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ASYMPTOTIC BEHAVIOR OF THE COUNTABLE DIMENSIONAL RENEWAL FUNCTION

(This paper is devoted to memory of our teacher prof. V.M.Surenkov)

Abstract

In this paper one fundamental results of the classik renewal theory is proved for countable dimensional renewal equation for lower semi-matrices.

The necessity in investigating transition processes for renewal equation appeared as a result of studies about branch processes close to critical, limit problems for random walk. The investigation of transition processes started with works of D.Silvestrov [1], O.Vjuhin [2], V.Shurenkov [3]. P.Kutsija [4] investigated these processes for equation of multidimensional renewal with matrix of absolute mass measure close to identity matrix that is, separable as much as possible .

The question about the properties of transition process for equation of countable dimensional renewal with matrix of absolute mass measure close to identity matrix that is, as much as possible separable remained open.

Here we investigate asymptotic behavior of countable dimensional renewal function.

Let $G^{\varepsilon}_{ij}(dx),\ i,j\in\mathbb{N}$, - be the family of complex measures on $\mathbb{R}_+$ such that $G^{\varepsilon}_{ij}(dx)=0$ for $j>i$, $V^{\varepsilon}_{ij}(dx)$ – variation of the measure $G^{\varepsilon}_{ij}(dx)$ and $V_{\varepsilon}(dx)=(V^{\varepsilon}_{ij}(dx))^{\infty}_{i,j=1}$, $G_{\varepsilon}(dx)=(G^{\varepsilon}_{ij}(dx))^{\infty}_{i,j=1}$.

Define $H_{\varepsilon}(dx)=\sum\limits_{k=0}^{\infty}G^{k*}_{\varepsilon}(dx)$ the renewal matrix, where $G^{0*}_{\varepsilon}([0,x])=I\ (x\geq 0)$, and $G^{(k+1)*}_{\varepsilon}([0,x])=G^{k*}_{\varepsilon}*G_{\varepsilon}([0,x])=\int\limits_0^x G^{k*}_{\varepsilon}([0,x-y])G_{\varepsilon}(dy)$.

Denote $m_i=\int\limits_0^{\infty}xG_{ii}(dx)$ and assume that measures $G^{\varepsilon}_{ij}(dx)$ satisfy the following conditions: weak limit of $G^{\varepsilon}_{ij}(dx)$ is equal to $G_{ij}(dx)$ when $\varepsilon\to 0$, that is

$$G^{\varepsilon}_{ij}(dx)\underset{\varepsilon\to 0}{\longrightarrow}G_{ij}(dx); \tag{1}$$

$$G_{ij}(dx)=0\ (i\neq j),\ G_{ii}(dx)\geq 0,\ G_{ii}([0,\infty))=1; \tag{2}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(I-V_{\varepsilon}([0,\infty)))=D=\{d_{ij}\}^{\infty}_{i,j=1},\ d_{ij}<\infty,\ i,j\in\mathbb{N}; \tag{3}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(I-G_{\varepsilon}([0,\infty)))=C=\{c_{ij}\}^{\infty}_{i,j=1},\ c_{ij}<\infty,\ i,j\in\mathbb{N}; \tag{4}$$

$$\sup_{\varepsilon>0}\int\limits_T^{\infty}xV^{\varepsilon}_{ij}(dx)\underset{T\to\infty}{\longrightarrow}0. \tag{5}$$

$$\inf_{i\in\mathbb{N}} m_i > 0, \ \sup_{i\in\mathbb{N}} m_i < \infty. \tag{6}$$

$$\inf_{i\in\mathbb{N}} \sum_{j=1}^{\infty} d_{ij} > 0. \tag{7}$$

Note that condition (5) implies that $0 < m_i < \infty$ for all $i \in \mathbb{N}$. Let $G(dx) = (G_{ij}(dx))_{i,j=1}^{\infty}$, $V(dx) = (V_{ij}(dx))_{i,j=1}^{\infty}$, $M = diag\{m_1, m_2, ...\}$.

Define by L_1 the space of integrable on $(-\infty;\infty)$ complex functions and $\widehat{g}(\lambda)$ Fourier transformation of function $g \in L_1$ that is $\widehat{g}(\lambda) = \int\limits_{-\infty}^{\infty} e^{i\lambda x} g(x)dx$ and let $\Phi_\varepsilon(\lambda) = \int\limits_0^{\infty} e^{i\lambda x} G_\varepsilon(dx)$, $\Psi_\varepsilon(\lambda) = \int\limits_0^{\infty} e^{i\lambda x} V_\varepsilon(dx)$, $\Phi(\lambda) = \int\limits_0^{\infty} e^{i\lambda x} G(dx)$.

