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ON GENERALIZATION OF A CLASS OF THE FIRST PASSAGE TIME OF RANDOM WALK FOR THE LINEAR BOUNDARY

Abstract

In the paper we obtained the integral limit theorems for one closs of the first passage times of random walk for the linear boundary.

1. Introduction. Let ξ_n, $n \geq 1$ be a sequence of independent identically distributed random variable with finite mean $\nu = E\xi_1$ and let the Borel function $\Delta(x)$, $x \in (-\infty, \infty)$ be given. Assume

$$S_n = \sum_{k=1}^{n} \xi_k, \quad S_n = \frac{1}{n} S_n, \quad T_n = n\Delta(S_n)$$

$$\tau = \tau_a = \inf\{n \geq 1 : T_n \geq f_a(n)\}, \tag{1}$$

where $f_a(t)$, $a > 0$, $t > 0$ is some family of nonlinear boundaries there we'll assume that $\inf\{\oslash\} = \infty$.

A series of first passage time in theory of boundary crossing problems for random walks has form (1). For example, assuming in (1) $\Delta(x) = x$, we obtain the following first passage time

$$t_a = \inf\{n \geq 1 : S_n \geq f_a(n)\}$$

which was investigated in the papers [1], [2], [3].

For $f_a(t) = a$ from (1) we have the following form of the first passage time

$$\nu = \inf\left\{n \geq 1 : n\Delta\left(\bar{S}_n\right) \geq a\right\},$$

to whose investigation a lot papers ([4], [5]) were devoted.

Note that statistics in the form of $T_n = n\Delta\left(\bar{S}_n\right)$ arises in testing statistic hypotheses, at that τ_a is the number of observations (sample size) (see [5]).

In the present paper we study the integral limit theorems for τ_a at some suppositions for the functions $\Delta(x)$ and nonlinear boundary $f_a(t)$. Note that the similar problems for t_a have been studied in the paper [3] and for the first passage time ν_a in the papers [4], [5] and [6].

2. Conditions and notation. We'll assume that the function $\Delta(x)$ is positive, twice continuous-differentiable with respect to $x \in (-\infty, \infty)$, moreover $\mu = \Delta(\nu) > 0$ and $\Delta'(\nu) \neq 0$.

For the boundary $f_a(t)$ we'll assume that it satisfies the following conditions:

1) for each a the function $f_a(t)$ increases monotonically, is continuously differentiable for $t > 0$, moreover $f_a(t) \uparrow \infty$, $a \to \infty$.

2) $n = n(a) \to \infty$, $a \to \infty$. Thus $\frac{1}{n} f_a(n) \to \mu$ and $f_a(n) \to \theta$ for some $\theta \in [0, \mu)$.

3) For each a the function $f_a'(t)$ weakly oscillates at infinity, i.e.

$$\frac{f_a'(n)}{f_a'(m)} \to 1 \quad at \quad \frac{n}{m} \to 1, \quad n \to \infty.$$

Denote by $N_a = N_a(\mu)$ a solution of the equation $f_a(n) = n\mu$ which exists for sufficiently large a [3]. We also denote by $\Phi(x)$ and $G_\alpha(x)$ standard normal distribution and stable distribution with the exponent $\alpha \in (0, 2]$, respectively.

3. Formulation of the basic results.

Theorem 1. *Let ξ_n, $n \geq 1$ be a sequence of independent identically distributed random variables with $\sigma^2 = D\xi_2 < \infty$, $\nu = E\xi_1$ and let above mentioned conditions be satisfied for function $\Delta(x)$ and boundary $f_a(t)$.*

Then

$$\lim_{a\to\infty} P\left(\tau_a - N_a \leq \frac{rx}{\lambda}\sqrt{N_a}\right) = \Phi(x), \quad r = \left|\Delta'(\nu)\right|\sigma,$$

where $\lambda = \mu - \theta$.

