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ON ASYMPTOTICS OF PERIODICAL IN TIME SOLUTIONS OF SECOND ORDER NONLINEAR PARABOLIC EQUATIONS IN CYLINDRICAL DOMAINS

Abstract

In the paper we find the asymptotic behavior of solutions in spatial variables of nonlinear parabolic equations of the second order with periodic coefficients in time in cylindrical domains.

Let's consider the equations

$$Lu \equiv \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(\hat{x},t)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n-1} a_i(\hat{x},t)\frac{\partial u}{\partial x_i} = -|u|^{p-1}u(1-u) \qquad (1)$$

in the domain

$$G(0,\infty) = S(0,\infty)\times R_t^1, \text{ where } S(0,\infty) = \{x; \hat{x}\in\omega, 0<x_n<\infty\},$$

with the boundary condition

$$\frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{n} a_{ij}(\hat{x},t)\frac{\partial u}{\partial x_j}\nu_i = 0 \text{ on } \Gamma(0,\infty) = \sigma(0,\infty)\times R_t^1, \qquad (2)$$

where $x=(x_1,...x_{n-1},x_n)$, $\hat{x}=(x_1,...x_{n-1})$, ω is a bounded domain in$R_{\hat{x}}^{n-1}$ with Lipchitzian boundary,

$\sigma(0,\infty) = \{x : \hat{x}\in\partial\omega,\ 0<x_n<\infty\}$, $\nu = (\nu_1,...,\nu_n)$ is a unique normal to $\Gamma(0,\infty)$. $a_{ij}(\hat{x},t)$, $a_i(\hat{x},t)$ are such measurable, bounded, T-periodic with respect to t functions that

$$\nu_1|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(\hat{x},t)\xi_i\xi_j \le \nu_2|\xi|^2$$

for any $\xi\in R^n$, $\hat{x}\in\omega$, $t\in(0,T)$, $\nu_1\nu_2 = const > 0$, $p = const > 1$, $a_{ij} = a_{ji}$, $a_{in}(\hat{x},t)\equiv 0$ for $i<n$, $a_{nn}(\hat{x},t)\equiv 1$.

Let's denote:

$$G_T(a,b) = S(a,b)\times(0,T), \text{ where } S(a,b) = \{x; \hat{x}\in\omega,\ a<x_n<b\};$$

$$\Gamma_T(a,b) = \sigma(a,b)\times(0,T), \text{ where } \sigma(a,b) = \{x; \hat{x}\in\partial\omega,\ a<x_n<b\}.$$

Under $W_2^{1,\frac{1}{2}}(G_T(a,b))$ we'll understand the space of functions $u(x,t)$ such that

$$u(x,t+T) = u(x,t),\ u(x,t)\in W_2^{1,0}(G_T(a,b))$$

and

$$\sum_{k=-\infty}^{+\infty} |k| \int\limits_{S(a,b)} |u_k(x)|^2 dx < \infty, \text{ where } u_k(x) = \frac{1}{T}\int\limits_0^T u(x,t) e^{-ik\frac{2\pi}{T}} dt.$$

The norm in it is determined by the equality

$$\|u\|_{W_2^{1,\frac{1}{2}}(G_T(a,b))} = \left[\|u\|^2_{L_2(G_T(a,b))} + \|\nabla u\|^2_{L_2(G_T(a,b))} + \sum_{k=-\infty}^{+\infty} |k| \int\limits_{S(a,b)} |u_k(x)|^2 dx\right]^{\frac{1}{2}}.$$

Weak solution of problem (1), (2) we'll call the function $u(x,t)$ if for any $k>0$ $u(x,t) \in W_2^{1,\frac{1}{2}}(G_T(0,k)) \cap L_\infty(G_T(0,k))$ and

$$2\pi \sum_{k=-\infty}^{+\infty} (ik) \int\limits_{S(0,k)} u_k(x)\varphi_{-k}(x)\,dx + \int\limits_{G_T(0,k)} \sum_{i,j=1}^{n} a_{ij}(\hat{x},t)\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i} dxdt+$$

$$+\int\limits_{G_T(0,k)} \sum_{i=1}^{n} a_i(\hat{x},t)\frac{\partial u}{\partial x_i}\varphi dxdt = -\int |u|^{p-1} u(1-u)\varphi dxdt$$

for any $\varphi(x,t) \in W_2^{1,\frac{1}{2}}(G_T(0,k))$, $\varphi(\hat{x},0,t)=0$, $\varphi(\hat{x},k,t)=0$.

