MATHEMATICS

Ziyatkhan S. ALIYEV

THE BASIS PROPERTIES IN THE SPACE L_2 OF THE SYSTEMS OF ROOT FUNCTIONS ON A FOURTH ORDER SPECTRAL PROBLEM

Abstract

In this paper the fourth order spectral problem with spectral parameter in the boundary condition is considered. This problem is interpreted as a problem on eigenvalues of some J-self-adjoined operator in Pontragin's space Π_1. The necessary and sufficient conditions of Riesz basicity in the space L_2 of the root functions system of considered problem are proved.

Let's consider the spectral problem

$$y^{(4)}(x) - \left(q(x)\, y'(x)\right)' = \lambda y(x), \qquad 0 < x < l \tag{1}$$

$$y'(0)\cos\alpha - y''(0)\sin\alpha = 0, \tag{2.a}$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{2.b}$$

$$y'(l)\cos\gamma + y''(l)\sin\gamma = 0, \tag{2.c}$$

$$(a\lambda + b)\, y(l) - (c\lambda + d)\, Ty(l) = 0, \tag{2.d}$$

where $Ty \equiv y''' - qy'$, $\lambda \in \mathbb{C}$ is a spectral parameter, q is positive and absolutly continuous function on interval $[0,\ l]$, $\alpha, \beta, \gamma, a\ , b, c, d$ are real constants, moreover, $\alpha, \beta, \gamma \in [0, \pi/2]$, $\sigma = bc - ad \neq 0$.

The present paper is dedicated to investigation of basis properties in space $L_2(0,l)$ of root functions systems of boundary value problem (1), (2).

As is known (see [1,2] and bibliography that available there), the spectral problem for the ordinary differential operator with spectral parameter that polynomially included in boundary conditions, is reduced by standard way to a spectral problem for the linear operator in suitable space. In particular, the considered problem (1), (2) is reduced to a problem on eigenvalues for the linear operator L in Hilbert space $H = L_2(0,l) \oplus \mathbb{C}$ with scalar product

$$(\hat{y}, \hat{u}) = (\{y, m\}, \{u, s\}) = (y, u)_{L_2} + |\sigma|^{-1} m\bar{s},$$

where $(\cdot,\cdot)_{L_2}$ is scalar product in $L_2(0,l)$,

$$L\hat{y} = L\{y, m\} = \left\{(Ty(x))', dTy(l) - by(l)\right\},$$

definitional domain

$$D(L) = \left\{\{y, m\} \in H :\ y(x) \in W_2^4(0,l),\ \ (Ty(x))' \in L_2(0,l)\right.$$

$$\left. y(x) \text{ satisfies boundary conditions (2.a,b,c)},\ m = ay(l) - cTy(l)\right\}$$

that everywhere dense in H (see [1]). Hence, problem (1), (2) takes the form of

$$L\hat{y} = \lambda\hat{y}, \tag{3}$$

i.e. the eigenvalues of operator L and of problem (1), (2) coincide together with their multiplicities, and between elements of eigen- and associated vectors chain of operator L and elements of eigen- and associated functions chain of problem (1), (2) which respond to the same eigenvalue, it is can be assigned the one-to-one correspondence (here the first components of chain of eigen- and associated vectors of operator L form chain of eigen- and associated functions of problem (1), (2)).

Problem (1), (2) is strenuously regular in view of [1], in particular this problem has discrete spectrum.

In case of $\sigma > 0$ the operator L will be self-adjoint discrete semi-bounded from below operator in H and so possesses by system of eigenvectors $\{y_n(x), m_n\}$, $n \in \mathbb{N}$ that forms orthogonal basis in H, where $y_n(x), n \in \mathbb{N}$, are eigenfunctions of problem (1), (2) and $m_n = ay_n(l) - cTy_n(l)$.

Let's note that problem (1), (2) in case of $\sigma > 0$ is considered in [3] (see also [4]), where the basicity in space $L_p(0,l)$, $1 < p < \infty$, of system of eigenfunctions with one arbitrary removed function is determined.

In case of $\sigma < 0$, L is nonself-adjoint closed operator in H with compact resolvent.

Let's define operator $J : H \to H$ in the following way: $J\{y,m\} = \{y,-m\}$. The operator J is unitary and symmetric in H, with spectrum consists of two eigenvalues: -1 with multiplicity 1; $+1$ with infinite multiplicity. Hence, this operator generates a Pontragin space $\Pi_1 = L_2(0,l) \oplus \mathbb{C}$ with inter product (J-metric)

$$[\hat{y}, \hat{u}] = [\{y,m\}, \{u,s\}] = (y,u)_{L_2} + \sigma^{-1}m\bar{s}.$$

Theorem 1. *Operator L is J-self-adjoint in Π_1.*

The proof is carried by scheme of proving of proposition 1 from [5] (see also [6]).

Let λ is eigenvalue of operator L of algebraic multiplicity ν. Let's suppose $\rho(\lambda)$ is equal to ν if $\operatorname{Im}\lambda \neq 0$, and equal to whole part $\nu/2$ if $\operatorname{Im}\lambda = 0$.