Lemma 1. *Let conditions (1)-(6) hold and let matrix D be positive definite, then for all sufficiently small $\varepsilon > 0$ and for any bounded function $g \in L_1$ such that $\widehat{g}(\lambda) = 0$, for $\lambda \notin [a,b]$ it holds*

$$g * H_\varepsilon(x) = \frac{1}{2\pi} \int\limits_a^b e^{-i\lambda x} \widehat{g}(\lambda)[I - \Phi_\varepsilon(\lambda)]^{-1} d\lambda. \tag{8}$$

Proof. Let $\sigma_1, \sigma_2, ...$ be the eigenvalues of matrix D. Positive definiteness of matrix D implies that for all $k \in \mathbb{N}$ there exists $\sigma > 0$ such that $Re\sigma_k \geq \sigma$. This fact implies that all eigenvalues of matrix e^{-D} lies in a circle with radius $e^{-\sigma} < 1$. Let us show that for all sufficiently small $\varepsilon > 0$ matrix series $\sum\limits_{k=0}^{\infty} [\Phi_\varepsilon(\lambda)]^k$ is convergent absolutely and uniformly on λ.

Taking into account condition (3) we get $|\Phi_\varepsilon(\lambda)^k| \leq \Psi_\varepsilon(0)^k = (I - \varepsilon D + \circ(\varepsilon))^k$, $\varepsilon \to 0$. Let us choose $k = k_\varepsilon = [\frac{1}{\varepsilon}]$. Then $(I - \varepsilon D + \circ(\varepsilon))^{k_\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} e^{-D}$ that is $(\forall \delta > 0)(\exists \varepsilon(\delta) > 0)(\forall \varepsilon \in (0; \varepsilon(\delta)))\ (\forall \lambda \in \mathbb{R})[|\Psi_\varepsilon(\lambda)^{k_\varepsilon}| \leq (1+\delta) \cdot e^{-D}]$. Let us choose $\delta > 0$ such that $r = (1+\delta)e^{-\sigma} < 1$ and let $\varepsilon > 0$ such that previous inequality holds. The fact that all eigenvalues of matrix e^{-D} lie in a circle with radius $e^{-\sigma}$ implies that all eigenvalues of the matrix $\Phi_\varepsilon(\lambda)^{k_\varepsilon}$ have a form $\rho_j^{k_\varepsilon}$, where ρ_j $(j \in \mathbb{N})$, eigenvalues of matrix $\Phi_\varepsilon(\lambda)$ which lie in a circle with radius $r^{\frac{1}{k_\varepsilon}} < 1$. Hence for sufficiently small $\varepsilon > 0$ spectral radiuses of matrices $\Phi_\varepsilon(\lambda)$ are uniformly on λ smaller than one and this fact proves necessary convergent of the series $\sum\limits_{k=0}^{\infty} [\Phi_\varepsilon(\lambda)]^k$. The sum of the series is equal to $(I - \Phi_\varepsilon(\lambda))^{-1}$.

Using the property of Fourier transformation we get $\int\limits_{-\infty}^{\infty} e^{i\lambda x} g * G_\varepsilon^{k*}(x)dx = \widehat{g}(\lambda) \cdot \Phi_\varepsilon^k(\lambda)$. Under existing conditions on function g we get that $\widehat{g}(\lambda) \cdot \Phi_\varepsilon^k(\lambda)$ is continuous and $\widehat{g}(\lambda) \cdot \Phi_\varepsilon^k(\lambda) = 0$ for $\lambda \notin [a;b]$. Hence the function $\widehat{g}(\lambda) \cdot \Phi_\varepsilon^k(\lambda)$ is absolutely integrable. Taking the sum of both sides of the equality $g * G_\varepsilon^{k*}(x) = \frac{1}{2\pi} \int\limits_a^b e^{-i\lambda x} \widehat{g}(\lambda) \Phi_\varepsilon^k(\lambda) d\lambda$ completes the proof.

Let us denote the Banach space $\mathbf{K}$ of matrix functions $Q(x)=(Q_{ij}(x))_{i,j=1}^{\infty}$ with norm $|||Q|||=\int\limits_{-\infty}^{\infty}\|Q(x)\|dx<\infty$, where $\|A\|=\sup\limits_{i\in\mathbb{N}}\sum\limits_{j=1}^{\infty}|a_{ij}|$ is a norm of matrix A. For $Q\in\mathbf{K}$ define $\widehat{Q}(\lambda)=\int\limits_{-\infty}^{\infty}e^{i\lambda x}Q(x)dx$.

Lemma 2. *Let the sequence $\{Q_n\}_{n=1}^{\infty}\subset\boldsymbol{K}$ be convergent in norm $|||\cdot|||$ to $Q\in\boldsymbol{K}$ and for $\lambda\in[a;b]$ there exists $\widehat{Q}(\lambda)^{-1}$. Then there exists the sequence $\{F_n\}_{n=1}^{\infty}\subset\boldsymbol{K}$ such that for all sufficiently large n and for $\lambda\in[a;b]$*

$$|||F_n-F|||\underset{n\to\infty}{\longrightarrow}0,\ \ \widehat{Q}(\lambda)^{-1}=\widehat{F}(\lambda),\ \ \widehat{Q}_n(\lambda)^{-1}=\widehat{F}_n(\lambda).$$

The proof of Lemma 2 is analogical to finite dimensional case([4, p. 32-37]).