Note that from the classic results on smoothness of solutions of linear problems [1,2] it follows that $u(x,t)$ is continuous (besides, it satisfies Hölder condition) in $G_T(0,k)$ at any $k>0$.

First, we'll state a series of properties of solutions of linear problems (see [3,4]) which are useful for investigation of nonlinear equation (1).

Let's consider the problem

$$Lu + q(x,t)u = \sum_{i,j=1}^{n} \frac{\partial f_i}{\partial x_j} + f_0 \text{ in } G_T(-\infty,+\infty) \tag{3}$$

$$u(x,t+T) \equiv u(x,t), \ \frac{\partial u}{\partial \nu} \text{ on } \Gamma_T(-\infty,+\infty) \tag{4}$$

with the condition

$$J_h(f_i) \equiv \int\limits_{G_T(-\infty,+\infty)} e^{2hx_n} |f_i|^2 dxdt < \infty,$$

where $f_i(x,t)$, $i=0,1,...,n$, $q(x,t)$ are T-periodic with respect to t functions and h is a such constant that the problem

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i}\left(a_{ij}(\hat{x},t)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n-1} a_i(\hat{x},t)\frac{\partial u}{\partial x_i} + \lambda^2 u = 0 \text{ in } \omega\times(0,T) \tag{5}$$

$$u(x,t+T) \equiv u(x,t), \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\omega \times (0,T) \tag{6}$$

has no eigenvalues such that $\operatorname{Im}\lambda = h$.

I. Let's assume that $q(x,t) \equiv 0$. Then problem (3), (4) has a unique solution in $G_T(-\infty,+\infty)$ such that

$$T_h(u) \equiv \int\limits_{G_T(-\infty,+\infty)} \left[|u|^2 + |\nabla u|^2\right] e^{2hx_n} dxdt \le C_h \sum_{i=0}^{n} J_h(f_i),$$

where $C_h = const$ is independent of $u(x,t)$, $f_i(x,t)$, $i = 0,1,...,n$.

This assertion in more general form is proved in [3,4]. It can be established by means of Fourier transformation with respect to x_n.

II. There will be found such constant $\varepsilon < 0$ (which doesn't depend on f_i, $i = 0,1,...,n$) that if $|q(x,t)| < \varepsilon$ in $G_T(-\infty,+\infty)$ then there exists a unique solution of problem (3), (4) such that

$$T_h(u) \le C_h \sum_{i=0}^{n} J_h(f_i)$$

Assertion II is the corollary of assertion I and Banach theorem from invertibility of operator little different in the norm from invertibile one.

III. Let's assume that $q(x,t) \equiv 0$, $u(x,t)$ are solutions of problem (3), (4) satisfying the relation $T_{h_2}(u) < \infty$

$$J_{h_1}(f_i) < \infty,\ J_{h_2}(f_i) < \infty,\ i = 0,1,...,n$$

and in the band $h_1 < \operatorname{Im}\lambda < h_2$ there is only one eigenvalue λ_0 of problem (5), (6) $h_1 < \operatorname{Im}\lambda_0 < h_2$.

Then

$$u(x,t) = \sum_{j=0}^{k} C_j x_n^j \Phi_j(\hat{x},t) e^{i\lambda_0 x_n} + u_1(x,t),$$

$$T_{h_1}(u_1) \le C \sum_{j=0}^{n} J_{h_1}(f_i),\ C, C_j = const,$$

where $\Phi_0, \Phi_1, ..., \Phi_k$ are eigen and adjoint functions of problem (5), (6) corresponding to the eigenvalue $\lambda = \lambda_0$.

Now we'll find the asymptotics of solution of problem (1), (2) as $x_n \to \infty$.

Let's prove the following auxiliary lemma.

