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FINDING THE BEST APPROXIMATION ORDER BY THE QUASIPOLYNOMIALS IN A MIXED NORM SPACE

Abstract

In the paper we establish the best approximation order of many variable functions by the polynomials of class Π *in the space* $L_{\overline{p}}(Q)$*, where the* Q *may differ from parabellelepiped. Moreover, bilateral inequalities for the best approximation* $E(f,\Pi)_{L_{\overline{p}}(Q)}$ *were obtained with obvious constants.*

Let $\delta \subset \overline{n} = \{1,...,n\}$, $\mathcal{D}$ be a set of subsets $\overline{n}, x_\delta = \{x_i | \, i \in \delta\}$. We similarly determine $\mu_\delta, \alpha_\delta$ where $\mu_i, \alpha_i \in N$; assume $\hat{x}_j \overset{def}{=} (x_1,...,x_{j-1},x_{j+1},...x_n)$. We similarly determine $\hat{x}_\delta$. Let's denote δ in the following way $\delta = \left\{\delta^{(1)},...,\delta^{(|\delta|)}\right\}$, where $|\delta|$ is the quantity of elements of the set δ.

Let $\sum\limits_{\alpha_\delta=0}^{\mu_\delta-1} = \sum\limits_{\alpha_\delta(1)=0}^{\mu_\delta(1)-1} ... \sum\limits_{\alpha_\delta(|\delta|)=0}^{\mu_\delta(|\delta|)-1}$ and $x_\delta^{\alpha_\delta} = \prod\limits_{i\in\delta} x_i^{\alpha_i}$.

By $P_{\mu_\delta}(x)$ we denote a polynomial with respect to a group of variables x_δ of power $\mu_\delta - 1$ i.e. with respect to the variable x_i of power $\mu_i - 1$, $i \in \delta$ with coefficients that depend on other variables:

$$P_{\mu_\delta}(x) = \sum_{\alpha_\delta=0}^{\mu_\delta-1} c_{\alpha_\delta}(\hat{x}_\delta)\, x_\delta^{\alpha_\delta}.$$

The expression

$$\Pi_\mu^{\mathcal{D}} = \Pi_\mu^{\mathcal{D}}(x) = \sum_{\delta\in\mathcal{D}} P_{\mu_\delta}(x) \tag{1}$$

said to be a generalized quasipolynomial of order $\mu_{\mathcal{D}} - 1$ [1].

Let a red function $f = f(x)$, $x = (x_1,...,x_n)$ be determined in the bounded set Q of n-dimensional Euclidean space $R_n = R_n(x_1,...,x_n)$.

Denote $x_{\overline{i,n}} = (x_i, x_{i+1},...,x_n)$ and the set Q be given by inequalities

$$\begin{cases} \alpha_1\left(x_{\overline{2,n}}\right) \le x_1 \le \beta_1\left(x_{\overline{2,n}}\right) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_i\left(x_{\overline{i+1,n}}\right) \le x_i \le \beta_i\left(x_{\overline{i+1,n}}\right) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_n = \alpha \le x_n \le b = \beta_n \end{cases} \qquad i = \overline{1,n-1} \tag{2}$$

where $\alpha_i\left(x_{\overline{i+1,n}}\right), \beta_i\left(x_{\overline{i+1,n}}\right)$ are fixed real functions.

Let's consider a norm of many variable functions $f(x)$ with respect to one variable in the space $L_{p_i(x_i)}$

$$\|f\|_{p_i} \overset{def}{=} \left(\int\limits_{\alpha_i\left(x_{\overline{i+1,n}}\right)}^{\beta_i\left(x_{\overline{i+1,n}}\right)} |f(x)|^{p_i}\, dx_i \right)^{1/p_i}$$

and denote by $L_{\overline{p}}(Q)$, $p=(p_1,...,p_n)$ a space of functions $f=f(x)$ for which the integral [2]

$$\|f\|_{\overline{p}} \overset{def}{=} \left\| \left(\left\| ... \left(\|f\|_{p_1} \right) ... \right\|_{p_{n-1}} \right) \right\|_{p_n}.$$

exists and finite.