Corollary 1. *Let the conditions of the theorem be fulfilled and $n = n(a) \to \infty$ as $a \to \infty$ such that*

$$c_n = \frac{f_a(n) - n\mu}{r\sqrt{n}} = O(1).$$

Then

$$\lim_{a\to\infty} \left[P(\tau_a \leq n) - \Phi(-c_n)\right] = 0.$$

Theorem 1 admits the following generalization;

Theorem 2. *Let ξ_n, $n \geq 1$ be a sequence of independent identically distributed random variables with $E|\xi_1| < \infty$, for which there exists normalizing constant $A(n) > 0$ such that*

$$\lim_{n\to\infty} P(S_n - nv \leq xA(n)) = G_\alpha(x), \qquad \alpha \in (1, 2].$$

Then

$$\lim_{a\to\infty} P\left(\frac{\tau_a - N_a}{|\Delta'(v)|A([N_a])} \leq x\right) = 1 - G_\alpha(-\lambda x),$$

where [ ] is a sign of an entire part.

Note that without loss of generality as a sequence $A(n)$ we can assume $A(n) = n^{1/\alpha}L_n$, where $L(x)$, $x > 0$ is some slowly varying function at infinity [8].

Corollary 2. *Let the conditions of theorem 2 be satisfied also as $n = n(a) \to \infty$, $a \to \infty$ such that*

$$c_n = \frac{f_a(n) - n\mu}{|\Delta'(\nu)|A(n)} = O(1).$$

Then

$$\lim_{a\to\infty} \left[P(\tau_a \leq n) - G_\alpha(-c_n)\right] = 0.$$

4. Proof of the basic results. We first note that by virtue of conditions for the function $\Delta(x)$ we have

$$T_n = Z_n + \varepsilon_n, \quad n \geq 1$$

where

$$Z_n = n\Delta(\nu) + n\Delta'(\nu)\left(\overline{S}_n - \nu\right),$$

$$\varepsilon_n = \frac{n}{2}\Delta"(\nu_n)\left(\overline{S}_n - \nu\right)^2$$

and ν_n is an intermediate point between ν and $\overline{S}_n$, $n \geq 1$.

It is obvious Z_n, $n \geq 1$ is one-dimensional random walk with the step

$$X_i = \Delta(\nu) + \Delta'(\nu)(\xi_i - \nu), \quad i \geq 1,$$

i.e.

$$Z_n = X_1 + ... + X_n, \quad EX_1 = \Delta(\nu) > 0.$$

The following concept is very important in theory of boundary crossing problems for random walks with perturbation [5].

Definition. *They say that a sequence of random variables η_n, $n \geq 1$ is slowly changing if the following conditions are satisfied*

$$\frac{1}{n}\max\{|\eta_1|, |\eta_2|, ..., |\eta_n|\} \xrightarrow{P} 0, \quad n \to \infty \tag{2}$$

and for any $\gamma > 0$ there exists $\delta = \delta(\gamma) > 0$ such that

$$P\left\{\max_{1\leq k\leq n\delta}\left|\eta_{n+k} - \eta_n\right| ...\gamma\right\} \to 0, \quad n \to \infty. \tag{3}$$

Observe that (2) holds if $\frac{\eta_n}{n} \to 0$ w.p.1 as $n \to \infty$ and (3) holds if η_n converges to a finite w.p.1 as $n \to \infty$.

For the proof of the theorem we'll need the following lemmas.

Lemma 1. *By fulfilling the condition of the theorem the sequence ε_n, $n \geq 1$ is slowly changing.*

Proof. We should show that the convergences (2) and (3) are satisfied for the sequence ε_n, $n \geq 1$.

It's easy to see that convergence (2) is satisfied, since $\Delta"(\nu_n) \xrightarrow{\text{a.e}} \Delta"(\nu)$ and by virtue of strong law of large numbers

$$\frac{\varepsilon_n}{n} = \frac{1}{2}\Delta"(\nu_n)\left(\overline{S}_n - \nu\right) \xrightarrow{\text{a.e}} 0$$

as $n \to \infty$.

Further we note that the sequence

$$\eta_n = \left(\overline{S}_n - \nu\right)^2 \left(\frac{\overline{S}_n - n\nu}{\sqrt{n}}\right)^2, \quad n \geq 1,$$

as is shown in [5], is uniformly continuous in probability, i.e. for it (3) is satisfied.

Then it follows from lemma 1.4 from the paper [5, p.10] that for the sequence ε_n, $n \geq 1$, relation (3) is also satisfied.