Lemma 1. *If $u(x,t)$ is a solution of problem (1), (2) such that*

$$0 < u(x,t) < \frac{p}{p+1},$$

then

$$\lim_{x_n \to \infty} u(x,t) = 0 \tag{7}$$

Proof. Let's consider the function $\upsilon_h(x_n)$ such that

$$\upsilon_h'' = |\upsilon_h|^{p-1}\upsilon_h(1-\upsilon_n), \quad \upsilon_h(h)=\varepsilon>0,\ \upsilon_h'(h)=0,\ x_n>h \tag{8}$$

For solutions of equation (8)

$$\left(\frac{p+2}{2}\right)^{\frac{1}{2}}\int\limits_{\varepsilon}^{\upsilon_h(x_n)}\left(\left(\frac{p+2}{p+1}-y\right)y^{p+1}-\left(\frac{p+2}{p+1}-\varepsilon\right)\varepsilon^{p+1}\right)^{-\frac{1}{2}}dy = x_n-h \tag{9}$$

is true.

From (8) and (9) it follows that there exists such $k(\varepsilon)$ not depending on h that $\upsilon_h(x_n)=\dfrac{p}{p+1}$ at $x_n=h+k(\varepsilon)$. Putting $\upsilon_h(x_n)=\upsilon_n(2h-x_n)$ let's continue the function $\upsilon_h(x_n)$ to the semi-interval $h-k(\varepsilon)\le x_n<h$.

If (7) isn't fulfilled then there exists the sequence $h_n\to\infty$ such that

$$|u(\hat{x}_m,h_m,t_m)|>\varepsilon,\quad \varepsilon=const>0,$$

Then for $u_m(x,t)=u(x,t)-\upsilon_{h_m}(x_n)$ we have:
$\frac{\partial u_m}{\partial\gamma}=0$ on $\Gamma_T(h_m-k(\varepsilon),h_m+k(\varepsilon))$, $u_m(\hat{x}_m,h_m,t_m)>0$ if

$$u(\hat{x}_m,h_m,t_m)>\varepsilon. \tag{10}$$

$$u_m(x,t)<0 \text{ at } x_n=h_m+k(\varepsilon) \text{ and } x_n=h_m-k(\varepsilon) \tag{11}$$

$$Lu_m+\left(|u|^{p-1}u(1-u)-|\upsilon_{h_m}|^{p-1}(1-\upsilon_{h_m})\right)=0 \tag{12}$$

From (10), (11), (12) it follows that $u_m(x,t)$ attains positive maximum inside of $G_T(h_m-k(\varepsilon),h_m+k(\varepsilon))$. This contradicts the strict maximum principle.

The lemma is proved.

Remark. Assertion of the lemma is also true for any solutions which satisfy the condition

$$0<|u(x,t)|<\frac{p}{p+1}.$$

Let's consider the equation

$$y''=y^p(1-y) \text{ in } (0,+\infty) \tag{13}$$

It is easy to prove that equation (13) has a positive solution such that $\lim\limits_{x_n\to\infty}y(x_n)=0.$

Lemma 2. *If solution of equation (13) is such that* $\lim\limits_{x_n\to\infty}y(x_n)=0$ *then for any* $\varepsilon>0$ *there exist the constants* $M_\varepsilon,\ N_\varepsilon, C_\varepsilon$ *such that*

$$(C_p-\varepsilon)(x_n+M_\varepsilon)^{\frac{2}{1-p}}<y(x_n)<(C_p+\varepsilon)(x_n+N_\varepsilon)^{\frac{2}{1-p}}$$

for any $x_n>C_\varepsilon$, *where* $C_p=\left[\dfrac{2(1+p)}{(1-p)^2}\right]^{\frac{1}{p-1}}$, *and* $M_\varepsilon,\ N_\varepsilon, C_\varepsilon$ *depend on* $y(x_n)$.