Denote

$$M_\varepsilon=\int\limits_0^{\infty}xG_\varepsilon(dx),\ \ L_\varepsilon=M_\varepsilon^{-1}(I-G_\varepsilon([0;\infty))),$$

$$Q_\varepsilon(x)=e^{-xL_\varepsilon}-\int\limits_0^{x}G_\varepsilon(dy)e^{-(x-y)L_\varepsilon}.$$

Lemma 3. *Let $G_{ij}^{\varepsilon}(dx)$ satisfy the conditions (1)-(7). Then $Q_\varepsilon(x)\in\boldsymbol{K}$ and it holds that $Q_\varepsilon(x)\underset{\varepsilon\to0}{\longrightarrow}Q(x)=V([x;\infty))$ in the norm of $\boldsymbol{K}$.*

Proof. The conditions (6) and (7) imply that $\sigma=\inf\limits_{i}\sum\limits_{j=1}^{\infty}\frac{d_{ij}}{m_i}>0$, therefore $\|e^{-tM^{-1}D}\|\le e^{-t\sigma}$. Since $\lim\limits_{\varepsilon\to0}\|e^{-x\frac{1}{\varepsilon}L_\varepsilon}\|=\|e^{-xM^{-1}D}\|$ then using previous inequality we get that $\|e^{-xL_\varepsilon}\|\le e^{-\varepsilon x\sigma}$. Next,

$$\int\limits_0^{+\infty}\left(\int\limits_0^{x}V_\varepsilon(dy)e^{-(x-y)L_\varepsilon}\right)dx=\int\limits_0^{+\infty}V_\varepsilon(dy)\int\limits_y^{+\infty}e^{-(x-y)L_\varepsilon}dx=$$

$$=\int\limits_0^{+\infty}V_\varepsilon(dy)e^{yL_\varepsilon}\left(e^{-xL_\varepsilon}L_\varepsilon^{-1}\Big|_{+\infty}^{y}\right)=\int\limits_0^{+\infty}L_\varepsilon^{-1}V_\varepsilon(dy)=V_\varepsilon([0,+\infty))L_\varepsilon^{-1}.$$

Previous equality and the fact that $\|e^{-xL_\varepsilon}\|\le e^{-\varepsilon x\sigma}$ imply that $Q_\varepsilon(x)\in\mathbf{K}$.

Denote $\overline{V}_\varepsilon(y)=V_\varepsilon([y;\infty))$. Then $d\overline{V}_\varepsilon(y)=-V_\varepsilon(dy)$ and

$$Q_\varepsilon(x)=e^{-xL_\varepsilon}+\int\limits_0^{x}d\overline{V}_\varepsilon(y)\cdot e^{-(x-y)L_\varepsilon}=e^{-xL_\varepsilon}+\overline{V}_\varepsilon(y)e^{-(x-y)L_\varepsilon}|_0^x-$$

$$-\int\limits_0^{x}\overline{V}_\varepsilon(y)e^{-(x-y)L_\varepsilon}dyL_\varepsilon=-\overline{V}_\varepsilon(0)e^{-xL_\varepsilon}-\int\limits_0^{x}\overline{V}_\varepsilon(y)e^{-(x-y)L_\varepsilon}dyL_\varepsilon+$$

$$+e^{-xL_\varepsilon}+\overline{V}_\varepsilon(x)=\overline{V}_\varepsilon(x)+(I-\overline{V}_\varepsilon(0))e^{-xL_\varepsilon}-$$

$$-\int_0^x \overline{V}_\varepsilon(y)e^{-(x-y)L_\varepsilon}L_\varepsilon dy = \overline{V}_\varepsilon(x) + N_\varepsilon L_\varepsilon e^{-xL_\varepsilon} - \int_0^x \overline{V}_\varepsilon(y)e^{-(x-y)L_\varepsilon}L_\varepsilon dy.$$

Let us prove that

$$\overline{V}_\varepsilon(x)\underset{\varepsilon\to 0}{\longrightarrow}Q(x) = V([x;\infty)), \tag{9}$$

$$\int_0^\infty \|N_\varepsilon L_\varepsilon e^{-xl_\varepsilon} - \int_0^x \overline{V}_\varepsilon(y)e^{-(x-y)L_\varepsilon}L_\varepsilon dy\|dx\underset{\varepsilon\to 0}{\longrightarrow}0. \tag{10}$$

For $i \neq j$ it holds

$$\int_0^\infty \overline{V}^\varepsilon_{ij}(x)dx = \int_0^\infty V^\varepsilon_{ij}([x;\infty))dx = \int_0^\infty xV^\varepsilon_{ij}(dx) + xV^\varepsilon_{ij}([x;\infty))|_0^\infty =$$

$$= \int_0^c xV^\varepsilon_{ij}(dx) + \int_c^\infty xV^\varepsilon_{ij}(dx) \leq c\cdot V^\varepsilon_{ij}([0;c]) + \int_c^\infty xV^\varepsilon_{ij}(dx).$$

The conditions (3) and (5) give us $c\cdot V^\varepsilon_{ij}([0;c])\underset{\varepsilon\to 0}{\longrightarrow}0$ and $\int\limits_c^\infty xV^\varepsilon_{ij}(dx)\underset{c\to\infty}{\longrightarrow}0$.