Theorem 2 [7] *(see also [2,8]). The eigenvalues of operator L arrange symmetrical with regard to real axis.*

$$\sum_{k=1}^{n} \rho(\lambda_k) \leq 1$$

for any system $\{\lambda_k\}_{k=1}^{n}$ $(n \leq \infty)$ of eigenvalues with non-negative imaginary parts. This estimate is accurate.

From theorem 2 it follows that either all the eigenvalues of boundary value problem (1), (2) is simple, or problem (1), (2) has one multiple eigenvalue with multiplicity no greater than three. In case when all eigenvalues of problem (1), (2) is simple or all they are real, or this problem has one pair conjugate nonreal eigenvalues.

Let $\{\lambda_n\}_{n=1}^{\infty}$ $(|\lambda_1| \leq |\lambda_2| \leq ... < |\lambda_n| < ...)$ be sequence of eigenvalues of operator L, $\{\hat{y}_n\}_{n=1}^{\infty}$ is system of root vectors of operator L corresponding to system $\{\lambda_n\}_{n=1}^{\infty}$, i.e. $L\hat{y}_n = \lambda_n\hat{y}_n + \theta_n\hat{y}_{n-1}$, where number θ_n either equals to 0 (in that case $\hat{y}_n$ is eigenvector), or equals to 1 (in that case $\lambda_n = \lambda_{n-1}$ and $\hat{y}_n$ is associated vector) [9].

Let's denote by L^* operator conjugate with operator L in H. By [8; §3] $L^* = JLJ$.

Let $\{\hat{v}_n\}_{n=1}^{\infty}$ $(\hat{v}_n = \{v_n(x), s_n\})$ be system of root vectors of operator L^*, i.e. $L^*\hat{v}_n = \overline{\lambda}_n \hat{v}_n + \theta_{n+1}\hat{v}_{n+1}$. Without loosing generality, we can consider that system $\{\hat{v}_n\}_{n=1}^{\infty}$ is biorthogonally conjugate to the system $\{\hat{y}_n\}$ (for example, in case when all eigenvalues of operator L are simple, $\hat{v}_n = \delta_n^{-1} J\overline{\hat{y}}_n$, where $\delta_n = \int\limits_0^l y_n^2(x)\,dx + \sigma^{-1} m_n^2$, $\overline{\hat{y}}_n = \{\bar{y}(x), \bar{m}_n\}$), i.e. $(\hat{y}_n, \hat{v}_k) = \delta_{nk}$, δ_{nk} is Kroneker's symbol.

Theorem 3. *The system of eigen- and associated vectors $\{\hat{y}_n\}_{n=1}^{\infty}$ of operator L forms Riesz basis in H.*

Proof. Let μ be regular value of operator L. Then problem (3) is adequate to the following problem on eigenvalues

$$(L - \mu I)^{-1}\hat{y} = (\lambda - \mu)^{-1}\hat{y}, \qquad \hat{y} \in D(L).$$

Since L is J-self-adjoint operator in Π_1, $(L - \mu I)^{-1}$ is completely continuous J-self-adjoint operator in Π_1. Then in view of [10] system of root vectors of operator $(L - \mu I)^{-1}$ (hence of operator L) forms Riesz basis in H. Theorem 3 is proved.

Since system $\{\hat{y}_n\}_{n=1}^{\infty}$ is Riesz basis in H, system $\{\hat{v}_n\}_{n=1}^{\infty}$ is also Riesz basis in H [11]. Then for any vector $\hat{f} = \{f, \tau\} \in H$ it holds the expansion

$$\hat{f} = \{f, \tau\} = \sum_{n=1}^{\infty} \left(\hat{f}, \hat{y}_n\right)\hat{v}_n = \sum_{n=1}^{\infty} (\{f, \tau\}, \{y_n, m_n\})\{v_n, s_n\} =$$

$$= \sum_{n=1}^{\infty} \left((f, y_n)_{L_2} + |\sigma|^{-1}\tau\overline{m}_n\right)\{v_n, s_n\},$$

whence the equalities

$$f = \sum_{n=1}^{\infty} \left((f, y_n)_{L_2} + |\sigma|^{-1}\tau\overline{m}_n\right)v_n, \tag{4}$$

$$\tau = \sum_{n=1}^{\infty} \left((f, y_n)_{L_2} + |\sigma|^{-1}\tau\overline{m}_n\right)s_n, \tag{5}$$

follow.