Proof. Let

$$\theta(x_n) = (C_p + \varepsilon)(x_n + N_\varepsilon)^{\frac{2}{1-p}}$$

$$\theta''(x_n) = (C_p + \varepsilon)\frac{2(1+p)}{(1-p)^2}(x_n + N_\varepsilon)^{\frac{2p}{1-p}} =$$

$$= (C_p + \varepsilon) C_p^{p-1}(x_n + N_\varepsilon)^{\frac{2p}{1-p}} = \left(\frac{C_p}{C_p + \varepsilon}\right)^{p-1}\theta^p.$$

Since

$$\lim_{x_n \to \infty} y(x_n) = 0 \text{ and } \left(\frac{C_p}{C_p + \varepsilon}\right)^{p-1} < 1,$$

then for any $\varepsilon > 0$ there exists C_ε such that for any $x_n > C_\varepsilon$.

$$1 - y(x_n) > \left(\frac{C_p}{C_p + \varepsilon}\right)^{p-1}$$

Then

$$y'' - \theta'' = y^p(1-y) - \left(\frac{C_p}{C_p + \varepsilon}\right)^{p-1}\theta^p >$$

$$> \left(\frac{C_p}{C_p + \varepsilon}\right)^{p-1} y^p - \left(\frac{C_p}{C_p + \varepsilon}\right)^{p-1}\theta^p =$$

$$= \left(\frac{C_p}{C_p + \varepsilon}\right)^{p-1}(y^p - \theta^p) = \left(\frac{C_p}{C_p + \varepsilon}\right)^{p-1}(y - \theta)\, w(x_n),$$

where $w(x_n) > 0$.

Let's take N_ε such that there be

$$y(x_n) < (C_p + \varepsilon)(x_n + N_\varepsilon)^{\frac{2}{1-p}} \text{ at } x_n = C_\varepsilon.$$

From maximum principle we get that $y(x_n) < (C_p + \varepsilon)(x_n + N_\varepsilon)^{\frac{2}{1-p}}$ for $x_n > C_\varepsilon$.

The lover estimation is analogously obtained.

The lemma is proved.

For solutions of equation (1), (2) satisfying the condition $0 < u(x,t) < \dfrac{p}{p+1}$ from maximum principle the following inequalities are obtained

$$u(x,t) < y(x_n + \delta_1) \text{ for } x_n > X_{\delta_1}, \tag{14}$$

$$u(x,t) > y(x_n + \mu_1) \text{ for } x_n > X_{\mu_1}, \tag{15}$$

moreover δ_1 and μ_1 depend on $u(x,t)$.

Theorem. *Let $u(x,t)$ be a solution of problem (1), (2) in $G_T(0,\infty)$ and $0 < u(x,t) < \dfrac{p}{p+1}$. Then*

$$u(x,t) = y(x_n + \mu) + O\left(e^{-\alpha x_n}\right) \text{ at } x_n \to \infty,$$

where $y(x_n)$ is a solution of equation (13), $\alpha, \mu = const > 0$, moreover μ depends on $u(x,t)$.

Proof. From (14) it follows that the set of nonnegative δ such that

$$u(x,t) \le y(x_n+\delta) \text{ for } x_n > X_\delta \text{ is not empty}$$

Let's denote $\delta_0 = \sup \delta$. From (15) it follows that $\delta_0 \le \mu_1 < \infty$. From determination of δ_0 it is obtained that

$$u(x,t) \le y(x_n+\delta_0-\varepsilon)$$

for any $\varepsilon > 0$ and for $x_n > X_\varepsilon$ and

$$u(x^\varepsilon, t^\varepsilon) \ge y(x_n^\varepsilon + \delta_0 + \varepsilon), \tag{16}$$

where $x_n^\varepsilon \to \infty$.

Let's consider the function

$$\upsilon(x,t) = u(x,t) - y(x_n+\delta_0).$$

For $\upsilon(x,t)$ we have

$$\upsilon(x,t) \le y(x_n+\delta_0-\varepsilon) - y(x_n+\delta_0) = y'\left(x_n+\delta_0-\varepsilon'\right)\varepsilon = \overline{\overline{o}}\left(x_n^{\frac{1+p}{1-p}}\right).$$

Since

$$y' = -\sqrt{\frac{2}{p+1}y^{p+1} - \frac{2}{p+2}y^{p+2}},$$

then

$$\upsilon_+(x,t) = \overline{\overline{o}}\left(x_n^{\frac{1+p}{1-p}}\right), \text{ as } x_n \to \infty.$$