For $i = j$

$$\int_0^\infty |\overline{V}^\varepsilon_{ii}(x) - \overline{V}_{ii}(x)|dx = \int_0^c |\overline{V}^\varepsilon_{ii}(x) - \overline{V}_{ii}(x)|dx + \int_c^\infty |\overline{V}^\varepsilon_{ii}(x) - \overline{V}_{ii}(x)|dx \leq$$

$$\leq \int_0^c |\overline{V}^\varepsilon_{ii}(x) - \overline{V}_{ii}(x)|dx + \int_c^\infty V^\varepsilon_{ii}([x;\infty))dx + \int_c^\infty V_{ii}([x;\infty))dx \leq$$

$$\leq \int_0^c |V^\varepsilon_{ii}([0,x]) - V_{ii}([0,x])|dx + c|1 - V^\varepsilon_{ii}([0,\infty))| + \int_c^\infty xV^\varepsilon_{ii}(dx) + \int_c^\infty xV_{ii}(dx).$$

The condition (5) implies that integrals $\int\limits_c^\infty xV^\varepsilon_{ii}(dx)$ and $\int\limits_c^\infty xV_{ii}(dx)$ are convergent to 0 uniformly on ε. The conditions (4) and (1) imply that $c|1 - V^\varepsilon_{ii}([0,\infty))|\underset{\varepsilon\to 0}{\longrightarrow}0$ and $|V^\varepsilon_{ii}([0,x]) - V_{ii}([0,x])|\underset{\varepsilon\to 0}{\longrightarrow}0$. And we get (9).

Let us denote

$$\overline{V}^c_\varepsilon(y) = \begin{cases} \overline{V}_\varepsilon(y), & y \leq c \\ 0, & y > c \end{cases}, \qquad \overline{R}^c_\varepsilon(y) = \begin{cases} 0, & y \leq c \\ \overline{V}_\varepsilon(y), & y > c \end{cases}.$$

We have that $\overline{V}_\varepsilon(y) = \overline{V}^c_\varepsilon(y) + \overline{R}^c_\varepsilon(y)$ and hence,

$$\int_0^\infty \left|N_\varepsilon L_\varepsilon e^{-xL_\varepsilon} - \int_0^x \overline{V}_\varepsilon(y)e^{-(x-y)L_\varepsilon}L_\varepsilon dy\right| dx \leq$$

$$\leq \int_0^\infty \left| N_\varepsilon L_\varepsilon e^{-xL_\varepsilon} - \int_0^x \overline{V}_\varepsilon^c(y) e^{-(x-y)L_\varepsilon} L_\varepsilon dy \right| dx + \int_0^\infty \left| \int_0^x \overline{R}_\varepsilon^c(y) e^{-(x-y)L_\varepsilon} L_\varepsilon dy \right| dx \leq$$

$$\leq \int_0^c \left| N_\varepsilon L_\varepsilon e^{-xL_\varepsilon} - \int_0^x \overline{V}_\varepsilon^c(y) e^{-(x-y)L_\varepsilon} L_\varepsilon dy \right| dx +$$

$$+ \int_c^\infty \left| N_\varepsilon L_\varepsilon e^{-xL_\varepsilon} - \int_0^x \overline{V}_\varepsilon^c(y) e^{-(x-y)L_\varepsilon} L_\varepsilon dy \right| dx + \int_c^\infty |\overline{V}_\varepsilon(y)| dy \int_0^\infty |e^{-xL_\varepsilon} L_\varepsilon| dx.$$

We can write $\int_0^\infty \|e^{-xL_\varepsilon} L_\varepsilon\| dx = \int_0^\infty \|e^{-\frac{x}{\varepsilon} L_\varepsilon} \frac{1}{\varepsilon} L_\varepsilon\| dx \leq \|\frac{1}{\varepsilon} L_\varepsilon\| \int_0^\infty e^{-\sigma x} dx = \frac{1}{\sigma} \cdot \|\frac{1}{\varepsilon} L_\varepsilon\| \underset{\varepsilon\to 0}{\longrightarrow} \frac{1}{\sigma} \|M^{-1} D\|$ and integral $\int_c^\infty |\overline{V}_\varepsilon(y)| dy$ can be made anyhow small with the choice of c. That is why $\int_c^\infty |\overline{V}_\varepsilon(y)| dy \int_0^\infty |e^{-xL_\varepsilon} L_\varepsilon| dx \to 0$ when $\varepsilon \to 0$.

Next,

$$\int_c^\infty \left| N_\varepsilon L_\varepsilon e^{-xL_\varepsilon} - \int_0^x \overline{V}_\varepsilon^c(y) e^{-(x-y)L_\varepsilon} L_\varepsilon dy \right| dx \leq$$

$$\leq \left| N_\varepsilon - \int_0^c \overline{V}_\varepsilon(y) e^{yL_\varepsilon} dy \right| \int_c^\infty \left| e^{-xL_\varepsilon} L_\varepsilon \right| dx \leq \left| N_\varepsilon - \int_0^c \overline{V}_\varepsilon(y) dy \right| \int_c^\infty \left| e^{-xL_\varepsilon} L_\varepsilon \right| dx$$

and analogically

$$\int_0^c \left| N_\varepsilon L_\varepsilon e^{-xL_\varepsilon} - \int_0^x \overline{V}_\varepsilon^c(y) e^{-(x-y)L_\varepsilon} L_\varepsilon dy \right| dx \leq$$

$$\leq \left| N_\varepsilon - \int_0^c \overline{V}_\varepsilon(y) e^{yL_\varepsilon} dy \right| \int_0^c \left| e^{-xL_\varepsilon} L_\varepsilon \right| dx \leq \left| N_\varepsilon - \int_0^c \overline{V}_\varepsilon(y) dy \right| \int_0^c \left| e^{-xL_\varepsilon} L_\varepsilon \right| dx.$$