Let's consider the best approximation of the function $f \in L_{\overline{p}}(Q)$ by the set Π all quasipolynomials $\Pi_\mu^{\mathcal{D}}$:

$$E(f,\Pi)_{L_{\overline{p}}(Q)} = \inf_{\Pi_\mu^{\mathcal{D}} \in \Pi} \left\| f - \Pi_\mu^{\mathcal{D}} \right\|_{L_{\overline{p}}(Q)}.$$

Let s, s_k, i_s, j_s be natural numbers

Let's tak $\eta \subset \mathcal{D}, \overline{\eta} \overset{def}{=} \{s | \exists \delta \in \eta, s \in \delta\}$, $x_\eta = \{x_s, s \in \overline{\eta}\}$.

Let $x_s^{(i_s)} \in \Pi p_{x_s} Q$, $1 \le i_s \le \mu_s$ $x_s^{(1)} + (i_s - 1) h_s, h_s > 0$, $s \in \overline{\mathcal{D}}$.

In sequel, we'll need the denotation

$$\eta = \left\{ \eta^{(1)}, ..., \eta^{(|\overline{\eta}|)} \right\},$$

$$\sum_{k_\eta=1}^{\mu_\eta} \overset{def}{=} \sum_{k_{\eta^{(1)}}=1}^{\mu_{\eta^{(1)}}} ... \sum_{k_{\eta^{(|\overline{\eta}|)}}=1}^{\mu_{\eta^{(|\overline{\eta}|)}}};$$

$$|k_\eta| \sum_{s \in \overline{\eta}} k_s; f\left(\breve{x}_j^{(1)}\right) \overset{def}{=} f\left(x_1, ..., x_{j-1}, x_j^{(1)}, x_{j+1}, ..., x_n\right)$$

$f\left(\breve{x}_\eta^{(1)}\right)$ is determined similar to $f\left(\breve{x}_j^{(1)}\right)$.

To the set $\mathcal{D}$ we associate a family of operators

$$\nabla^*_{\mathcal{D}_{\mu,\theta}} : X(Q) \to X(Q) \quad \text{for}$$

$$\theta = \left(x^{(1)}_{\mathcal{D}^{(1)}}, ..., x^{(\mu_1)}_{\mathcal{D}^{(1)}}, ..., x^{(1)}_{\mathcal{D}^{(|\overline{\mathcal{D}}|)}}, ..., x^{\left(\mu_{(|\overline{\mathcal{D}}|)}\right)}_{\mathcal{D}^{(|\overline{\mathcal{D}}|)}} \right) \in R^{\sum_{i=1}^{|\overline{\mathcal{D}}|} \mu_i},$$

determined in the following way

$$\nabla^*_{\mathcal{D}_{\mu,\theta}} f = \sum_{|\eta|=0}^{|\mathcal{D}|} \sum_{\eta \subset \mathcal{D}} \sum_{k_\eta=1}^{\mu_\eta} (-1)^{|\mu_\eta| - |k_\eta| + |\eta|} f\left(\breve{x}_\eta^{(k_\eta)}\right) \prod_{s \in \overline{\eta}} k_s \binom{\mu_s}{k_s} \times$$

$$\times \prod_{\tau_s=1}^{\mu_s} \left(x_s - x_s^{(\tau_s)} \right) \prod_{s \in \overline{\mathcal{D}}/\overline{\eta}} \mu_s! h_s^{\mu_s - 1} \tag{3}$$

The following theorem show that a family of operators $\nabla^*_{\mathcal{D}_{\mu,\theta}}$ is an exact annihilator of a class of Π quasipolynomials $\Pi_\mu^{\mathcal{D}}$ in the space $L_{\overline{p}}(Q)$.

Theorem 1. *For the function* $f \in L_{\overline{p}}(Q)$ *almost for all*

$$f(x) = \Pi_\mu^{\mathcal{D}}(x), \nabla^*_{\mathcal{D}_{\mu,\theta}} f = 0, \text{almost for all } (x,\theta) \in R^{n + \sum_{i=1}^{|\overline{\mathcal{D}}|} \mu_i} \tag{4}$$

Proof. Let's consider the expression $\nabla^{*}_{\mathcal{D}_{\mu,\theta}}$ determined in (3). First of all we should be convinced that for $\forall\delta$ and $0 \leq \alpha_i \leq \mu_i, i \in \delta$ a family of operators $\nabla^{*}_{\mathcal{D}_{\mu,\theta}} f$ are annihilated on the function $a\left(\hat{x}_\delta\right) x_\delta^{\alpha_i}$. Assume $\delta = \delta_1$. By $\nabla^{|\eta|}$ we denote addends in (3) that correspond to the values $0 < |\eta| \leq \mathcal{D}$, i.e.

