as is known from [2], the problems on definitionof functions $\psi(x_1, x_2)$ and $\Phi(x_1, x_2)$ are also Dirichlet problems for Laplace equation. At solution of the noted problems the complex torsion function [2] can be also applied.

Examples. 1. **Elliptic cross-section**. Let a and b be semi-axes of an ellipse. Let's use solution of the corresponding elastic problem [2]:

$$u_1 = -\frac{M\left(a^2+b^2\right)x_2x_3}{G\pi a^3b^3}; \quad u_2 = \frac{M\left(a^2+b^2\right)x_1x_3}{G\pi a^3b^3}; \quad u_3 = M\frac{\left(b^2-a^2\right)x_1x_2}{G\pi a^3b^3};$$

$$\sigma_{13}^e = -\frac{2M}{\pi ab^3}x_2, \quad \sigma_{23}^e = -\frac{2M}{\pi a^3b}x_1,$$

where G is the shear modulus of body's material, M is the torque.

Having replaced G by G_0, M by $M_T(t)$ in the last expressions, we shall have expressions for quantities u_1', u_2', u_3', σ_{13}', σ_{23}'. Taking into account the received expressions in transition formulas (19) we shall write down the solution problem of the viscoelastic prismatic torsion body with elliptic cross-section:

$$u_1 = -\frac{\left(a^2+b^2\right)x_2x_3}{G_0\pi a^3b^3}M_T^*(t); \quad u_2 = \frac{\left(a^2+b^2\right)x_1x_3}{G_0\pi a^3b^3}M_T^*(t);$$

$$u_3 = \frac{\left(b^2-a^2\right)x_1x_2}{G_0\pi a^3b^3}M_T^*(t); \quad \sigma_{13} = -\frac{2M_T(t)}{\pi ab^3}x_2, \quad \sigma_{23} = -\frac{2M_T(t)}{\pi a^3b}x_1.$$

Here

$$M_T^*(t) = M_T(t) + \int_0^t \Gamma(t-\tau)\,M_T(\tau)\,d\tau. \tag{26}$$

For $a = b$ the received solution corresponds to the solution problem on viscoelastic prismatic body's torsion with circular cross-section. In this case $u_3 = 0$, that testifies to deplanation absence.

2. Circular prismatic bar with half-round limiting recess. According to [2, 4] the solution of the elastic problem is represented as:

$$u_1^e = -\frac{Mx_2x_3}{2GDa^4}; \quad u_2^e = \frac{Mx_1x_3}{2GDa^4}; \quad u_3^e = -\frac{Mb^2x_2}{2GDa^3\left(x_1^2+x_2^2\right)};$$

$$\sigma_{13}^e = \frac{M}{2Da^4}\left[\frac{2ab^2x_1x_2}{\left(x_1^2+x_2^2\right)^2} - x_2\right], \quad \sigma_{23}^e = \frac{M}{2Da^4}\left[-\frac{ab^2\left(x_1^2-x_2^2\right)}{\left(x_1^2+x_2^2\right)^2} + x_1 - a\right].$$

Here, as in the previous problem, G is the shear modulus of material, M is the torque; a is radius of a bar's circle , b is radius of recesses circle. Besides

$$\begin{aligned} D \;=\; & \frac{1}{24}\left(\sin 4\alpha + 8\sin 2\alpha + 12\alpha\right) - \frac{1}{2}\left(\frac{b}{a}\right)^2\left(\sin 2\alpha + 2\alpha\right) + \\ & + \frac{4}{3}\left(\frac{b}{a}\right)^3\sin\alpha - \frac{1}{4}\left(\frac{b}{a}\right)^4\alpha, \end{aligned}$$

where

$$\alpha = \arccos\frac{b}{2a}. \tag{27}$$

Now, to receive the problem's solution of torsion of the round viscoelastic bar with semicircular longitudinal recess in the represented solution of the elastic problem we shall replace G by G_0, M by $M_T(t)$ and we shall use formulas (19). Thus we shall receive:

$$u_1 = -\frac{x_2 x_3}{2G_0 D a^4} M_T^*; \quad u_2 = \frac{x_1 x_3}{2G_0 D a^4} M_T^*;$$
$$u_3 = -\frac{b^2 x_2}{2G_0 D a^3 \left(x_1^2 + x_2^2\right)} M_T^*;$$

$$\sigma_{13} = \frac{M_T(t)}{2Da^4}\left[\frac{2ab^2 x_1 x_2}{\left(x_1^2+x_2^2\right)^2} - x_2\right], \quad \sigma_{23} = \frac{M_T(t)}{2Da^4}\left[-\frac{ab^2\left(x_1^2 - x_2^2\right)}{\left(x_1^2+x_2^2\right)^2} + x_1 - a\right].$$

Quantity D included in these relations is expressed by formula (27), the operator M_T^* is of the form (26).

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