Since $\int_0^\infty \overline{V}(y) dy = N$ and $|N_\varepsilon - \int_0^c \overline{V}_\varepsilon(y) dy| \underset{\varepsilon\to 0}{\longrightarrow} |N - \int_0^c \overline{V}(y) dy|$ we can find c such that (10) holds. Lemma 3 is proved.

Lemma 4. *Let $G_{ij}^\varepsilon(dx)$ satisfy conditions (1)-(7). If matrix $(\Phi(\lambda) - I)$ is invertible then for all $-\infty < a < b < \infty$ and for all sufficiently small $\varepsilon > 0$ there exist functions $F \in \boldsymbol{K}$, $F_\varepsilon \in \boldsymbol{K}$ such that for all $\lambda \in [a,b]$ it holds*

$$\widehat{F}_\varepsilon^{-1}(\lambda) = (I - \Phi_\varepsilon(\lambda))(L_\varepsilon - i\lambda I)^{-1},$$
$$\widehat{F}^{-1}(\lambda) = \frac{i}{\lambda}(I - \Phi(\lambda))$$
$$F_\varepsilon(x) \underset{\varepsilon\to 0}{\rightarrow} F(x)$$

in the norm of $\boldsymbol{K}$.

Proof. Using the proposition of Lemma 2 and Lemma 3 it is sufficient to show that the Fourier transformation of the matrix $Q(x) = G([x;\infty))$ is invertible and to find the explicit form of matrices $\widehat{Q}(\lambda)$ and $\widehat{Q}_\varepsilon(\lambda)$. The equality $\widehat{Q}(\lambda) = \int\limits_0^\infty e^{i\lambda x} G([x,\infty))dx = \int\limits_0^\infty e^{i\lambda x}dx \int\limits_x^\infty G(dy) = \int\limits_0^\infty G(dy) \int\limits_0^y e^{i\lambda x}dx = \int\limits_0^\infty G(dy)\frac{e^{i\lambda x}-1}{i\lambda} = \frac{1}{i\lambda}(\Phi(\lambda) - I)$ implies that the matrix $\widehat{Q}(\lambda)$ is invertible. Let us find the explicit form of matrices $\widehat{Q}_\varepsilon(\lambda)$.

$$\widehat{Q}_\varepsilon(\lambda) = \int\limits_0^\infty e^{i\lambda x} e^{-xL_\varepsilon}dx - \int\limits_0^\infty e^{i\lambda x}\left(\int\limits_0^x G_\varepsilon(dy)e^{-(x-y)L_\varepsilon}\right)dx =$$
$$(L_\varepsilon - i\lambda I)^{-1} - \Phi_\varepsilon(\lambda)(L_\varepsilon - i\lambda I)^{-1} = (I - \Phi_\varepsilon(\lambda))(L_\varepsilon - i\lambda I)^{-1}$$

and it completes the proof.

Theorem 1.*Let $G^\varepsilon_{ij}(dx)$ satisfy conditions (1)-(6), matrix D is positive definite and matrix $(\Phi(\lambda) - I)$ is invertible, where $\Phi(\lambda) = \int\limits_0^\infty e^{i\lambda x}G(dx)$. Then for any $s \geq 0$ it holds that*

$$\lim_{\substack{\varepsilon\to 0\ x\to\infty \\ \varepsilon x \to t}} H_\varepsilon([x; x+s]) = se^{-tM^{-1}C}M^{-1}.$$

Proof Let us assume that condition (7) holds. Lemma 4 implies that for $\lambda \in [a;b]$ it holds that $(I - \Phi_\varepsilon(\lambda))^{-1} = (L_\varepsilon - i\lambda I)^{-1}\cdot \widehat{Q}_\varepsilon(\lambda)^{-1} = (L_\varepsilon - i\lambda I)^{-1}\cdot \widehat{F}_\varepsilon(\lambda)$. Hence, using Lemma 1, equality $(L_\varepsilon - i\lambda I)^{-1} = \int\limits_0^\infty e^{i\lambda x}e^{-xL_\varepsilon}dx$ and properties of Fourier transformation ([5, p. 433]) we get

$$g * H_\varepsilon(x) = \int\limits_{-\infty}^x e^{-(x-y)L_\varepsilon}F_\varepsilon * g(y)dy.$$

Define the family of matrix functions

$$T_{\varepsilon,x}(y) = \begin{cases} e^{-(x-y)L_\varepsilon}, & y \leq x \\ 0, & y > x \end{cases}.$$