Since $\upsilon(x,t) = u(x,t) - y(x_n+\delta_0) \ge -y(x_n+\delta_0)$, then

$$\upsilon_-(x,t) = O\left(x_n^{\frac{2}{1-p}}\right), \text{ as } x_n \to \infty.$$

Let's determine the function $z(x,t)$ by the following way

$$z(x,t) = \upsilon(x,t)(x_n+\delta_0)^{-\frac{2}{1-p}}.$$

The function $z(x,t)$ satisfies the equation

$$\frac{\partial z}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial z}{\partial x_j}\right) + \sum_{i=1}^{n-1} a_i z + b_1(x_n)\frac{\partial z}{\partial x_n} + b_2(x_n) z = 0,$$

where $b_1(x_n) = O\left(\frac{1}{x_n}\right)$, $b_2(x_n) = O\left(\frac{1}{x_n^2}\right)$ as $x_n \to \infty$.

From determination of $z(x,t)$ we can get the following estimations:

$$z(x,t) = \upsilon(x,t)(x_n+\delta_0)^{-\frac{2}{1-p}} =$$

$$=\frac{1}{(x_n+\delta_0)^{\frac{2}{1-p}}}\left[u(x,t)-y(x_n+\delta_0)\right]\geq$$

$$\geq (x_n+\delta_0)^{-\frac{2}{1-p}}\left[y(x_n+\gamma_1)-y(x_n+\delta_0)\right]=$$

$$=(x_n+\delta_0)^{-\frac{2}{1-p}}(\gamma_1-\delta_0)\,y'\left(x_n+\delta'\right)=O\left(\frac{1}{x_n}\right),$$

$$z(x,t)=(x_n+\delta_0)^{-\frac{2}{1-p}}\left[u(x,t)-y(x_n+\delta_0)\right]\leq$$

$$\leq (x_n+\delta_0)^{-\frac{2}{1-p}}\left[y(x_n+\delta_0+\varepsilon)-y(x_n+\delta_0)\right]=\overline{\overline{o}}\left(\frac{1}{x_n}\right).$$

So,

$$|z(x,t)|\leq O\left(\frac{1}{x_n}\right),\ z_+(x,t)=\overline{\overline{o}}\left(\frac{1}{x_n}\right). \tag{17}$$

Let's prove that $z_-(x^\varepsilon,t^\varepsilon)=\overline{\overline{o}}\left(\frac{1}{x_n^\varepsilon}\right)$. Allowing for (16) we obtain

$$z(x^\varepsilon,t^\varepsilon)=\left[u(x^\varepsilon,t^\varepsilon)-y(x_n^\varepsilon+\delta_0)\right](x_n^\varepsilon+\delta_0)^{-\frac{2}{1-p}}\geq$$

$$\geq\left[y(x_n^\varepsilon+\delta_0+\varepsilon)-y(x_n^\varepsilon+\delta_0)\right](x_n^\varepsilon+\delta_0)^{-\frac{2}{1-p}}=$$

$$=y'\left(x_n^\varepsilon+\delta_0+\varepsilon'\right)(+\varepsilon)(x_n^\varepsilon+\delta_0)^{-\frac{2}{1-p}}.$$

So,

$$z_-(x^\varepsilon,t^\varepsilon)=\overline{\overline{o}}\left(\frac{1}{x_n^\varepsilon}\right)$$

Let's prove that $|z(x,t)|=\overline{\overline{o}}\left(x_n^{-1}\right)$, as $x_n\to\infty$.

Let's consider the function

$$\omega(x,t)=-z(x,t)+\frac{\varepsilon}{x_n}.$$

Since $z_+(x,t)=\overline{\overline{o}}\left(x_n^{-1}\right)$, then

$$\omega(x,t)\geq\frac{\varepsilon}{2x_n}\ \text{for}\ x_n>X_\varepsilon.$$

For the sequence $(x^\varepsilon,t^\varepsilon)$ we have

$$\omega(x^\varepsilon,t^\varepsilon)\leq 2\varepsilon\left(x_n^\varepsilon\right)^{-1}.$$

The function $\omega(x,t)$ satisfies the equation

$$L\omega+b_1\frac{\partial\omega}{\partial x_n}+b_2\omega+2\varepsilon x_n^{-3}+b_1\varepsilon x_n^{-2}-b_2\varepsilon x_n^{-1}=0 \tag{18}$$

in $G_T(0,\infty)$ and $\frac{\partial\omega}{\partial\gamma}=0$ on $\Gamma_T(0,\infty)$, $\omega(x,t+T)\equiv\omega(x,t)$.