$$\nabla_{\mathcal{D}_{\mu,\theta}} f = \sum_{|\eta|=0}^{\left|\overline{\mathcal{D}}\right|} \nabla^{|\eta|} f \tag{5}$$

It is known that divided difference of one variable function of order r is an annihilator of a polynomial with respect to this variable of power $\leq r-1$. Then $\forall s \in \overline{\mathcal{D}}$

$$\left[x_s^{(0)} x_s^{(1)} ... x_s^{(\mu_s)}\right] P_{\mu_s - 1} \overset{def}{=} \sum_{k_s=0}^{\mu_s} P_{\mu_s-1}\left(x_s^{(k_s)}\right)\left[\prod_{\substack{r_s=0 \\ r_s \neq k_s}}^{\mu_s}\left(x_s^{(k_s)} - x_s^{(\tau_s)}\right)\right]^{-1} = 0.$$

Hence, $\forall\ 0 \leq \alpha_s \leq \mu_s - 1$,

$$\sum_{k_s=0}^{\mu_s}\left(x_s^{(k_s)}\right)^{\alpha_s}\left[\prod_{\substack{r_s=0 \\ r_s \neq k_s}}^{\mu_s}\left(x_s^{(k_s)} - x_s^{(\tau_s)}\right)\right]^{-1} \prod_{0 \leq i_s \leq j_s \leq \mu_s}\left(x_s^{(i_s)} - x_s^{(j_s)}\right) = 0.$$

Since $x_s^{(0)} \overset{def}{=} x_s$ we have

$$\sum_{k_s=1}^{\mu_s} (-1)^{k_s - 1}\left(x_s^{(k_s)}\right)^{\alpha_s} \prod_{\substack{0 \leq i_s \leq j_s \leq \mu_s \\ i_s, j_s \neq k_s}}\left(x_s^{(i_s)} - x_s^{(j_s)}\right) =$$

$$= x_s^{\alpha_s} \prod_{1 \leq i_s \leq j_s \leq \mu_s}\left(x_s^{(i_s)} - x_s^{(j_s)}\right) \tag{6}$$

Determine a result of action of each addend $\nabla^{|\eta|}$ on the function

$$\varphi = \varphi(x) \overset{def}{=} a\left(\hat{x}_\delta\right) x_\delta^{\alpha_s}$$

For $\eta = \oslash$

$$\nabla^{|\oslash|} a\left(\hat{x}_\delta\right) x_\delta^{\alpha_s} = \left(\hat{x}_\delta\right) x_\delta^{\alpha_s} \left(\Pi x\right)_{\overline{\mathcal{D}}},$$

where $(\Pi x)_{\overline{\mathcal{D}}} \overset{def}{=} \prod_{s \in \overline{\eta}} \prod_{1 \leq i_s < j_s \leq \mu_s}\left(x_s^{(i_s)} - x_s^{(j_s)}\right)$.

For all values $|\eta|, \eta \neq \oslash$

$$\nabla_I^{|\eta|} \varphi \overset{def}{=} \sum_{\substack{\eta \subset \mathcal{D} \\ \delta_1 \in \eta}} \sum_{k_\eta = 1} (-1)^{|k_\eta| - |\overline{\eta}| + |\eta|} \varphi\left(\hat{x}_\eta, x_\eta^{(k_\eta)}\right)\left(\prod_{\neq k} x\right)_\eta (\Pi x)_{\overline{\mathcal{D}} \setminus \overline{\eta}},$$

where $\left(\prod_{\neq k} x\right)_\eta \overset{def}{=} \prod_{s \in \overline{\eta}} \prod_{0 \leq i_s < j_s \leq \mu_s}\left(x_s^{(i_s)} - x_s^{(j_s)}\right)$.