We can write $g * H_\varepsilon(x) = \int\limits_{-\infty}^\infty T_{\varepsilon,x}(y)F_\varepsilon * g(y)dy$. The inequality $\|e^{-xL_\varepsilon}\| \leq e^{-\varepsilon x\sigma}$ implies that $\|T_{\varepsilon,x}(y)\| \leq e^{-\varepsilon(x-y)\sigma} \leq 1$ for $y \leq x$, and we get $\sup\limits_{-\infty<y<\infty} \|T_{\varepsilon,x}(y)\| \leq 1$. Let $\varepsilon \to 0$, $x \to \infty$, $x\varepsilon \to t$ then $\limsup \|T_{\varepsilon,x}(y)\| \leq 1$, $T_{\varepsilon,x}(y) \to e^{-tM^{-1}C}$. Since $F_\varepsilon(x)$ converges to $F(x)$ in the norm of $\mathbf{K}$ that is $\int\limits_{-\infty}^\infty \|F_\varepsilon * g(y) - F * g(y)\|dy \underset{\varepsilon\to 0}{\longrightarrow} 0$, then for all band-limited functions $g \in L^1$ it holds that $g * H_\varepsilon(x) \to e^{-tM^{-1}C}\cdot \int\limits_{-\infty}^\infty F * g(y)dy = e^{-tM^{-1}C}M^{-1}\int\limits_{-\infty}^\infty g(y)dy$ when $\varepsilon \to 0,\ x \to \infty,\ \varepsilon x \to t$.

Let us show that for any $s \geq 0$ it holds that $\limsup \|H_\varepsilon([x;x+s])\| < \infty$ when $\varepsilon \to 0$, $x \to \infty$, $\varepsilon x \to t$. Define by $R_\varepsilon(dy)$ the renewal matrix of $V_\varepsilon(dy)$ that is $R_\varepsilon(dy) = \sum\limits_{k=0}^{\infty} V_\varepsilon^{k*}(dy)$. We get that $\|H_\varepsilon([x;x+s])\| \leq \|R_\varepsilon([x;x+s])\|$. Denote $\overrightarrow{v}_\varepsilon(x)$ vector function with coordinates $v_k^\varepsilon(x) = \sum\limits_{j=1}^{\infty} V_{kj}^\varepsilon([x;\infty))$. Then it holds that $R_\varepsilon * \overrightarrow{v}_\varepsilon(x) = \overrightarrow{1} - R_\varepsilon([0;x])(\overrightarrow{1} - \overrightarrow{v}_\varepsilon(0))$ and $\sum\limits_{j=1}^{\infty} V_{ij}^\varepsilon([0;\infty)) \leq 1$ for all sufficiently small $\varepsilon > 0$. Hence, the coordinates of the vector $\overrightarrow{v}_\varepsilon(0)$ are nonnegative. This fact implies that $R_\varepsilon * \overrightarrow{v}_\varepsilon(x) \leq \overrightarrow{1}$ for all $\varepsilon > 0$ and $x \geq 0$.

Condition (3) implies that $\lim\limits_{\varepsilon \to 0} \sum\limits_{j=1}^{\infty} V_{ij}^\varepsilon([0;\infty)) = 1$ for all $j \in \mathbb{N}$ then there exists $\delta > 0$ such that for all sufficiently small $\varepsilon > 0$ it holds that $v_k^\varepsilon(x) \geq \delta$ for $0 \leq x \leq \delta$. Using previous inequality and the fact that $R_\varepsilon * \overrightarrow{v}_\varepsilon(x) \leq \overrightarrow{1}$ we get $\delta R_\varepsilon([x;x+\delta])\overrightarrow{1} \leq R_\varepsilon * \overrightarrow{v}_\varepsilon(x) \leq \overrightarrow{1}$ for all $x \geq 0$ and for all sufficiently small $\varepsilon > 0$. It implies that $\limsup \|R_\varepsilon([x;x+\delta])\| \leq \frac{1}{\delta}$ when $\varepsilon \to 0$, $x \to \infty$, $\varepsilon x \to t$ and hence $\limsup \|R_\varepsilon([x;x+s])\| \leq \frac{s+\delta}{\delta^2}$ for all $s > \delta$

The fact that $\limsup \|H_\varepsilon([x;x+s])\| < \infty$ when $\varepsilon \to 0$, $x \to \infty$, $\varepsilon x \to t$ implies that $\limsup \|\chi * H_\varepsilon(x)\| < \infty$ for any bounded measurable scalar function $\chi(x)$ with compact carrier. Let us fix such function $\chi(x)$ and denote by $\widetilde{L}^1(\chi)$ the set of functions $g \in L^1$ for which

$$g * \chi * H_\varepsilon(x) \underset{\substack{\varepsilon \to 0\ x \to \infty \\ \varepsilon x \to t}}{\longrightarrow} e^{-tM^{-1}C} M^{-1} \int\limits_{-\infty}^{\infty} g(y)dy \int\limits_{-\infty}^{\infty} \chi(y)dy \tag{11}$$

holds.