Let ω_0 be T-periodic with respect to t solution of equation (18) in $G_T(k-2,k+2)$ with boundary conditions

$$\omega_0(\hat{x},k-2,t)=0,\ \omega_0(\hat{x},k+2,t)=0,\ \frac{\partial\omega_0}{\partial\gamma}=0 \text{ on } \Gamma_T(k-2,k+2). \tag{19}$$

At large k such solution ω_0 exists, since $b_2(x_n)\to 0$ as $x_n\to\infty$.

Let's consider the function $w(x,t)=\omega(x,t)-\omega_0(x,t)$.in $G_T(k-2,k+2)$.

For $w(x,t)$ we have

$$Lw+b_1\frac{\partial w}{\partial x_n}+b_2w=0 \text{ in } G_T(k-2,k+2)$$

$$\frac{\partial w}{\partial\gamma}=0 \text{ on } \Gamma_T(k-2,k=2),\ w=\omega \text{ for } x_n=k-2,k+2.$$

Since $b_2(x_n)$ is sufficiently small in $G_T(k-2,k+2)$ at large k then from theory of linear parabolic equations (see [1]) the following estimation is known for solution ω_0 of problem (18), (19):

$$|\omega_0(x,t)|\le C\left(\max_{\substack{x_n=k+2\\ x_n=k-2}}|u|+C_1\sup_{G_T(k-2,k=2)}|f|\right),\ C,C_1=const>0,$$

where $f(x,t)=-2\varepsilon x_n^{-3}-b_1\varepsilon x_n^{-2}+b_2\varepsilon_n^{-1}$.

Then

$$w(x,t)\ge\frac{\varepsilon}{2x_n}-\overline{\overline{o}}\left(x_n^{-3}\right)\ge\frac{\varepsilon}{4x_n} \text{ in } G_T(k-2,k+2)$$

So,

$$w(x,t)>0 \text{ in } G_T(k-2,k+2).$$

Let's apply to $w(x,t)$ the Harnack inequality (see [1]). Then we'll get that

$$w(x,t)\le Cw(x^\varepsilon,t^\varepsilon)\le C_1\frac{\varepsilon}{x_n^3} \text{ for any } (x,t)\in G_T(k-2,k+2)\cap\{x_n=x_n^\varepsilon\}.$$

Since $\omega=w+\omega_0$, then

$$|\omega|\le C_1\varepsilon\left(x_n^\varepsilon\right)^{-1}+C_2\left(x_n^3\right)^{-3}\le\overline{\overline{o}}\left(x_n^{-1}\right) \text{ for } x_n=X_n^\varepsilon$$

Therefore

$$|\omega|\le C_1\varepsilon x_n^{-1}+C_2\overline{\overline{o}}\left(x_n^{-\varepsilon}\right)^{-3}\le\overline{\overline{o}}\left(x_n^{-1}\right) \text{ for } x_n>X_\varepsilon.$$

Since

$$z(x,t)=\frac{\varepsilon}{x_n}-\omega(x,t),$$

then

$$z(x,t)\ge\frac{\varepsilon}{x_n}-\overline{\overline{o}}\left(x_n^{-1}\right)=\overline{\overline{o}}\left(x_n^{-1}\right).$$

Hence

$$z_-(x,t)=\overline{\overline{o}}\left(x_n^{-1}\right). \tag{20}$$

From (17) and (20) we conclude that

$$|z(x,t)| = \overline{\overline{o}}\left(x_n^{-1}\right).$$

Hence

$$v(x,t) = u(x,t) - y(x_n + \delta_0) = z(x,t)(x_n + \delta_0)^{\frac{2}{1-p}} = \overline{\overline{o}}\left(x_n^{\frac{1+p}{1-p}}\right). \tag{21}$$

The function $v(x,t)$ satisfies the equation

$$Lv = -g(x,t)\,v,$$

where $g(x,t) = \dfrac{|u|^{p-1}u(1-u) - y^p(1-y)}{u-y}$.