Using (5) we get:

$$\nabla_I^{|\eta|}\varphi \overset{def}{=} \sum_{\substack{\eta\subset\mathcal{D}\\ \delta_1\in\eta}} \left[\sum_{k_{\delta_1}}^{\mu_{s_1}} (-1)^{\left|k_{\delta_1}\right|-|\delta_1|+1} \left(x_{\delta_1}^{\left(k_{\delta_1}\right)}\right)^{\alpha_{\delta_1}} \left(\prod_{\neq k} x\right)_{\delta_1}\right]\times$$

$$\times\left[\sum_{k_{\eta\setminus\delta_1}=1}^{\mu_{\bar\eta\setminus\delta_1}} (-1)^{\left|k_{\bar\eta\setminus\delta_1}\right|-|\bar\eta\setminus\delta_1|+|\eta\setminus\delta_1|}\, a\left(\hat{x}_\eta, x_{\bar\eta\setminus\delta_1}^{\left(k_{\bar\eta\setminus\delta_1}\right)}\right)^{\alpha_{\delta_1}} \left(\prod_{\neq k} x\right)_{\bar\eta\setminus\delta_1}\right]\times$$

$$\times\left(\Pi x\right)_{\overline{\mathcal{D}}\setminus\bar\eta} \overset{def}{=} -\sum_{\substack{\eta\subset\mathcal{D}\\ \delta_1\in\eta}} x_{\delta_1}^{\alpha_{\delta_1}}\,[]_1 \left(\Pi x\right)_{\overline{\mathcal{D}}\setminus(\bar\eta\setminus\delta_1)}.$$

Further we introduce the denotation

$$V_1 \overset{def}{=} \delta_1\cap\left(\delta_{\eta_1}\cup\ldots\cup\delta_{\eta_{|\eta|}}\right),$$

$$V_2 \overset{def}{=} \left(\delta_{\eta_1}\cup\ldots\cup\delta_{\eta_{|\eta|}}\right)\setminus\delta_1;$$

$$V_3 \overset{def}{=} \delta_1\setminus\left(\delta_{\eta_1}\cup\ldots\cup\delta_{\eta_{|\eta|}}\right).$$

For all values $|\eta|\,,\eta\neq\oslash$,

$$\nabla_{II}^{|\eta|}\varphi \overset{def}{=} \sum_{\substack{\eta\subset\mathcal{D}\\ \delta_1\notin\eta}} \sum_{k_{\delta_1}}^{\mu_{s_1}} (-1)^{|k_\eta|-|\bar\eta|+|\eta|}\,\varphi\left(\hat{x}_\eta, x_\eta^{(k_\eta)}\right)\left(\prod_{\neq k} x\right)_\eta \left(\Pi x\right)_{\overline{\mathcal{D}}\setminus\bar\eta} =$$

$$= \sum_{\substack{\eta\subset\mathcal{D}\\ \delta_1\in\eta}} \left[\sum_{k_{\delta_1}}^{\mu_{s_1}} (-1)^{\left|k_{V_1}\right|-\left|\overline{V_1}\right|} x_{V_1}^{\left(k_{V_1}\right)} \left(\prod_{\neq k} x\right)_{M_1}\right]\times$$

$$\times\left[\sum_{k_{\delta_1}}^{\mu_{V_2}} (-1)^{\left|k_{V_2}\right|-\left|\overline{V_2}\right|+|V_2|}\, a\left(\hat{x}_\delta, x_{V_2}^{\left(k_{\delta_2}\right)}\right)\left(\prod_{\neq k} x\right)_{V_2}\right]_2\times$$

$$\times x_{V_3}^{\alpha_{V_3}}\left(\Pi x\right)_{V_3}\cdot\left(\Pi x\right)_{\overline{\mathcal{D}}\setminus(\bar\eta\cup\delta_1)} = \sum_{\substack{\eta\in\mathcal{D}\\ \delta_1\notin\bar\eta}} x_{\delta_1}^{\alpha_{\delta_1}}\,[]_2\left(\left(\Pi x\right)_{\overline{\mathcal{D}}\setminus\bar\eta}\right).$$

For each $\oslash\neq\eta\subset\mathcal{D}$

$$\sum_{\eta\subset\mathcal{D}} = \sum_{\substack{\eta\subset\mathcal{D}\\ \delta_1\in\eta}} + \sum_{\substack{\eta\subset\mathcal{D}\\ \delta_1\notin\eta}} \tag{7}$$

By (6) we have

$$\nabla^{|\eta|}\varphi = \nabla_I^{|\eta|}\varphi + \nabla_{II}^{|\eta|}\varphi$$

Assuming $\|\eta\| = 1,...,|\mathcal{D}|-1$ we can get

$$\nabla_{II}^{|\oslash|}\varphi = -\nabla_I^{|\eta|+1}\varphi;\quad \nabla_{II}^{|\mathcal{D}|-1}\varphi = \nabla^{|\mathcal{D}|}\varphi.$$