The set $\widetilde{L}^1(\chi)$ contains all band-limited functions $g \in L^1$, is closed in the sense of convergent in norm of L^1 and is linear set, hence $\widetilde{L}^1(\chi)$ is a closed subspace of L^1 which contains a dense set which consists of band-limited functions, that is why $\widetilde{L}^1(\chi) = L^1$ or in other words previous formula holds for all $g \in L^1$ and for all bounded measurable functions $\chi(x)$.

Let a function $\chi(x)$ be infinitely differentiable and its carrier lies inside interval $(0;\infty)$, $l(y) = \begin{cases} e^{-y}, & y \geq 0 \\ 0, & y < 0 \end{cases}$. Then

$$\chi(x) = \int\limits_0^x \chi(x-y)e^{-y}dy + \int\limits_0^x \chi'(x-y)e^{-y}dy = l * \chi(x) + l * \chi'(x).$$

The formula (11) implies that

$$\chi * H_\varepsilon(x) = l * \chi * H_\varepsilon(x) + l * \chi' * H_\varepsilon(x) \underset{\substack{\varepsilon \to 0\ x \to \infty \\ \varepsilon x \to t}}{\longrightarrow} e^{-tM^{-1}C} M^{-1} \int\limits_0^{\infty} l(y)dy \times$$

$$\times \int\limits_0^{\infty} \chi(y)dy + e^{-tM^{-1}C} M^{-1} \int\limits_0^{\infty} l(y)dy \int\limits_0^{\infty} \chi'(y)dy = e^{-tM^{-1}C} M^{-1} \int\limits_0^{\infty} \chi(y)dy$$

for any infinitely differentiable function $\chi(x)$ with carrier inside interval $(0;\infty)$.

The indicator of interval $[1;1+s]$ can be approximate anyhow precisely in the norm of L^1 by means of infinitely differentiable functions with carrier inside interval $(0;\infty)$ that is for any $\delta>0$ there exist the pair of infinitely differentiable band-limited functions $\chi_-(x)$, $\chi_+(x)$ with carriers inside $(0;\infty)$ such that $\chi_-(x)\le 1_{[1;1+s]}(x)\le\chi_+(x)$ and $\int\limits_0^\infty|\chi_+(x)-\chi_-(x)|dx<\delta$. It implies that $\chi_\pm *H_\varepsilon(x)\xrightarrow[\varepsilon\to0\ x\to\infty\ \varepsilon x\to t]{} e^{-tM^{-1}C}M^{-1}\int\limits_0^\infty\chi_\pm(y)dy$ and $|\int\limits_0^\infty\chi_\pm(y)dy-s|<\delta$. Hence,

$$H_\varepsilon([x-1-s;x-1])=1_{[1;1+s]}*H_\varepsilon(x)=(1_{[1;1+s]}-\chi_-)*H_\varepsilon(x)+$$
$$+\chi_-*H_\varepsilon(x)\underset{\substack{\varepsilon\to0\ x\to\infty\\ \varepsilon x\to t}}{\to} se^{-tM^{-1}C}M^{-1}.$$

In order to avoid the condition (7) we do analogically to [4, p. 58]. Let us fix $k>0$ which we will choose later and define $U_\varepsilon(x)=e^{-k\varepsilon x}H_\varepsilon[x-s,x]$, $\widetilde{G}_\varepsilon(dy)=e^{-k\varepsilon y}G_\varepsilon(dy)$, $\widetilde{V}_\varepsilon(dy)=e^{-k\varepsilon y}V_\varepsilon(dy)$ matrix of variations of $\widetilde{G}_\varepsilon(dy)$ and $\widetilde{H}_\varepsilon(dy)$ renewal matrix of $\widetilde{G}_\varepsilon(dy)$. We get that $U_\varepsilon(x)=\int\limits_0^x e^{-k\varepsilon(x-y)}1_{[0,s]}(x-y)\widetilde{H}_\varepsilon(dy)$.

Let us show that matrices $\widetilde{G}_\varepsilon(dy)$ and $\widetilde{V}_\varepsilon(dy)$ satisfy the conditions (1)-(6).

Clearly that conditions (1) and (2) hold.

Let us consider

$$\widetilde{V}_\varepsilon([0,\infty))=\int\limits_0^\infty e^{-k\varepsilon y}V_{ij}^\varepsilon(dy)=V_{ij}^\varepsilon([0,\infty))-\int\limits_0^\infty(1-e^{-k\varepsilon y})V_{ij}^\varepsilon(dy)=$$
$$1-d_{ij}\varepsilon+\circ(\varepsilon)-\int\limits_0^\infty k\varepsilon yV_{ij}^\varepsilon(dy)+\int\limits_0^\infty(k\varepsilon y-1+e^{-k\varepsilon y})V_{ij}^\varepsilon(dy).$$

The condition (5) implies that $\int\limits_0^\infty yV_{ij}^\varepsilon(dy)\underset{\varepsilon\to0}{\to}m_i\delta_{ij}$, then

$$\widetilde{V}_\varepsilon([0,\infty))=1-\widetilde{d}_{ij}\varepsilon+\circ(\varepsilon)+\int\limits_0^\infty(k\varepsilon y-1+e^{-k\varepsilon y})V_{ij}^\varepsilon(dy),$$

where $\widetilde{d}_{ij}=d_{ij}+km_i\delta_{ij}$. We get

$$\int\limits_0^\infty(k\varepsilon y-1+e^{-k\varepsilon y})V_{ij}^\varepsilon(dy)=\int\limits_0^T(k\varepsilon y-1+e^{-k\varepsilon y})V_{ij}^\varepsilon(dy)+\int\limits_T^\infty(k\varepsilon y-1+e^{-k\varepsilon y})V_{ij}^\varepsilon(dy).$$