Let $\tau = 0$. Then from (4) and (5) we get, respectively,

$$f = \sum_{n=1}^{\infty} (f, y_n)_{L_2} v_n, \tag{6}$$

$$0 = \sum_{n=1}^{\infty} (f, y_n)_{L_2} s_n. \tag{7}$$

If $s_r \neq 0$, where r is certain natural number, then in view of (4) we have

$$(f, y_r)_{L_2} s_r = -\sum_{\substack{n=1\\ n\neq r}}^{\infty} (f, y_n)_{L_2} s_n. \tag{8}$$

Taking into account (8), from (6) we get

$$f=\sum_{n=1}^{\infty}(f,y_n)_{L_2}v_n=\sum_{\substack{n=1\\n\neq r}}^{\infty}(f,y_n)_{L_2}v_n+(f,y_r)_{L_2}v_r=$$

$$=\sum_{\substack{n=1\\n\neq r}}^{\infty}(f,y_n)_{L_2}v_n-\sum_{\substack{n=1\\n\neq r}}^{\infty}(f,y_n)_{L_2}\frac{s_n}{s_r}v_r=$$

$$=\sum_{\substack{n=1\\n\neq r}}^{\infty}(f,y_n)_{L_2}\left(v_n-\frac{s_n}{s_r}v_r\right).$$

Thus for any $f\in L_2(0,l)$ it holds the expansion

$$f=\sum_{\substack{n=1\\n\neq r}}^{\infty}(f,y_n)_{L_2}\left(v_n-\frac{s_n}{s_r}v_r\right). \tag{9}$$

We have

$$\left(y_n,v_k-s_r^{-1}s_kv_r\right)_{L_2}=(y_n,v_k)_{L_2}-\bar{s}_r^{-1}\bar{s}_k(y_n,v_r)=$$

$$(\hat{y}_n,\hat{v}_k)-|\sigma|^{-1}m_n\bar{s}_k-\bar{s}_r^{-1}\bar{s}_k(\hat{y}_n,\hat{v}_r)+\bar{s}_r^{-1}\bar{s}_k|\sigma|^{-1}m_n\bar{s}_r=\delta_{nk},\ n,k\neq r,$$

i.e. the system $\left\{v_n(x)-s_r^{-1}s_nv_r(x)\right\}_{n=1,n\neq r}^{\infty}$ is biorthogonally conjugate to the system $\{y_n(x)\}_{n=1,n\neq r}^{\infty}$. Hence, the system $\left\{v_n(x)-s_r^{-1}s_nv_r(x)\right\}_{n=1,n\neq r}^{\infty}$ is Riesz basis in $L_2(0,l)$ (see [12]). Then system $\{y_n(x)\}_{n=1,n\neq r}^{\infty}$ is also Riesz basis in $L_2(0,l)$.

Let now $s_r=0$. Since $(\hat{y}_n,\hat{v}_r)=0, n\in\mathbb{N}$, we have $(y_n,v_r)_{L_2}=(\hat{y}_n,\hat{v}_r)-|\sigma|^{-1}m_n\bar{s}_r=0$. So, the function $v_r(x)$ is orthogonal with respect to all functions of system $\{y_n(x)\}_{n=1,n\neq r}^{\infty}$, i.e. system $\{y_n(x)\}_{n=1,n\neq r}^{\infty}$ is not complete in $L_2(0,l)$.

Let $\lambda_p, p\neq r$ be simple eigenvalue of operator L. By condition $\sigma\neq 0$ we have $m_p\neq 0$. Whence, $s_p=-\delta_p^{-1}\bar{m}_p\neq 0$. Then on the base of (9) we have

$$y_p=\sum_{\substack{n=1\\n\neq p}}^{\infty}\left(y_p,v_n-\frac{s_n}{s_p}v_p\right)_{L_2}y_n=\sum_{\substack{n=1\\n\neq p,r}}^{\infty}\left(y_p,v_n-\frac{s_n}{s_p}v_p\right)_{L_2}y_n+$$

$$+\left(y_p,v_r-\frac{s_r}{s_p}v_p\right)_{L_2}y_r=\sum_{\substack{n=1\\n\neq p,r}}^{\infty}\left((y_p,v_n)_{L_2}-\frac{\bar{s}_n}{\bar{s}_p}(y_p,v_p)_{L_2}\right)y_n=$$

$$=\sum_{\substack{n=1\\n\neq p,r}}^{\infty}\left(-|\sigma|^{-1}m_p\bar{s}_n-\frac{\bar{s}_n}{\bar{s}_p}\left(1-|\sigma|^{-1}m_p\bar{s}_p\right)\right)y_n=-\sum_{\substack{n=1\\n\neq p,r}}^{\infty}\frac{\bar{s}_n}{\bar{s}_p}y_n,$$

i.e. system $\{y_n(x)\}_{n=1,n\neq r}^{\infty}$ is nonminimal in $L_2(0,l)$.

Thus, we proved the following

Theorem 4. *If $s_r\neq 0$, the system of root functions $\{y_n(x)\}_{n=1,n\neq r}^{\infty}$ of problem (1), (2) forms a Riesz basis in $L_2(0,l)$. If $s_r=0$, the system $\{y_n(x)\}_{n=1,n\neq r}^{\infty}$ is noncomlete and nonminimal in $L_2(0,l)$.*

Let's note papers [12-15] connected with present paper.

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Ziyatkhan S. Aliyev
Baku State University.
23, Z.Khalilov str., Baku, Azerbaijan.
Tel.: (99412) 439 05 49 (off.)
E-mail: *z_aliyev@mail.ru.*

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