By (5)these relations lead us to the following statement

$$\nabla^*_{\mathcal{D}_{\mu,\theta}} a\left(\hat{x}_\delta\right) x_\delta^{\alpha_s} = 0.$$

Finally, using linearity of the operator $\nabla_{\mathcal{D}_{\mu,\theta}} f$ and relation (6) almost for all $(x,\theta) \in R^{n+\sum\limits_{i=1}^{|\overline{\mathcal{D}}|}\mu_i}$ we get

$$\nabla^*_{\mathcal{D}_{\mu,\theta}} \Pi_\mu^{\mathcal{D}}(x) = \sum_{s=1}^{|\overline{\mathcal{D}}|} \nabla_{\mathcal{D}_{\mu,\theta}} P_{\mu_{\delta_s}}(x) = \sum_{s=1}^{|\mathcal{D}|} + \sum_{\alpha_{\delta_s}=0}^{\mu_{\delta_s}-1} \nabla_{\mathcal{D}_{\mu,\theta}} a\left(\hat{x}_\delta\right) x_\delta^{\alpha_s} = 0.$$

Necessity is proved.

Relation almost for all $\theta \in R^{\sum\limits_{i=1}^{|\overline{\mathcal{D}}|}\mu_i}$,$\nabla_{\mathcal{D}_{\mu,\theta}} f = 0 \Rightarrow f = \Pi_\mu^{\mathcal{D}}(x)$ is proved by the determination $\nabla^*_{\mathcal{D}_{\mu,\theta}} f$ and quasipolynomial. Now, let $\nabla^*_{\mathcal{D}_{\mu,\theta}} f = 0$. By (5)

$$\nabla^{|\oslash|} f + \sum_{|\eta|=1}^{|\overline{\mathcal{D}}|} \nabla^{|\eta|} f = 0$$

or

$$f(x) \prod_{s\in\overline{\mathcal{D}}} \mu_s! h_s^{\mu_s-1} + \sum_{|\eta|=1}^{|\overline{\mathcal{D}}|} \sum_{\eta\subset\mathcal{D}} \sum_{k_\eta=1}^{\mu_\eta} (-1)^{|\mu_\eta|-|k_\eta|+|\eta|} f\left(\breve{x}_\eta^{(k_\eta)}\right) \prod_{s\in\overline{\eta}} k_s \binom{\mu_s}{k_s} \times$$

$$\times \prod_{\tau_s=1}^{\mu_s} \left(x_s - x_s^{(\tau_s)}\right) \prod_{s\in\overline{\mathcal{D}}/\overline{\eta}} \mu_s! h_s^{\mu_s-1} = 0 \Rightarrow$$

$$f(x) \prod_{s\in\overline{\mathcal{D}}/\overline{\eta}} \mu_s! h_s^{\mu_s-1} = \Pi_\mu^{\mathcal{D}}(x) + \sum_{|\eta|=2}^{|\overline{\mathcal{D}}|} \nabla^{|\eta|} f. \tag{8}$$

where the first addend is a quasipolynomial determined in (1).

It is clear from determination of $\nabla^{|\eta|} f$ that for $|\eta| \geq 2$ each addend in the second sum of (8) is a partial case of any $P_{\mu_\delta}(x)$, $\delta \in \mathcal{D}$. Therefore, relation (8) allows that $f(x)$ is a generalized quasipolynomial of order $\mu_{\mathcal{D}} - 1$. Theorem 1 is proved.

Theorem 2. *The best approximation of the function $f \in L_{\bar{p}}(Q)$ $0 < \bar{p} < \infty$ by means of the set of quasipolynomials $\Pi_\mu^{\mathcal{D}}(x)$ may be lower estimated in the following way*

$$c^*\left(\mu_{\mathcal{D}}, p\right) \sup_\theta \left\| \nabla^*_{\mathcal{D}_{\mu,\theta}} f \right\|_{L_{\bar{p}}(Q)} \leq E_{\bar{p}}(f, \Pi) \tag{9}$$

where $\left\{\nabla^*_{\mathcal{D}_{\mu,\theta}}\right\}$ is an exact annihilator of a class of quasipolynomials, and

$$c^*\left(\mu_{\mathcal{D}}, p\right) = \left(\prod_{s\in\overline{\mathcal{D}}} k_s \binom{\mu_s}{k_s} \sum_{\eta\subset\bar{D}} \prod_{s\in\overline{\eta}} \mu_s \right)^{-\gamma}; \qquad \gamma = \max\left(1, \frac{1}{p}\right).$$