Since the function $k\varepsilon y-1+e^{-k\varepsilon y}$ is nonnegative and increasing we get

$$0\le\int\limits_0^T(k\varepsilon y-1+e^{-k\varepsilon y})V_{ij}^\varepsilon(dy)\le(k\varepsilon T-1+e^{-k\varepsilon T})V_{ij}^\varepsilon([0,\infty)),$$
$$0\le\int\limits_T^\infty(k\varepsilon y-1+e^{-k\varepsilon y})V_{ij}^\varepsilon(dy)\le k\varepsilon\int\limits_T^\infty yV_{ij}^\varepsilon(dy).$$

Hence,

$$\frac{1}{\varepsilon}\int_0^\infty (k\varepsilon y-1+e^{-k\varepsilon y})V_{ij}^\varepsilon(dy)\le\left(kT-\frac{1-e^{-k\varepsilon T}}{\varepsilon}\right)V_{ij}^\varepsilon([0,\infty))+k\int_T^\infty yV_{ij}^\varepsilon(dy).$$

If $T>0$ is fixed we get $\lim\limits_{\varepsilon\to0}(kT-\frac{1-e^{-k\varepsilon T}}{\varepsilon})V_{ij}^\varepsilon([0,\infty))=0$, the condition (5) implies that choosing $T>0$ the expression $\int\limits_T^\infty yV_{ij}^\varepsilon(dy)$ can be made anyhow small, that is

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}(\delta_{ij}-\widetilde{V}_{ij}^\varepsilon([0,\infty)))=\widetilde{d}_{ij},$$

and it implies that condition (3) holds. Clearly that positive definiteness of matrix D implies that matrix $\widetilde{D}=\{\widetilde{d}_{ij}\}$ is positive definite too.

Analogically it can be shown that the condition (4)

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}(\delta_{ij}-\widetilde{G}_{ij}^\varepsilon([0,\infty)))=\widetilde{c}_{ij}=c_{ij}+km_i\delta_{ij}.$$

holds.

Since $\widetilde{G}_{ij}^\varepsilon(dx)\underset{\varepsilon\to0}{\to}G_{ij}(dx)$, then the conditions (5) and (6) hold.

Using the fact that $\inf\limits_{i\in\mathbb{N}}m_i>0$ we get that there exists $k>0$ such that $\inf\limits_{i\in\mathbb{N}}\sum\limits_{j=1}^\infty\widetilde{d}_{ij}=\inf\limits_{i\in\mathbb{N}}\sum\limits_{j=1}^\infty(d_{ij}+km_i)>0$, hence condition (7) holds.

The facts proved earlier imply that

$$\widetilde{H}_\varepsilon([x,x+s])\underset{\substack{\varepsilon\to0\ x\to\infty\\ \varepsilon x\to t}}{\to}se^{-tM^{-1}\widetilde{C}}M^{-1},$$

where $\widetilde{C}=C+kM$ and $\limsup\|\widetilde{R}_\varepsilon([x,x+s])\|<\infty$, when $\varepsilon\to0$, $x\to\infty$, $\varepsilon x\to t$, where $\widetilde{R}_\varepsilon(dy)$ renewal matrix of $\widetilde{V}_\varepsilon(dy)$. Hence,

$$\widetilde{H}_\varepsilon([x,x+s])\underset{\substack{\varepsilon\to0\ x\to\infty\\ \varepsilon x\to t}}{\to}se^{-kt}e^{-tM^{-1}C}M^{-1}.$$

The facts that $U_\varepsilon(x)=\widetilde{H}_\varepsilon([x-s,x])-\int\limits_{x-s}^x(1-e^{-k\varepsilon(x-y)})\widetilde{H}_\varepsilon(dy)$ and

$$\left\|\int_{x-s}^x(1-e^{-k\varepsilon(x-y)})\widetilde{H}_\varepsilon(dy)\right\|\le(1-e^{-k\varepsilon s})\|\widetilde{R}_\varepsilon([x-s,x])\|\underset{\substack{\varepsilon\to0\ x\to\infty\\ \varepsilon x\to t}}{\to}0$$

imply that

$$U_\varepsilon(x)\underset{\substack{\varepsilon\to0\ x\to\infty\\ \varepsilon x\to t}}{\to}se^{-kt}e^{-tM^{-1}C}M^{-1}.$$

Using the equality $U_\varepsilon(x)=e^{-k\varepsilon x}H_\varepsilon[x-s,x]$ we get

$$H_\varepsilon([x-s,x])\underset{\substack{\varepsilon\to0\ x\to\infty\\ \varepsilon x\to t}}{\to}e^{kt}se^{-kt}e^{-tM^{-1}C}M^{-1}=se^{-tM^{-1}C}M^{-1},$$

and it completes the proof.

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