Remark 1. For validity of lemma 1 it suffices to assume that the function $\Delta(x)$ is positive and twice continuously differentiable at some neighbourhood of $(\nu - \varepsilon,\ \nu + \varepsilon)$, $\varepsilon > 0$. Really, we assume

$$A_n = \left\{\omega : \left|\overline{S}_n - \nu\right| < \varepsilon\right\},$$

$$Z_n = n\Delta(\nu) + n\Delta'(\nu)\left(\overline{S}_n - \nu\right)$$

and

$$\varepsilon_n = T_n - Z_n, \quad n \geq 1$$

Then the sequence $\varepsilon_n J_{A_n}$, $n \geq 1$ varies slowly and $\varepsilon_n J_{\overline{A}_n} \overset{\text{a.e}}{\to} 0$, $n \to \infty$, since $J_{\overline{A}_n} \overset{\text{a.e}}{\to} 0$. Here J_A is the indicator of the event A.

Lemma 2. *Let the conditions of the theorem be fulfilled. Then in terms of convergence almost sure as $a \to \infty$ we have*

1) $\tau_a \to \infty$; 2) $\frac{\tau_a}{N_a} \to 1$ 3) $\frac{A(\tau_a)}{A([N_a])} \to 1$.

Proof. It is easy to see that

$$P(\tau_a > n) = P\left(\max_{1 \leq k \leq n}(T_n - f_a(n)) < 0\right) \geq P\left(\max_{1 \leq k \leq n} T_n < f_a(t)\right).$$

Hence it follows that for all $n \geq 1$

$$\lim_{a \to \infty} P(\tau_a > n) = 1$$

Taking into account that the process τ_a increases as a function of a, statement 1) follows from the last equation.

Further, by definition of τ_a we have

$$\frac{T_{\tau_a - 1}}{\tau_a} \leq \frac{f_a(\tau_a)}{\tau_a} \leq \frac{T_{\tau_a}}{\tau_a}$$

$$\frac{Z_{\tau_a - 1} + \varepsilon_{\tau_a}}{\tau_a} \leq \frac{f_a(\tau_a)}{\tau_a} \leq \frac{T_{\tau_a} + \varepsilon_{\tau_a}}{\tau_a}$$

By virtue of the strong law of large numbers

$$\frac{Z_n}{n} \overset{\text{a.e}}{\to} \mu \qquad \text{and} \qquad \frac{\varepsilon_n}{n} \overset{\text{a.e}}{\to} 0$$

as $n \to \infty$. Therefore, from the first part of lemma 2 and from the Richter lemma [7] we obtain that

$$\frac{f_a(\tau_a)}{\tau_a} \overset{\text{a.e}}{\to} \mu, \qquad a \to \infty.$$

Denote

$$\Delta(a) = \frac{f_a(\tau_a)}{\tau_a} - \frac{f_a(N_a)}{N_a}.$$

It is easy to see that

$$\Delta(a) = \frac{\lambda_a(\nu_a)}{\nu_a} \frac{N_a - \tau_a}{\nu_a}$$

where $\lambda_a(t) = f_a(t) - t f_a'(t)$ and ν_a is an intermediate point between τ_a and n_a.

Taking into account that $\Delta(a) \overset{\text{a.e}}{\to} 0$ and

$$\frac{\lambda_a(\nu_0)}{\nu_a^2} \overset{\text{a.e}}{\to} \mu - \theta > 0 \qquad \text{at} \qquad a \to \infty$$

we obtain

$$\delta_a = \frac{n_a - \tau_a}{\nu_a} \overset{\text{a.e}}{\to} 0 \qquad \text{at} \qquad a \to \infty$$

Hence, statement 2) of the proved lemma 2 follows. Statement 3) of lemma 2 follows from statement 2).