According to (21)

$$g(x,t) = \theta^{p-1}(p - (p+1)\theta), \text{ where } \theta = y + \overline{\overline{o}}\left(x_n^{\lambda}\right), \ \lambda = \frac{1+p}{1-p}.$$

$$g(x,t) = \left(y + \overline{\overline{o}}\left(x_n^{\lambda}\right)\right)^{p-1}\left(p - (p+1)\left(y + \overline{\overline{o}}\left(x_n^{\lambda}\right)\right)\right) =$$

$$= y^{p-1}\left(1 + \overline{\overline{o}}\left(x_n^{-1}\right)\right)^{p-1}\left(p - (p+1)y^{p-1}\left(1 + \overline{\overline{o}}\left(x_n^{-1}\right)\right)\right) =$$

$$= y^{p-1}\left(1 + \overline{\overline{o}}\left(x_n^{-1}\right)\right)\left(p - (p+1)y\left(1 + \overline{\overline{o}}\left(x_n^{-1}\right)\right)\right) =$$

$$= y^{p-1}(p - (p+1)y) + \overline{\overline{o}}\left(x_n^{-3}\right).$$

Let's consider the function $V(x,t) = S(x_n)\,v(x,t)$, where $S(s) = 0$ for $s < \tau, \tau = const > 0$, $S(s) = 1$ for $s > \tau + 1$. $S(s) \in C^{\infty}\left(R^1\right)$.

The function $V(x,t)$ satisfies the equation

$$L_1 V \equiv \frac{\partial V}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(\hat{x},t)\frac{\partial V}{\partial x_j}\right) +$$

$$+\sum_{i=1}^{n-1} a_i(\hat{x},t)\frac{\partial V}{\partial x_j} + g_1(x,t)\,V = f(x,t), \tag{22}$$

in $G_T(-\infty,+\infty)$, $\dfrac{\partial V}{\partial \gamma} = 0$ on $\Gamma_T(-\infty,+\infty)$, $V(x,t+T) \equiv V(x,t)$. $f(x,t)$ has a compact support and $g_1(x,t) = g(x,t)$ for $x_n > \tau + 1$.

Using assumptions II and III we get that equation (22) has solution $V_0(x,t)$ in $G_T(-\infty,+\infty)$ which satisfies the following relations

$$V_0(x,t) = O\left(e^{-\alpha x_n}\right), \ \alpha = const > 0 \text{ at } x_n \to \infty, \tag{23}$$

$$V_0(x,t) = a x_n + b + O\left(e^{\alpha x_n}\right), \ a, b, \alpha = const > 0 \text{ at } x_n \to -\infty. \tag{24}$$

Using for $V_0(x,t)$ the maximum estimation of modulus and the estimation $T_h(u) \le C_h \sum_{i=0}^{n} J_h(f_i)$, we get

$$\max_{G_T(h-1,h+1)} |V_0(x,t)|^2 \le C \int_{G_T(h-2,h+2)} |V_0|^2 \, dxdt \le$$

$$\le Ce^{-2\alpha(h-2)} \int_{G_T(h-2,h+2)} |V_0|^2 e^{2\alpha x_n} dxdt \le C_3 e^{-2\alpha h}.$$

Thus $|V_0(x,t)| \le C_4 e^{-2\alpha x_n}$, $C_1, C_3, C_4 = const > 0$.

Let's consider the function

$$V_1(x,t) = V(x,t) - V_0(x,t).$$

From (21), (23) it follows that $V_1(x,t) = \overline{\overline{o}}\left(x_n^{\lambda}\right)$ as $x_n \to +\infty$ and according to (24) $V_1(x,t) = ax_n + b + O\left(e^{\alpha x_n}\right)$ as $x_n \to -\infty$, $a, b, \alpha = const > 0$. We'll prove that $a = 0$, $b = 0$. Let's assume that $a < 0$. The function $V_1(x,t)$ satisfies the equation

$$L_1 V_1 = 0 \text{ in } G_T(-\infty, +\infty).$$

Since $a < 0$, there exists a constant $M < 0$ such that $V_1(x,t) > 0$ at $x_n < M$. Then from $\lim V_1(x,t) = 0$ as $x_n \to \infty$ and maximum principle $V_1(x,t) > 0$ in $G_T(-\infty, +\infty)$.