Proof of theorem 2. By theorem 1 the family $\left\{\nabla^*_{\mathcal{D}_{\mu,\theta}} f\right\}$ is of exact annihilator of class $\Pi^{\mathcal{D}}_{\mu}(x)$. Then for any $\Pi^{\mathcal{D}}_{\mu}(x) \in \Pi$ almost for all $\theta \in R^{\sum_{i=1}^{|\overline{\mathcal{D}}|} \mu_i}$, $\nabla^*_{\mathcal{D}_{\mu,\theta}} f\ \Pi^{\mathcal{D}}_{\mu}(x) = 0$ by the fact that the operators ∇ are linear, we have

$$\left\|\nabla^*_{\mathcal{D}_{\mu,\theta}} f\ \right\|_{\bar{p}} = \left\|\nabla^*_{\mathcal{D}_{\mu,\theta}} f\ - \nabla^*_{\mathcal{D}_{\mu,\theta}} \Pi^{\mathcal{D}}_{\mu}(x)\right\|_{\bar{p}} = \left\|\nabla^*_{\mathcal{D}_{\mu,\theta}} \left(f\ - \Pi^{\mathcal{D}}_{\mu}(x)\right)\right\|_{\bar{p}}.$$

Further

$$\left\|\nabla^*_{\mathcal{D}_{\mu,\theta}} f\ \right\| \le c^*(\mu_{\mathcal{D}}, p)\left\|f\ - \Pi^{\mathcal{D}}_{\mu}(x)\right\|.$$

The right hand side of the inequality is independent of θ, that gives

$$c^*(\mu_{\mathcal{D}}, p)\sup_{\theta}\left\|\nabla^*_{\mathcal{D}_{\mu,\theta}} f\right\|_{\bar{p}} \le \left\|f\ - \Pi^{\mathcal{D}}_{\mu}(x)\right\|_{\bar{p}}.$$

And the left hand side of the inequality is independent of $\Pi^{\mathcal{D}}_{\mu}(x)$ therefore. F

From the last relation we get

$$c^*(\mu_{\mathcal{D}}, p)\sup_{\theta}\left\|\nabla^*_{\mathcal{D}_{\mu,\theta}} f\right\|_{\bar{p}} \le \inf_{\Pi^{\mathcal{D}}_{\mu}(x)}\left\|f\ - \Pi^{\mathcal{D}}_{\mu}(x)\right\|_{\bar{p}}$$

whence

$$c^*(\mu_{\mathcal{D}}, p)\sup_{\theta}\left\|\nabla^*_{\mathcal{D}_{\mu,\theta}} f\right\|_{\bar{p}} \le E(\ f, \Pi)_{L_{\bar{p}}(Q)}$$

$$c^*(\mu_{\mathcal{D}}, p)\sup_{\theta}\left\|\nabla^*_{\mathcal{D}_{\mu,\theta}} f\right\|_{\bar{p}} \le E\left[\ f, \Pi^{\mathcal{D}}_{\mu}(x)\right]_{\bar{p}}$$

Theorem 2 is proved.

Theorem 3. *The best approximation of the function $f \in L_{\bar{p}}(Q), 0 < \bar{p} < \infty$ by means of the of quasipolynomials $\Pi^{\mathcal{D}}_{\mu}(x)$ may be upper estimated in the following way*

$$E_{\bar{p}(f,\Pi)} \le B^*(\mu_{\bar{D}})\sup_{h}\left\|\nabla^*_{\mathcal{D}_{\mu,\theta}} f\right\|_{L_{\bar{p}}(Q)} \tag{10}$$

where

$$B^*(\mu_{\bar{D}}) = \prod_{s\in\overline{\mathcal{D}}}(\mu_s!)^{-1}(\mu_s - 1)^{\mu_s - 1}.$$