Proof of theorem 1. Let

$$\tau_a^* = \frac{\tau_a - N_a}{\sqrt{N_a}}, \qquad \chi_a = T_{\tau_a} - f_a(\tau_a).$$

By definition of τ_a we have

$$\frac{Z_\tau - \mu\tau}{\sqrt{N_a}} = \frac{f_a(\tau) - \mu\tau}{\sqrt{N_a}} + \frac{\chi_a - \varepsilon_\tau}{\sqrt{N_a}} =$$

$$= \frac{f_a(N_a) - \mu\tau}{\sqrt{N_a}} + \frac{f_a(\tau) - f_a(N_a)}{\sqrt{N_a}} + \frac{\chi_a - \varepsilon_\tau}{\sqrt{N_a}} =$$

$$= -\mu\tau_a^* + f_a'(\nu_a)\,\tau_a^* + \frac{\chi_a - \varepsilon_\tau}{\sqrt{N_a}} = \tau^*\left(f_a'(\nu_a) - \mu\right) + \frac{\chi_a - \varepsilon_\tau}{\sqrt{N_a}} \tag{4}$$

where ν_a is some intermediate point between τ_a and N_a. It follows from the conditions of the proved theorem that

$$P(Z_n^* \leq x) \to \Phi(x), \qquad n \to \infty,$$

where

$$Z_n^* = \frac{Z_n - \mu n}{\sigma\,|\Delta'(\nu)|\,\sqrt{n}}.$$

Besides, at made suppositions Anscombe theorem [5] is satisfied, by which

$$P\left(Z_{\tau_a}^* \leq x\right) \to \Phi(x), \qquad a \to \infty, \tag{5}$$

holds. Then for obtaining the statement of the theorem from equality (4), it suffices to show that

$$\frac{\chi_a - \varepsilon_\tau}{\sqrt{N_a}} \xrightarrow{P} 0 \qquad a \to \infty. \tag{6}$$

Really we have

$$0 \leq \chi_a = Z_\tau + \varepsilon_\tau - f_a(\tau) \leq Z_\tau + \varepsilon_\tau - f_a(gr-1) \leq Z_\tau + \varepsilon_\tau - T_{\tau-1} \leq$$

$$\leq Z_\tau + \varepsilon_\tau - Z_{\tau-1} - \varepsilon_{\tau-1} = X_\tau + \Delta\varepsilon_\tau, \quad \Delta\varepsilon_\tau = \varepsilon_\tau - \varepsilon_{\tau=1}. \tag{7}$$

For proof of (6) we should show that

$$\frac{X_\tau}{\sqrt{N_a}} \xrightarrow{P} 0 \;\text{ and }\; \frac{\varepsilon_\tau}{\sqrt{N_a}} \xrightarrow{P} 0 \qquad a \to \infty. \tag{8}$$

It is easy to see that

$$X_n^* = Z_n^* - \sqrt{\frac{n-1}{n}}\,Z_{n-1}^*,$$

where

$$X_n^* = \frac{X_n - \mu}{\sigma\,|\Delta'(\nu)|\,\sqrt{n}}.$$

According to the paper [5] mentioned above the sequence $Z_n^*,\ n \geq 1$ is uniformly continuous in probability, and by virtue of lemma 1.4 from the paper [5], the sequence $X_n^*,\ n \geq 1$ is also uniformly continuous in probability.

Taking into account that $X_n^* \xrightarrow{P} 0,\ n \to \infty$ from the Anscombe theorem [5] we find

$$X_\tau^* \xrightarrow{P} 0, \qquad a \to \infty.$$

Further, $\frac{\varepsilon_n}{\sqrt{n}} \xrightarrow{P} 0$ is easy to be sure that, consequently by the Anscombe theorem $\dfrac{\varepsilon_\tau}{\sqrt{\tau}} \xrightarrow{P} 0,\ a \to \infty.$

Now, taking into account that $f_a'(\nu_a) \to \theta,\ a \to \infty$, from (4) we complete the proof of the theorem.

Proof of corollary 1. We have

$$P(\tau_a \leq n) = P\left(\frac{\lambda}{r}\tau_a^* \leq \frac{n - N_a}{\sqrt{N_a}}\frac{\lambda}{r}\right).$$

It follow from the theorem that

$$P(\tau_a \leq n) - \Phi(b_n) \to 0, \quad a \to \infty,$$

where

$$b_n = \frac{n - N_a}{\sqrt{N_a}}\frac{\lambda}{r}.$$

we prove that

$$\Phi(b_n) - \Phi(-c_n) \to 0, \quad a \to \infty.$$

Indeed

$$c_n = \frac{f_a(n) - n\mu}{r\sqrt{n}} = \frac{f_a(n) - f_a(N_a) - \mu(n - N_a)}{r\sqrt{n}} =$$

$$= \frac{(n - N_a)(f_a'(\gamma_a) - \mu)}{r\sqrt{n}},$$

where γ_a is some point between n and N_a.