Let's consider the function

$$Y_1(x_n) = Y(x_n) + N x_n^{\lambda - \frac{1}{2}}, \; N = const > 0,$$

where $Y(x_n) = x_n^{\lambda}$, $\lambda = \dfrac{1+p}{1-p}$. It is easy to check that $Y(x_n) = x_n^{\lambda}$ satisfies the equation

$$Y'' - pC_p^{p-1} x_n^{-2} Y = 0 \text{ at } x_n > 0.$$

For $Y_1(x_n)$ the inequality

$$L_1 Y_1 \le 0 \text{ in } G_T(0, \infty).$$

is true.

Really,

$$L_1 Y_1 = -Y'' - \left(N x_n^{\lambda - \frac{1}{2}}\right)'' + g_1(x,t) Y + g_1(x,t) N x_n^{\lambda - \frac{1}{2}} =$$

$$= -Y'' - N\left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{3}{2}\right) x_n^{\lambda - \frac{5}{2}} - y^{p-1}\left(p - (p+1) y\right) Y +$$

$$+ y^{p-1}\left(p - (p+1) y\right) N x_n^{\lambda - \frac{1}{2}} + O\left(x_n^{-3\lambda}\right) \le$$

$$\leq -Y'' - N\left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{3}{2}\right)x_n^{\lambda - \frac{5}{2}} +$$

$$+py^{p-1}Y + py^{p-1}Nx_n^{\lambda - \frac{1}{2}} + O\left(x_n^{-3+\lambda}\right) \leq$$

$$\leq -Y'' - N\left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{3}{2}\right)x_n^{\lambda - \frac{5}{2}} +$$

$$+pC_p^{p-1}x_n^{-2}Y + pC_p^{p-1}x_n^{-2}Nx_n^{\lambda - \frac{1}{2}} +$$

$$+O\left(x_n^{-3+\lambda}\right) \leq -NC_0\left(x_n^{\lambda - \frac{5}{2}}\right) + O\left(x_n^{-3+\lambda}\right) \leq 0,$$

if $x_n > 0$ and N is sufficiently large, $C_0 = const > 0$.

Let's consider the function

$$E = CV_1 - Y_1.$$

We have

$$L_1E = -L_1Y_1 \geq 0 \text{ and } \lim E(x,t) = 0 \text{ as } x_n \to \infty.$$

Having taken C sufficiently large we can get that $E > 0$ at $x_n = 1$ since $V_1(x,t) > 0$ in $G_T(-\infty, +\infty)$.

From maximum principle it follows that $E(x,t) > 0$ in $G_T(1, +\infty)$. Finally, we get

$$CV_1(x,t) \geq x_n^{\lambda} + Nx_n^{\lambda - \frac{1}{2}} \text{ in } G_T(1, +\infty).$$

This contradicts to $V_1(x,t) = \overline{\overline{o}}\left(x_n^{\lambda}\right)$ as $x_n \to \infty$. It means that inequality $a < 0$ isn't true. By the same way we'll get contradiction if we'll assume that $a > 0$ and $b \neq 0$. Since $a = 0$, $b = 0$ then $\lim V_1(x,t) = 0$ as $x_n \to -\infty$ and $x_n \to +\infty$. According to maximum principle we obtain that $V_1(x,t) \equiv 0$ in $G_T(-\infty, +\infty)$. So, $V(x,t) \equiv V_0(x,t)$. Since $V(x,t) \equiv \upsilon(x,t)$ at $x_n > \tau + 1$ and $V_0(x,t) = O\left(e^{-\alpha x_n}\right)$ as $x_n \to \infty$ then from $u(x,t) = \upsilon(x,t) + y(x_n + \delta_0)$ we obtain

$$u(x,t) = y(x_n + \delta_0) + O\left(e^{-\alpha x_n}\right), \text{ as } x_n \to \infty.$$

The theorem is proved.

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