Proof. We write relation (5) in the form

$$\nabla_{\mathcal{D}_{\mu,\theta}} f = \nabla^{|\oslash|} f + \sum_{|\eta|=1}^{|\overline{\mathcal{D}}|}\nabla^{|\eta|} f$$

and allowing for the structure of the operator $\nabla_{\mathcal{D}_{\mu,\theta}}$ we can easily see that almost for each fixed variable $x^{(i)}_s$ it holds the representation

$$\nabla_{\mathcal{D}_{\mu,\theta}} f = f(x)\prod_{s\in\overline{\mathcal{D}}}\mu_s! h_s^{\mu_s - 1} + \Pi^{*\mathcal{D}}_{\mu}(x),$$

where $\Pi_{\mu}^{*\mathcal{D}}(x)$ is finite quasipolynomial of order $\mu_{\mathcal{D}}-1$. Taking this into account we have

$$\left\|\nabla_{\mathcal{D}_{\mu,\theta}}^{*}f\right\|_{L_{\bar{p}}(Q)}=\left\|f(x)\cdot\prod_{s\in\mathcal{D}}\mu_s!h_s^{\mu_s-1}+\Pi_{\mu}^{*\mathcal{D}}(x)\right\|_{L_{\bar{p}}(Q)}\geq$$

$$\geq\inf_{\Pi_{\mu}^{\mathcal{D}}\in L_{\bar{p}}(Q)}\left\|f(x)\cdot\prod_{s\in\mathcal{D}}\mu_s!h_s^{\mu_s-1}-\Pi_{\mu}^{*\mathcal{D}}(x)\right\|_{L_{\bar{p}}(Q)}=$$

$$=E\left[f(x)\cdot\prod_{s\in\mathcal{D}}\mu_s!h_s^{\mu_s-1},\Pi\right]_{L_{\bar{p}}(Q)}.$$

Since for an arbitrary constent

$$E\left[cf(x),\Pi\right]_{L_{\bar{p}}(Q)}=cE\left[f(x),\Pi\right]_{L_{\bar{p}}(Q)},$$

we above-mentioned relation allows to write

$$\left\|\nabla_{\mathcal{D}_{\mu,\theta}}^{*}f\right\|_{L_{\bar{p}}(Q)}\geq\left|\prod_{s\in\overline{\mathcal{D}}}\mu_s!h_s^{\mu_s-1}\right|E\left[f(x),\Pi\right]_{L_{\bar{p}}(Q)}$$

whence if follows that or

$$\sup_h\prod_{s\in\overline{\mathcal{D}}}\mu_s!h_s^{\mu_s-1}E\left[f(x),\Pi\right]_{L_{\bar{p}}(Q)}\leq\sup_h\left\|\nabla_{\mathcal{D}_{\mu,\theta}}^{*}f\right\|_{L_{\bar{p}}(Q)}\Rightarrow$$

$$\prod_{s\in\overline{\mathcal{D}}}\mu_s!\frac{1}{(\mu_s-1)^{\mu_s-1}}E\left[f(x),\Pi\right]_{L_{\bar{p}}(Q)}\leq\sup_h\left\|\nabla_{\mathcal{D}_{\mu,\theta}}^{*}f\right\|_{L_{\bar{p}}(Q)}$$

or

$$E\left[f(x),\Pi\right]_{L_{\bar{p}}(Q)}\leq\prod_{s\in\overline{\mathcal{D}}}\frac{(\mu_s-1)^{\mu_s-1}}{\mu_s!}\sup_h\left\|\nabla_{\mathcal{D}_{\mu,\theta}}^{*}f\right\|_{L_{\bar{p}}(Q)}=$$

$$=B^{*}(\mu_{\bar{D}})\sup_h\left\|\nabla_{\mathcal{D}_{\mu,\theta}}^{*}f\right\|_{L_{\bar{p}}(Q)}\tag{11}$$

The theorem is proved.

The order of the best approximation of the function $f\in L_{\bar{p}}(Q)$ by the set of quasipolynomials $\Pi_{\mu}^{\mathcal{D}}(x)$ namely

$$E\left[f,\Pi_{\mu}^{\mathcal{D}}\right]_{L_{\bar{p}}(Q)}\asymp\sup_h\left\|\nabla_{\mathcal{D}_{\mu,\theta}}^{*}f\right\|_{L_{\bar{p}}(Q)}.\tag{12}$$

Follows from theorem 2 and 3.

References.

[1]. Babaev M.-B.A. *Estimates of the best approximation by the quasipolynomials.* Proc. If Saratov winter school 2., 1984, part 1, pp.91-98. (Russian)

[2]. Babaev A.M.-B. *On the lower bound estimation of the best approximation.* Proceedings of Institute Mathematics and Mechanics, vol. 17, Baku, 2002, pp.28-32.

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