It follows from the condition $c_n = O(1)$ that

$$\frac{f_a(n)}{n} - \mu = O\left(\frac{1}{\sqrt{n}}\right)$$

or

$$\frac{f_a(\gamma_a)}{\gamma_a} \to \mu \quad \text{to} \quad a \to \infty$$

Therefore as $a \to \infty$

$$c_n \sim \frac{(n - N_a)}{\sqrt{n}}\frac{\lambda}{r} = -b_n$$

and

$$\Phi(b_n) - \Phi(-c_n) = \Phi'(\delta_n)(b_n + c_n),$$

where δ_n is some point between b_n and c_n. Taking into account that the function $\Phi'^{(x)}$ is bounded, from the last relation we obtain (9).

Proof of theorem 2. The proof is conducted by the scheme of the proof of theorem 1. At that it suffices to note that for the sequence of normalizing sums

$$Z_n^* = \frac{Z_n - \mu n}{|\Delta' (\nu)| A(n)}, \quad n \geq 1$$

by virtue of the Moguorodi from the paper [4] the Anscombe theorem is satisfied.

In order to obtain the statement of theroem 2 from equality (4) we should prove that

$$\frac{X_a - \varepsilon_\tau}{A(\tau_a)} \xrightarrow{P} 0, \quad a \to \infty.$$

For this by virtue of (7) it sufficies to show that

$$\frac{X_\tau}{A(\tau_a)} \xrightarrow{P} 0, \quad \text{and} \quad \frac{\varepsilon_\tau}{A(\tau_a)} \xrightarrow{P} 0. \tag{9}$$

The first relation in (9) follows from lemma 1.4 of the paper [5] and from the equality

$$X_n^* = \frac{X_n - \mu}{|\Delta' (\nu)| A(n)} = Z_n^* - \frac{A(n-1)}{A(n)} Z_{n-1}^*,$$

since

$$X_n^* \xrightarrow{P} 0, \quad a \to \infty.$$

The second relation in (9) follows from the convergence

$$\frac{\varepsilon_n}{A(n)} = \frac{1}{2}\Delta" (\nu_n) \left(\frac{S_n - n\nu}{A(n)} \right)^2 \frac{A(n)}{n} \xrightarrow{P} 0$$

and from the fact that $\frac{\varepsilon_n}{A(n)}$ is uniformly continuous in probability.

Proof of corollary 2 is also conducted with the help of reasoning of the proof of Corollary 1, at that it is necessary to take into account that the density of stable distribution is bounded.

References

[1]. Novikov A.A. *On intersection time of one-sided nonlinear boundaries by sums of independent random variables.* Theoria veroyatnosti i primenenie. 1982, v.XXVII, No 4, pp.643-656. (Russian)

[2]. Zarang C. *A nonlinear renewal theory.* Ann. Probab., 1988, v.16, No2, p.793-824.

[3]. Ragimov F.G. *Asymptotic expansion of distribution of intersection time of nonlinear boundaries.* Teoria veroyatnosti i primemenie. 1992, v.37, pp.580-587. (Russian)

[4]. Ragimov F.G. *Limit theorems for the first passage time of processes with independent increments.* Ph.D. dissertation, M., 1985, 127 p. (Russian)

[5]. Woofroofe M. *Nonlinear renewal theory in sequential analysis.* SIAM, 1982, p.1195.

[6]. Siegmund D. *Sequential analysis and confidence interval.* New-York, Springer-Verlag, 1985.

[7]. Richter W. *Ubertroigung Von geren zanssagen für folgen vonzufallingen intizes.* Teoria veroyatnosti i primenenie, 1965, v.X, No1, pp.82-93.

[8]. Feller V. *Introduction to theory of probabilities and its applications.* M., 1984, v.2. (Russian).

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